On matchings in hypergraphs

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Abstract

We show that if the largest matching in a k-uniform hypergraph G on n vertices has precisely s edges, and $n > 3k^2 s/2\log k$, then H has at most $\binom{n}{k} - \binom{n-s}{k}$ edges and this upper bound is achieved only for hypergraphs in which the set of edges consists of all k-subsets which intersect a given set of s vertices.

A *k-uniform hypergraph* $G = (V, E)$ is a set of vertices $V \subseteq \mathbb{N}$ together with a family E of k-element subsets of V, which are called edges. In this note by $v(G) = |V|$ and $e(G) = |E|$ we denote the number of vertices and edges of $G = (V, E)$, respectively. By a *matching* we mean any family of disjoint edges of G, and we denote by $\mu(G)$ the size of the largest matching contained in E. Moreover, by $\nu_k(n, s)$ we mean the largest possible number of edges in a k-uniform hypergraph G with $v(G) = n$ and $\mu(G) = s$, and by $\mathcal{M}_k(n, s)$ we denote the family of the extremal hypergraphs for this problems, i.e. $H \in \mathcal{M}_k(n, s)$ if $v(H) = n$, $\mu(H) = s$, and $e(H) = \nu_k(n, s)$. In 1965 Erdős [2] conjectured that, unless $n = 2k$ and $s = 1$, all graphs from $\mathcal{M}_k(n, s)$ are either cliques, or belong to the family $\mathrm{Cov}_k(n, s)$ of hypergraphs on n vertices in which the set of edges consists of

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all k-subsets which intersect a given subset $S \subseteq V$, with $|S| = s$. This conjecture, which is a natural generalization of Erdős-Gallai result [3] for graphs, has been verified only for $k = 3$ (see [5] and [8]). For general k there have been series of results which state that

$$\mathcal{M}_k(n,s) = \mathrm{Cov}_k(n,s) \quad \text{for} \quad n \geqslant g(k)s, \tag{1}$$

where $g(k)$ is some function of k. The existence of such $g(k)$ was shown by Erdős [2], then Bollobás, Daykin and Erdős [1] proved that (1) holds whenever $g(k) \geqslant 2k^3$; Frankl and Füredi [6] showed that (1) is true for $g(k) \geqslant 100k^2$ and recently, Huang, Loh, and Sudakov [7] verified (1) for $g(k) \geqslant 3k^2$. The main result of this note slightly improves these bounds and confirms (1) for $g(k) \geqslant 2k^2/\log k$.

Theorem 1. *If $k \geqslant 3$ and*

$$n > \frac{2k^2 s}{\log k}, \tag{2}$$

then $\mathcal{M}_k(n,s) = Cov_k(n,s)$.

In the proof we use the technique of shifting (for details see [4]). Let $G = (V,E)$ be a hypergraph with vertex set $V = \{1,2,\ldots,n\}$, and let $1 \leqslant i < j \leqslant n$. The hypergraph $\mathbf{sh}_{i,j}(G)$ is obtained from G by replacing each edge $e \in E$ such that $j \in e$, $i \notin e$ and $e_{ij} = e \setminus \{j\} \cup \{i\} \notin E$, by e_{ij}. Let $\mathbf{Sh}(G)$ denote the hypergraph obtained from G by the maximum sequence of shifts, such that for all possible i,j we have $\mathbf{sh}_{ij}(\mathbf{Sh}(G)) = \mathbf{Sh}(G)$. It is well known and not hard to prove that the following holds (e.g. see [4] or [8]).

Lemma 2. *$G \in \mathcal{M}_k(n,s)$ if and only if $\boldsymbol{Sh}(G) \in \mathcal{M}_k(n,s)$.*

Lemma 3. *Let $G \in \mathcal{M}_k(n,s)$ and $n \geqslant 2k+1$. Then $G \in Cov_k(n,s)$ if and only if $\boldsymbol{Sh}(G) \in Cov_k(n,s)$.*

Thus, it is enough to show Theorem 1 for hypergraphs G for which $\mathbf{Sh}(G) = G$. Let us start with the following observation.

Lemma 4. *If G is a hypergraph on vertex set $[n]$ such that $\boldsymbol{Sh}(G) = G$ and $\mu(G) = s$, then*

$$G \subseteq \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup A_k\,,$$

where

$$\mathcal{A}_i = \{A \subseteq [n] : |A| = k,\ |A \cap \{1,2,\ldots,i(s+1)-1\}| \geqslant i\},$$

for $i = 1,2,\ldots,k$.

Proof. Note that the set $e_0 = \{s+1, 2s+2, \ldots, ks+k\}$ is not an edge of G. Indeed, in such a case each of the edges $\{i, i+s+1, \ldots, i+(k-1)(s+1)\}$, $i = 1,2,\ldots,s+1$, belongs to G due to the fact that $G = \mathbf{Sh}(G)$ and, clearly, they form a matching of size $s+1$. Now it is enough to observe that all sets which do not dominate e_0 must belong to $\bigcup_{i=1}^k \mathcal{A}_i$. □

The following numerical consequence of the above result is crucial for our argument.

Lemma 5. *Let G be a hypergraph with vertex set $\{1, 2, \dots, n\}$ such that $\boldsymbol{Sh}(G) = G$ and $\mu(G) = s$, where $n \geqslant k(s+1)-1$. Then all except at most $\frac{s(s+1)}{2}\binom{n-1}{k-2}$ edges of G intersect $\{1, 2, \dots, s\}$.*

Proof. Let $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_i$. Observe first that $|\mathcal{A}| = s\binom{n}{k-1}$, for $n \geqslant k(s+1)-1$. Indeed, it follows from an easy induction on k, and then on n. For $k = 1$ it is obvious. For $k \geqslant 1$ and $n = k(s+1)-1$ we have clearly $|\mathcal{A}| = \binom{n}{k} = s\binom{n}{k-1}$. Now let $k \geqslant 2$, $n \geqslant k(s+1)$ and split all the sets of $\mathcal{A}$ into those which contain n and those which do not. Then, the inductional hypothesis gives

$$|\mathcal{A}| = s\binom{n-1}{k-2} + s\binom{n-1}{k-1} = s\binom{n}{k-1}.$$

Observe also that $\binom{n}{k} = \sum_{i=1}^{s}\binom{n-i}{k-1} + \binom{n-s}{k}$, which is a direct consequence of the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Thus, using Lemma 4 and the above observation, the number of edges of G which do not intersect $\{1, 2, \dots, s\}$ can be bounded in the following way.

$$\begin{aligned}
|G| - |G \cap \mathcal{A}_1| \leqslant |\mathcal{A}| - |\mathcal{A}_1| &= s\binom{n}{k-1} - \left[\binom{n}{k} - \binom{n-s}{k}\right] \\
&= s\left[\sum_{i=1}^{s}\binom{n-i}{k-2} + \binom{n-s}{k-1}\right] - \sum_{i=1}^{s}\binom{n-i}{k-1} \\
&= s\sum_{i=1}^{s}\binom{n-i}{k-2} - \sum_{i=1}^{s}\sum_{j=1}^{s-i}\binom{n-i-j}{k-2} \\
&= s\sum_{i=1}^{s}\binom{n-i}{k-2} - \sum_{i=2}^{s}(i-1)\binom{n-i}{k-2} \\
&= \sum_{i=1}^{s}(s-i+1)\binom{n-i}{k-2} \leqslant \sum_{i=1}^{s} i\binom{n-1}{k-2} \\
&= \frac{s(s+1)}{2}\binom{n-1}{k-2}.
\end{aligned}$$

□

Proof of Theorem 1. Let us assume that (2) holds for $G \in \mathcal{M}_k(n, s)$. Then, by Lemma 2, the hypergraph $H = \mathbf{Sh}(G)$ belongs to $\mathcal{M}_k(n, s)$. We shall show that $H \in \mathrm{Cov}_k(n, s)$ which, due to Lemma 3, would imply that $G \in \mathrm{Cov}_k(n, s)$. Our argument is based on the following two observations. Here and below by the degree $\deg(i)$ of a vertex i we mean the number of edges containing i, and by V and E we denote the sets of vertices and edges of H respectively.

Claim 6. *If $s \geqslant 2$, then $\{1, ks+2, ks+3, \dots, ks+k\} \in E$.*

Proof. Let us assume that the assertion does not hold. We shall show that then H has fewer edges than the graph $H' = (V, E')$ whose edge set consists of all k-subsets intersecting $\{1, 2, \ldots, s\}$. Let $E_i = \{\{i\} \cup e' : e' \subset \{ks+2, \ldots, n\}, |e'| = k-1\}$, $i \in [s]$ and observe that the sets E_i are pairwise disjoint and $|E_i| = \binom{n-ks-1}{k-1}$ for every $i \in [s]$. Moreover, since $H = \mathbf{Sh}(H)$ and $\{1, ks+2, ks+3, \ldots, ks+k\} \notin E$, $E_1 \cap E = \emptyset$, and so $E_i \cap E = \emptyset$ for every $i \in [s]$. Thus,

$$\begin{aligned} |E' \setminus E| &\geqslant s\binom{n-ks-1}{k-1} \\ &\geqslant \frac{s(n-1)_{k-1}}{(k-1)!}\Big(1 - \frac{ks}{n-k+1}\Big)^{k-1}, \end{aligned} \tag{3}$$

while from Lemma 5 we get

$$\begin{aligned} |E \setminus E'| &\leqslant \frac{s(s+1)}{2}\binom{n-1}{k-2} = \frac{s(n-1)_{k-1}}{(k-1)!}\frac{(s+1)(k-1)}{2(n-k+1)} \\ &\leqslant \frac{s(n-1)_{k-1}}{(k-1)!}\frac{ks}{n-k+1}. \end{aligned} \tag{4}$$

Thus,

$$e(H') - e(H) \geqslant \frac{s(n-1)_{k-1}}{(k-1)!}\left(\Big(1 - \frac{ks}{n-k+1}\Big)^{k-1} - \frac{ks}{n-k+1}\right).$$

Let $x = ks/(n-k+1)$. It is easy to check that for all $k \geqslant 3$ and $x \in (0, 0.7\log k/k)$ we have

$$(1-x)^{k-1} > x\,.$$

Thus, $e(H') - e(H) > 0$ provided $k^2 s < 0.7\log k(n-k+1)$, which holds whenever $n \geqslant 2sk^2/\log k$. Thus, since clearly $\mu(H') = s$, we arrive at contradiction with the assumption that $H \in \mathcal{M}_k(n, s)$. □

Claim 7. *If $s \geqslant 2$ then $\deg(1) = \binom{n-1}{k-1}$. In particular, the hypergraph H^-, obtained from H by deleting the vertex 1 together with all edges it is contained in, belongs to $\mathcal{M}_k(n-1, s-1)$.*

Proof. Let us assume that there is a k-subset of V, which contains 1 and is not an edge in H. Then, in particular, $e = \{1, n-k+2, \ldots, n\} \notin E$. Let us consider hypergraph $\bar{H}$ obtained from H by adding e to its edge set. Since $H \in \mathcal{M}_k(n, s)$, there is a matching of size $s+1$ in $\bar{H}$ containing e. Hence, as $H = \mathbf{Sh}(H)$, there exists a matching M in H such that $M \subset \{2, \ldots, ks+1\}$. Note however that, by Claim 6, $f = \{1, ks+2, ks+3, \ldots, ks+k\} \in E$. But then $M' = M \cup \{f\}$ is a matching of size $s+1$ in H, contradicting the fact that $H \in \mathcal{M}_k(n, s)$. Hence, we must have $\deg(1) = \binom{n-1}{k-1}$. Since $n \geqslant ks$, the second part of the assertion is obvious. □

Now Theorem 1 follows easily from Claim 7 and the observation that, since $\frac{s-1}{n-1} \leqslant \frac{s}{n}$, if (2) holds then it holds also when n is replaced by $n-1$ and s is replaced by $s-1$. Thus, we can reduce the problem to the case when $s = 1$ and use Erdős-Ko-Rado theorem (note that then $n > 2k^2/\log k > 2k+1$). □

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