Binomial edge ideals of graphs[*][†]

Sara Saeedi Madani

Department of Pure Mathematics
Faculty of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic)
424, Hafez Ave., Tehran 15914, Iran

`sarasaeedi@aut.ac.ir`

Dariush Kiani

Department of Pure Mathematics
Faculty of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic)
424, Hafez Ave., Tehran 15914, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran

`dkiani@aut.ac.ir`

Abstract

We characterize all graphs whose binomial edge ideals have a linear resolution. Indeed, we show that complete graphs are the only graphs with this property. We also compute some graded components of the first Betti number of the binomial edge ideal of a graph with respect to the graphical terms. Finally, we give an upper bound for the Castelnuovo-Mumford regularity of the binomial edge ideal of a closed graph.

1 Introduction

Binomial edge ideals of graphs were introduced in [4]. Let G be a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Also, let $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring over a field k. Then the **binomial edge ideal** of G in S, denoted by J_G, is generated by binomials of the form $f_{ij} = x_i y_j - x_j y_i$, where $i < j$ and $\{v_i, v_j\}$ is an

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edge of G. This ideal also could be seen as the ideal generated by a collection of 2-minors of a $(2 \times n)$-matrix whose entries are all indeterminates. In [4], the authors characterized those graphs, which, for certain labeling of their edges, have a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. These graphs are called **closed** graphs. In [1], the authors studied the depth of classes of binomial edge ideals and classified all closed graphs whose binomial edge ideals are Cohen-Macaulay.

Associated to the graph G is a monomial ideal $I(G) = (x_i x_j : \{v_i, v_j\} \in E(G))$, in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k, called the **edge ideal** of G. In [2], Fröberg characterized all graphs whose edge ideals have a linear resolution. He showed that $I(G)$ has a linear resolution if and only if the complementary graph $\overline{G}$ is chordal. It is natural to ask a similar question about the binomial edge ideal of a graph. More precisely, one could ask wether there is a graphical characterization for binomial edge ideals to have a linear resolution or not. In this paper, we give the positive answer to this question. Actually, in Section 1, we prove that J_G has a linear resolution if and only if J_G has linear relations if and only if G is a complete graph. Moreover, it is well-known that if $\mathrm{in}_<(J_G)$ has a linear resolution, then J_G does too. Here, we show that the converse is also true. In addition, we show that these conditions are equivalent to the condition that $\mathrm{in}_<(J_G)$ is generated in degree 2 and has linear quotients. Also, in this section, we determine some Betti numbers of the binomial edge ideal of a graph. Precisely, we show that $\beta_{1,3}(J_G) = 2k_3(G)$, where $k_3(G)$ is the number of triangles of G. Moreover, we show that if G is a non-complete connected graph, then $\beta_{1,4}(J_G) \neq 0$.

In Section 2 of this paper, we give an upper bound for the Castelnuovo-Mumford regularity of the binomial edge ideal of a closed graph, by using corresponding results for edge ideals. Indeed, we show that the regularity of the binomial edge ideal of a closed graph G is less than or equal to $c(G) + 1$, where $c(G)$ is the number of maximal cliques of G.

Throughout the paper, we mean by a graph G, a simple graph with the vertex set $V(G)$ and edge set $E(G)$, with no isolated vertices. Also, by $<$, we mean the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$.

2   Binomial edge ideals with linear resolutions

In this section, we study the graded Betti numbers $\beta_{1,3}(J_G)$ and $\beta_{1,4}(J_G)$, and we characterize all graphs whose binomial edge ideals have a linear resolution. The following theorem is the main theorem of this section.

Theorem 2.1. *Let G be a graph. Then the following conditions are equivalent:*
(a) J_G has a linear resolution.
(b) J_G has linear relations.
(c) $\mathrm{in}_<(J_G)$ is generated in degree 2 and has linear quotients.
(d) $\mathrm{in}_<(J_G)$ has a linear resolution.
(e) G is a complete graph.

To prove this theorem, we need some facts that we will mention in the sequel. We denote the number of triangles (i.e. 3-cycles) of a graph G, by $k_3(G)$. In the next result, we determine the first initial Betti number of the binomial edge ideal of a graph:

Theorem 2.2. *Let G be a graph. Then we have*
(a) $\beta_{1,3}(J_G) = 2k_3(G)$.
(b) $\beta_{1,4}(J_G) \neq 0$, *if G is non-complete and connected.*
(c) $\beta_{i,j}(J_G) = 0$, *for $j > 2i$, if G is closed. In particular, $\beta_{1,j}(J_G) = 0$, for $j \neq 3, 4$, if G is closed.*
(d) $\beta_{i,j}(J_G) = 0$, *for $j > 2n$.*

Proof. (a) Suppose that

$$\cdots \longrightarrow S^{|E(G)|}(-2) \xrightarrow{\psi} S \longrightarrow S/J_G \longrightarrow 0$$

is the minimal graded free resolution of S/J_G, in which $\psi(e_{ij}) = f_{ij}$. We first observe that J_G is $\mathbb{Z}^n$-graded, if we set $\deg(x_i) = \deg(y_i) = \varepsilon_i$, where ε_i denotes the i-th canonical basis vector of $\mathbb{Z}^n$. Thus, $\deg e_{ij} = \deg f_{ij}$. Let Z_1 be the relation module of J_G, and consider a relation $r = \sum g_{ij}e_{ij}$ of degree 3 (in the standard grading), that is, an element in $Z_1' = (Z_1)_3$. Since J_G is $\mathbb{Z}^n$-graded, it follows that Z_1' is also $\mathbb{Z}^n$-graded, and hence is generated by multihomogeous elements. Thus we may assume that r is multihomogeneous, say of degree $a \in \mathbb{Z}^n$. Then all nonzero summands $g_{ij}e_{ij}$ are of degree a, in which $|a| = 3$ (here $|a|$ is the sum of the components of a). Let $g_{ij}e_{ij} \neq 0$. Then $a = \deg(g_{ij}) + \varepsilon_i + \varepsilon_j$. Therefore, $\deg(g_{ij}) = \varepsilon_k$ for some k. If $k = i$ or $k = j$, then there is only one summand in r with this multidregree and r can not be a relation. If $k \neq i, j$, then r has exactly three summands and hence $r = g_{ij}e_{ij} + g_{ik}e_{ik} + g_{jk}e_{jk}$. Thus r is a relation of the ideal (f_{ij}, f_{ik}, f_{jk}), which is the ideal of 2-minors of the matrix $\begin{bmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}$. So, the generating relations are $x_k e_{ij} - x_j e_{ik} - x_i e_{jk}$ and $y_k e_{ij} - y_j e_{ik} - y_i e_{jk}$, by Hilbert-Burch theorem.

(b) Since G is not complete, it contains a path over three vertices, as an induced subgraph. Let $\{v_i, v_j, v_k\}$ be the vertices of this induced subgraph of G with edges $\{v_i, v_j\}$ and $\{v_j, v_k\}$. We may assume that $i < j < k$. We show that the degree 4 element $r = f_{ij}e_{jk} - f_{jk}e_{ij}$ of Z_1 can not be reduced by elements of Z_1'. Then we have $\beta_{1,4}(J_G) > 0$. Note that the relation r has multidegree $\varepsilon_i + 2\varepsilon_j + \varepsilon_k$. If it is not a minimal relation, it must be reduced by generating relations of degree 3. Since their multidegree is of the form $\varepsilon_s + \varepsilon_t + \varepsilon_l$, only relations of multidegree $\varepsilon_i + \varepsilon_j + \varepsilon_k$ can be in expression of r. But, this is impossible, since the path with edges $\{v_i, v_j\}$ and $\{v_j, v_k\}$ is an induced subgraph of G, so that $\{v_i, v_k\}$ is not an edge of G.

(c) Notice that if G is a closed graph, then we have $\text{in}_<(J_G) = (x_i y_j : i < j, \{v_i, v_j\} \in E(G))$. Thus, it can be seen as the edge ideal of a bipartite graph over the vertex set $V = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. We denote this bipartite graph by $\text{in}_<(G)$. So, we have $\text{in}_<(J_G) = I(\text{in}_<(G))$. On the other hand, we have $\beta_{i,j}(J_G) \leqslant \beta_{i,j}(\text{in}_<(J_G))$, for all i, j,

by [3, Corollary 3.3.3]. So, if G is a closed graph, then $\beta_{i,j}(J_G) = 0$, for all $j > 2i$, by [5, Lemma 2.2].

(d) By [4, Theorem 2.1], $\text{in}_<(J_G)$ is a squarefree monomial ideal in S. Thus, the result follows by Hochster's formula, since $\beta_{i,j}(J_G) \leqslant \beta_{i,j}(\text{in}_<(J_G))$. □

Corollary 2.3. *If $k_3(G) = 0$, then $\beta_{i,i+2}(J_G) = 0$, for all i. In particular, for any bipartite graph G, one has $\beta_{i,i+2}(J_G) = 0$, for all i.*

Remark 2.4. Whenever G is a closed graph, we use **consecutive cancellations** to show that $\beta_{1,3}(J_G) = \beta_{1,3}(\text{in}_<(J_G)) = 2k_3(G)$. Actually, we have $\beta_{0,3}(J_G) = \beta_{0,3}(\text{in}_<(J_G)) = 0$ and $\beta_{2,3}(J_G) = \beta_{2,3}(\text{in}_<(J_G)) = 0$, by minimality of the free resolutions. On the other hand, by [6, Theorem 22.12], the sequence of graded Betti numbers of J_G is obtained from the sequence of graded Betti numbers of $\text{in}_<(J_G)$ by **consecutive cancellations**. So, we have $\beta_{1,3}(J_G) = \beta_{1,3}(\text{in}_<(J_G))$. A sequence $q_{i,j}$ of numbers is said to be obtained from a sequence $p_{i,j}$ by a consecutive cancellation if there exist indices s and r such that $q_{s,r} = p_{s,r} - 1$, $q_{s+1,r} = p_{s+1,r} - 1$ and $q_{i,j} = p_{i,j}$ for all other values of i, j. Note that, more generally, in [1], the authors conjectured that for a closed graph G, all the graded Betti numbers of J_G and $\text{in}_<(J_G)$ coincide.

Remark 2.5. The third part of Theorem 1.2 may not be true without the assumption that G is closed. For example, consider the cycle over five vertices, C_5, which is not closed. One can see by *CoCoA* that $\beta_{1,5}(J_{C_5}) = 4$. More generally, according to our computations by *CoCoA*, it seems that $r = \sum_{i=1}^{n}(\frac{y_1 \cdots y_n}{y_i y_{i+1}})e_{i,i+1}$ is a minimal relation of degree n of J_{C_n}, for all n. So that $\beta_{1,n}(J_{C_n}) \neq 0$, for all n.

Now, recall that a homogeneous ideal I whose generators all have degree d is said to have a d-**linear resolution** (or simply linear resolution) if for all $i \geqslant 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

Also, a graded ideal I is said to have **linear quotients**, if there exists a minimal system of homogeneous generators $g_1, g_2, \ldots, g_m$ of I such that the colon ideal $(g_1, \ldots, g_{i-1}) : g_i$ is generated by linear forms for all i.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b) is trivial, (c) $\Rightarrow$ (d) follows by [3, Proposition 8.2.1], and (d) $\Rightarrow$ (a) follows by [3, Corollary 3.3.3].

(b) $\Rightarrow$ (e) Note that G is a connected graph, since J_G has linear relations. Now, suppose on the contrary that G is not complete. Assume that H is a maximal clique of G. So, G has a vertex v_k which is not in H and is adjacent to a vertex v_j of H, since G is connected. Moreover, there is a vertex of H, say v_i, which is non-adjacent to v_k, because H is a maximal clique. So, the induced subgraph of G on $\{v_i, v_j, v_k\}$ is a path over 3 vertices. Thus, by Theorem 1.2, we have $\beta_{1,4}(J_G) > 0$, which is a contradiction, since J_G has linear relations. Therefore, G is a complete graph.

(e) $\Rightarrow$ (c) Suppose that G is complete. Then $\mathrm{in}_<(J_G)$ has $\binom{n}{2}$ quadratic square-free minimal generators, more precisely, we have

$$\mathrm{in}_<(J_G) = (x_s y_{m_s} \ : \ 1 \leqslant s \leqslant n-1 \ , \ s+1 \leqslant m_s \leqslant n),$$

since G is closed. We order the generators by the lexicographic order with $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. So, we have

$$x_1 y_2 > x_1 y_3 > \cdots > x_1 y_n > x_2 y_3 > x_2 y_4 > \cdots > x_2 y_n > \cdots > x_{n-1} y_n.$$

We consider $u_1, \ldots, u_{\binom{n}{2}}$, the generators of $\mathrm{in}_<(J_G)$, as above, that is $u_1 > \ldots > u_{\binom{n}{2}}$. We should show that for each i, the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a set of variables. Note that the set $\{\frac{u_j}{\gcd(u_j, u_i)} \ : \ 1 \leqslant j \leqslant i-1\}$ is a set of monomial generators of $(u_1 \ldots, u_{i-1}) : u_i$. It is enough to consider two following cases. For each $1 \leqslant l \leqslant n-2$, the ideal $(x_1 y_2, \ldots, x_1 y_n, x_2 y_3 \ldots, x_2 y_n, \ldots, x_l y_{l+1}, \ldots, x_l y_n) : x_{l+1} y_{l+2}$ is generated by the set $\{x_1, \ldots, x_l\}$. Also, for each $1 < l \leqslant n-2$ and $l \leqslant t \leqslant n$, the ideal $(x_1 y_2, \ldots, x_1 y_n, x_2 y_3 \ldots, x_2 y_n, \ldots, x_l y_{l+1}, \ldots, x_l y_t) : x_l y_{t+1}$ is generated by the set $\{x_1, \ldots, x_{l-1}, y_{l+1}, \ldots, y_t\}$. Thus, J_G has linear quotients. $\quad\square$

3 The castelnuovo-mumford regularity of binomial edge ideals

In this section, we focus on the Castelnuovo-Mumford regularity of the binomial edge ideals of graphs. Actually, we give an upper bound for the regularity of the binomial edge ideal of a connected closed graph. In order to prove the main theorem of this section, we need some facts which we will mention in the following.

A graph G is **chordal** if every induced cycle in G has length 3, and G is **co-chordal** if the complement graph $\overline{G}$ is chordal. The **co-chordal cover number** of a graph G, denoted by $\mathrm{cochord}(G)$, is the minimum number of subgraphs $H_1, \ldots, H_s$ of G such that every H_i is cochordal and $\bigcup_{i=1}^{s} E(H_i) = E(G)$.

In [7], Woodroofe gave an upper bound for the regularity of the edge ideal of a graph. Indeed, he showed:

Theorem 3.1. *[7, Theorem 11] For any graph G, we have* $\mathrm{reg}(I(G)) \leqslant \mathrm{cochord}(G) + 1$.

We denote by $c(G)$, the number of maximal cliques of the graph G. We mean by a maximal clique of G, an induced subgraph of G which is a complete graph and is also maximal with this property. Now, we are ready for the main theorem of this section:

Theorem 3.2. *For any closed graph G, we have* $\mathrm{reg}(J_G) \leqslant c(G) + 1$.

Proof. By [3, Corollary 3.3.4], we have $\mathrm{reg}(J_G) \leqslant \mathrm{reg}(\mathrm{in}_<(J_G)) = \mathrm{reg}(I(\mathrm{in}_<(G)))$. Therefore, it is enough to show that $\mathrm{reg}(I(\mathrm{in}_<(G))) \leqslant c(G) + 1$. By Theorem 2.1, we have $\mathrm{reg}(I(\mathrm{in}_<(G))) \leqslant \mathrm{cochord}(\mathrm{in}_<(G)) + 1$. Now, we show that $\mathrm{cochord}(\mathrm{in}_<(G)) \leqslant c(G)$. Let

H be a maximal clique of G. Then $\mathrm{in}_<(H)$ is an induced subgraph of $\mathrm{in}_<(G)$. By Theorem 1.1, $I(\mathrm{in}_<(H))$ has a linear resolution. Hence, by Fröberg's theorem, [2, Theorem 1], the complement graph of $\mathrm{in}_<(H)$ is chordal. On the other hand, all maximal cliques of G, say $H_1, \ldots, H_{c(G)}$, cover all edges of G. So, clearly, $\mathrm{in}_<(H_1), \ldots, \mathrm{in}_<(H_{c(G)})$ cover all edges of $\mathrm{in}_<(G)$. Thus, by definition, we have $\mathrm{cochord}(\mathrm{in}_<(G)) \leqslant c(G)$. $\qquad\square$

Remark 3.3. Theorem 2.2 could be seen as a generalization of the fact that if G is a complete graph, then it has a linear resolution. In this case, we have $c(G) = 1$ and hence $\mathrm{reg}(J_G) = 2$. So, clearly, in Theorem 2.2, equality holds for complete graphs. There are some other classes of closed graphs with equality in Theorem 2.2. For example, let P_n be the path over n vertices. Then we have $\mathrm{reg}(J_{P_n}) = \mathrm{reg}(\mathrm{in}_<(J_{P_n})) = c(P_n) + 1 = n$, since S/J_{P_n} is Cohen-Macaulay and $\mathrm{in}_<(J_{P_n})$ is the edge ideal of $n - 1$ disjoint edges, (see [1, Corollary 1.2] and [1, Proposition 3.2]). As an other example with this property, consider any closed graph G which has exactly two maximal cliques. Then we have $\mathrm{reg}(J_G) = 3$.

Remark 3.4. The inequality of Theorem 2.2 might be strict. For example, consider the graph G with vertex set $V = \{v_1, \ldots, v_6\}$ and edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$, $\{v_4, v_5\}$, $\{v_4, v_6\}$, $\{v_5, v_6\}$. We have G is closed and $c(G) = 4$. But, one can see, by $CoCoA$, that $\mathrm{reg}(J_G) = 4$.

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