Alternating permutations with restrictions and standard Young tableaux

Yuexiao Xu
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
xyx1985@163.com

Sherry H. F. Yan*
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
huifangyan@hotmail.com

Submitted: May 2, 2012; Accepted: Jun 21, 2012; Published: Jun 28, 2012
Mathematics Subject Classifications: 05A05, 05C30

Abstract

Abstract. In this paper, we establish bijections between the set of 4123-avoiding down-up alternating permutations of length $2n$ and the set of standard Young tableaux of shape (n, n, n), and between the set of 4123-avoiding down-up alternating permutations of length $2n-1$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ via an intermediate structure of Yamanouchi words. Moreover, we show that 4123-avoiding up-down alternating permutations of length $2n+1$ are in one-to-one correspondence with standard Young tableaux of shape $(n+1, n, n-1)$, and 4123-avoiding up-down alternating permutations of length $2n$ are in bijection with shifted standard Young tableaux of shape $(n+2, n, n-2)$.

Keywords: alternating permutation; pattern avoiding; Yamanouchi word; standard Young tableau; shifted standard Young tableau.

1 Introduction

A permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ of length n on $[n] = \{1, 2, \ldots, n\}$ is said to be an *up-down alternating* permutation if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Similarly, π is said to be a *down-up* alternating permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. We denote by $\mathcal{UD}_n$ and $\mathcal{DU}_n$ the set of up-down and down-up alternating permutations of length n, respectively. Note that the *complement* map $\pi = \pi_1\pi_2 \ldots \pi_n \longmapsto (n + 1 - \pi_1)(n + 1 - \pi_2) \ldots (n + 1 - \pi_n)$ is a bijection between the set $\mathcal{UD}_n$ and the set $\mathcal{DU}_n$.

Denote by $\mathcal{S}_n$ the set of all permutations on $[n]$. Given a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1\tau_2 \ldots \tau_k \in \mathcal{S}_k$, we say that π contains the *pattern* τ if there

*Corresponding author.

exists a subsequence $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ of π that is order-isomorphic to τ. Otherwise, π is said to *avoid* the pattern τ or be τ-*avoiding*.

Pattern avoiding permutations have been extensively studied over last decade. For a thorough summary of the current status of research, see Bóna's book [1] and Kitaev's book [5]. Analogous to the ordinary permutations, Mansour [8] initiated the study of alternating permutations avoiding a given pattern. For any pattern of length 3, the number of alternating permutations of a given length avoiding that pattern is given by Catalan numbers, see [8, 10]. Recently, Lewis [6] considered the enumeration of alternating permutations avoiding a given pattern of length 4. Let $\mathcal{UD}_n(\tau)$ and $\mathcal{DU}_n(\tau)$ be the set of τ-avoiding up-down and down-up alternating permutations of length n, respectively. Lewis [6] provided bijections between the set $\mathcal{UD}_{2n}(1234)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(1234)$ and the set of standard Young tableaux of shape $(n+1, n, n-1)$. By applying the hook length formula for standard Young tableaux [9], the number of 1234-avoiding up-down alternating permutations of length $2n$ is given by $\frac{2(3n)!}{n!(n+1)!(n+2)!}$, and the number of 1234-avoiding up-down alternating permutations of length $2n+1$ is given by $\frac{16(3n)!}{(n-1)!(n+1)!(n+3)!}$. Using the method of generating trees, Lewis [7] constructed recursive bijections between the set $\mathcal{UD}_{2n}(2143)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(2143)$ and the set of shifted standard Young tableaux of shape $(n+2, n+1, n)$. Using computer simulations, Lewis [7] came up with several conjectures on the enumeration of alternating permutations avoiding a given pattern of length 4 and 5. Recently, Bóna [2] proved generalized versions of some conjectures of Joel Lewis on the number of alternating permutations avoiding certain patterns. He showed that $|\mathcal{UD}_n(12\ldots k)| = |\mathcal{UD}_n(21\ldots k)|$ and $|\mathcal{UD}_{2n}(12\ldots(k-1)k)| = |\mathcal{UD}_{2n}(12\ldots k(k-1))|$ for all n and all k.

In this paper, we are concerned with the enumeration of 4123-avoiding down-up and up-down alternating permutations of even and odd length. We establish bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ via an intermediate structure of Yamanouchi words. Consequently, we prove the conjectures, posed by Lewis [7], that $|\mathcal{UD}_{2n}(1432)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(1432)| = |\mathcal{UD}_{2n+1}(2143)|$ in the sense that $|\mathcal{UD}_n(1432)| = |\mathcal{DU}_n(4123)|$ by the operation of complement.

Applying the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux, we show that 4123-avoiding up-down alternating permutations of length $2n+1$ are in one-to-one correspondence with standard Young tableaux of shape $(n+1, n, n-1)$, and 4123-avoiding up-down alternating permutations of length $2n$ are in bijection with shifted standard Young tableaux of shape $(n+2, n, n-2)$. By the hook length formula for shifted standard Young tableaux [4], we derive that the number of shifted standard Young tableaux of shape $(n+2, n, n-2)$ is equal to $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. As a result, we deduce that $|\mathcal{UD}_{2n}(4123)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(4123)| = |\mathcal{UD}_{2n+1}(1234)|$, as conjectured by Lewis [7].

The paper is organized as follows. In Section 2, we introduce the bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n,n,n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1,n,n-1)$. In Section 3, we construct bijections between the set $\mathcal{UD}_{2n+1}(4123)$ and the set of standard Young tableaux of shape $(n+1,n,n-1)$, and between the set $\mathcal{UD}_{2n}(4123)$ and the set of shifted standard Young tableaux of shape $(n+2,n,n-2)$.

2 4123-avoiding down-up alternating permutations

In this section, we aim to establish bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n,n,n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1,n,n-1)$.

2.1 Preliminaries

In this subsection, we give some definitions and notations that will be used throughout the rest of the paper. Moreover, we provide two lemmas that will be essential in the construction of the bijections.

Given a word $w = w_1w_2\ldots w_n$ on the alphabet $\{1,2,\ldots\}$, we define c_i to be the number of occurrences of the letter i in w and the *type* of the word w to be the sequence $(c_1,c_2,c_3,\ldots)$. The subword $w_1w_2\ldots w_j$ is said to be a *left* subword for $1 \leqslant j \leqslant n$. Similarly, the subword $w_{n+1-j}w_{n+2-j}\ldots w_n$ is said to be a *right* subword for $1 \leqslant j \leqslant n$. The word w is said to be a *Yamanouchi* word if every left subword of w does not contain more occurrences of the letter $i+1$ than that of i for every $i \geqslant 1$. If every subword of w contains more occurrences of the letter i than that of $i+1$ for every $i \geqslant 1$, then the word w is said to be a *shifted Yamanouchi* word. Similarly, the word w is said to be a *skew Yamanouchi* word if every right subword of w contains more occurrences of the letter $i+1$ than that of i for every $i \geqslant 1$. For instance, it is easy to check that the word $w = 11231223$ is a Yamanouchi word of type $(3,3,2)$, the word $u = 1121213123$ is a shifted Yamanouchi word of type $(5,3,2)$, and the word $v = 1231323233$ is a skew Yamanouchi word of type $(2,3,5)$.

A *partition* λ of a positive integer n is defined to be a sequence $(\lambda_1,\lambda_2,\ldots,\lambda_m)$ of nonnegative integers such that $\lambda_1+\lambda_2+\ldots\lambda_m = n$ and $\lambda_1 \geqslant \lambda_2\ldots \geqslant \lambda_m$. Given a partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_m)$, the *(ordinary) Young diagram* of shape λ is the left-justified array of $\lambda_1+\lambda_2+\ldots+\lambda_m$ boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. If λ is a partition with distinct parts then the *shifted* Young diagram of shape λ is an array of cells with m rows, each row indented by one cell to the right with respect to the previous row, and λ_i cells in row i.

If λ is a Young diagram with n boxes, a *standard Young tableau* of shape λ is a filling of the boxes of λ with $[n]$ so that each element appears in exactly one box and entries increase along rows and columns. We identify boxes in Young diagrams and tableaux using matrix coordinates. For example, the box in the first row and second column is numbered $(1,2)$.

Let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 \geqslant n_2 \ldots \geqslant n_k$. There exists a bijection χ between the set of standard Young tableaux of shape $(n_1, n_2, \ldots, n_k)$ and the set of Yamanouchi words of type $(n_1, n_2, \ldots, n_k)$ [3]. Given a standard Young tableau T, we associate T with a word $\chi(T)$ by letting the j-th letter be the row index of the box of T containing the number j. On the other hand, given a Yamanouchi word w, it is straightforward to recover the corresponding tableaux $\chi^{-1}(w)$ by letting the i-th row contain the indices of letters of w that are equal to i. For example, the associated standard Young tableau of the Yamanouchi word 112311223 is illustrated as follows:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

Clearly, for any shifted standard Young tableau T of shape $(n_1, n_2, \ldots, n_k)$, where $n_1 > n_2 \ldots > n_k$, the word $\chi(T)$ is a shifted Yamanouchi word of type $(n_1, n_2, \ldots, n_k)$. More precisely, the map χ is a bijection between the set of shifted standard Young tableaux of shape $(n_1, n_2, \ldots, n_k)$ and the set of shifted Yamanouchi words of type $(n_1, n_2, \ldots, n_k)$. For example, let T be a shifted standard Young tableau shown in Figure 1. By applying the map χ, we obtain the shifted Yamanouchi word $\chi(T) = 1121213123$ of type $(5, 3, 2)$. Note that the map χ is not a bijection between skew Yamanouchi words and skew standard Young tableaux.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 8 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 9 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & 10 & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}$$

Figure 1: The shifted standard Young tableau T.

Let $w = w_1 w_2 \ldots w_n$ be a word on the alphabet $\{1, 2, 3\}$. The left subword $w_1 w_2 \ldots w_j$ is called the *initial run* of w if w_{j+1} is the leftmost letter of w that is equal to 3. Similarly, the right subword $w_{n+1-j} w_{n+2-j} \ldots w_n$ is said to be the *final run* of w if w_{n-j} is the rightmost letter equal to 1. As usual, the *length* of any word v is defined to be the number of entries of v. For instance, the word $w = 121211231323233$ has the initial run of length 7 and the final run of length 6. Denote by $\alpha(w)$ the length of the initial run of w.

In order to establish the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux, we consider the following two sets. Given a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathcal{S}_n$ and a word $w = w_1 w_2 \ldots w_n$ on the alphabet $\{1, 2, 3\}$, let

$$\mathcal{A}(\pi) = \{0\} \cup \{k \mid \exists i < j \ \text{ s.t. } \pi_i = k, \pi_j = k+1 \, \text{and } k \leqslant \pi_1 - 2\},$$

and

$$\mathcal{B}(w) = \{0\} \cup \{k \mid w_k w_{k+1} = 12 \text{ and } k \leqslant \alpha(w) - 2\},$$

respectively. For example, let $\pi = 658397(10)142$ and $w = 121211231323233$. We have $\mathcal{A}(\pi) = \{0,1,3\}$ and $\mathcal{B}(w) = \{0,1,3\}$.

Let $\pi = \pi_1\pi_2\dots\pi_n$ be a permutation. Suppose that $\mathcal{A}(\pi) = \{a_0, a_1, \dots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 \dots < a_p$. Define a_{p+1} to be π_1. Similarly, given a word w, if $\mathcal{B}(w) = \{a_0, a_1, \dots, a_p\}$, then we define a_{p+1} to be $\alpha(w)$. Obviously, according to the definitions of $\mathcal{A}(\pi)$ and $\mathcal{B}(w)$, we have $a_p < a_{p+1}$.

Given a permutation $\sigma \in \mathcal{S}_n$ and an element $a \in [n+1]$, there is a unique permutation $\pi = \pi_1\pi_2\dots\pi_{n+1} \in \mathcal{S}_{n+1}$ such that $\pi_1 = a$ and the word $\pi_2\pi_3\dots\pi_{n+1}$ is order-isomorphic to σ. We denote this permutation by $a \to \sigma$. Let $a, b \in [n+2]$ with $b < a$. Denote by $(a,b) \to \sigma$ the permutation $\pi = \pi_1\pi_2\dots\pi_{n+2}$ such that $\pi_1 = a$, $\pi_2 = b$ and $\pi_3\pi_4\dots\pi_{n+2}$ is order-isomorphic to σ. More precisely, the permutation π is defined by

$$\pi_i = \begin{cases} a, & i = 1, \\ b, & i = 2, \\ \sigma_{i-2}, & \sigma_{i-2} < b, \\ \sigma_{i-2}+1, & b \leqslant \sigma_{i-2} < a-1, \\ \sigma_{i-2}+2, & \sigma_{i-2} \geqslant a-1. \end{cases}$$

Note that the permutation $(a,b) \to \sigma$ is identical with the permutation $a \to (b \to \sigma)$.

We conclude this subsection with two lemmas that will be essential in the construction of the bijections between 4123-avoiding down-up alternating permutations and standard Young tableaux. First, we present the following simple observation that will be of use in the subsequent proofs of lemmas.

Observation 1. *Let $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathcal{S}_n$ with $\mathcal{A}(\sigma) = \{a_0, a_1, \dots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 < a_2 \dots < a_p < a_{p+1} = \sigma_1$. For any integers r and s with $a_j < r < s \leqslant a_{j+1}$, suppose that $\sigma_\ell = r$ and $\sigma_m = s$. Then we have $\ell > m$.*

Lemma 2. *Let $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathcal{DU}_n(4123)$ with $\mathcal{A}(\sigma) = \{a_0, a_1, \dots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 < a_2 \dots < a_p < a_{p+1} = \sigma_1$. If $\pi = (a,b) \to \sigma$ is a permutation in $\mathcal{DU}_{n+2}(4123)$, then $b \leqslant \sigma_1$ and there exists an integer j such that $a_{j+1}+2 \geqslant a > b \geqslant a_j+1$.*

Proof. Let $\pi = \pi_1\pi_2\dots\pi_{n+2}$. Recall that $\pi_1 = a$ and $\pi_2 = b$. Since π is a down-up alternating permutation and $\pi = (a,b) \to \sigma$, we have $b \leqslant \sigma_1 = a_{p+1}$. Choose the integer j such that $a_{j+1} \geqslant b \geqslant a_j+1$. We claim that $a \leqslant a_{j+1}+2$. Suppose that $a > a_{j+1}+2$. Then we have two cases. If $j = p$, then the subsequence $ab(a_{p+1}+1)(a_{p+1}+2)$ is order-isomorphic to 4123 in π. If $j < p$, then according to the definition of $\mathcal{A}(\sigma)$, there exists integers l and m with $l < m$ such that $\sigma_l = a_{j+1}$ and $\sigma_m = a_{j+1}+1$. Note that $\pi_{l+2} = \sigma_l + 1 = a_{j+1}+1$ and $\pi_{m+2} = \sigma_m + 1 = a_{j+1}+2$. Then the subsequence $\pi_1\pi_2\pi_{l+2}\pi_{m+2}$ forms a 4123 pattern in π. This yields a contradiction. Hence, we conclude that $a_j + 1 \leqslant b < a \leqslant a_{j+1}+2$. This completes the proof. □

Lemma 3. *Let $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathcal{DU}_n(4123)$ with $\mathcal{A}(\sigma) = \{a_0, a_1, \dots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 < a_2 \dots < a_p < a_{p+1} = \sigma_1$. Let a, b be two integers such that $a_{j+1}+2 \geqslant a > b \geqslant a_j+1$ and $b \leqslant \sigma_1$. Then $\pi = (a,b) \to \sigma$ is a permutation in $\mathcal{DU}_{n+2}(4123)$ satisfying that*

(i) *if* $b = a_j + 1$ *and* $j \geqslant 1$*, then we have* $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_{j-1}, b\}$ *when* $a > b + 1$ *and* $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_{j-1}\}$ *when* $a = b + 1$*;*

(ii) *otherwise, we have* $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_j, b\}$ *when* $a > b + 1$ *and* $\mathcal{A}(\pi) = \{a_0, a_1, \ldots, a_j\}$ *when* $a = b + 1$*.*

Proof. Since $b \leqslant \sigma_1$ and $a > b$, the permutation π is a down-up alternating permutation. Now we proceed to show that π avoids the pattern 4123. Let $\pi = \pi_1\pi_2 \ldots \pi_{n+2}$. Suppose that there is a subsequence $\pi_k\pi_l\pi_m\pi_q$ with $k < l < m < q$ which is order-isomorphic to 4123. Note that the subsequence $\pi_3 \ldots \pi_{n+2}$ is order-isomorphic to σ. This implies that the subsequence $\pi_3 \ldots \pi_{n+2}$ avoids the pattern 4123. So we have either $k = 1$ or $k = 2$. Suppose that $k = 2$. Since $\pi_2 < \pi_3$, the subsequence $\pi_3\pi_l\pi_m\pi_q$ is order-isomorphic to the pattern 4123. This contradicts with the fact that the subsequence $\pi_3 \ldots \pi_{n+2}$ avoids the pattern 4123. Hence, we must have $k = 1$. Suppose that $k = 1$ and $l > 2$. Since $\pi_3 \geqslant \sigma_1+1$ and $\pi_1 = a \leqslant a_{j+1} + 2 \leqslant a_{p+1} + 2 = \sigma_1 + 2$, the subsequence $\pi_3\pi_l\pi_m\pi_q$ is an instance of 4123, which contradicts with the fact that the subsequence $\pi_3 \ldots \pi_{n+2}$ is 4123-avoiding. Thus, it follows that $k = 1$ and $l = 2$. Recall that $\pi_1 = a$, $\pi_2 = b$, which implies that $b < \pi_m < \pi_q < a$. From this, we deduce that $a_j \leqslant b-1 < \pi_m-1 < \pi_q-1 < a-1 \leqslant a_{j+1}+1$. Note that $\pi_m = \sigma_{m-2}+1$ and $\pi_q = \sigma_{q-2}+1$. So we have $a_j+1 \leqslant \sigma_{m-2} < \sigma_{q-2} \leqslant a_{j+1}$ and $m < q$. This contradicts with Observation 1. Thus, the permutation π is in $\mathcal{DU}_{n+2}(4123)$.

It remains to prove that the permutation π verifies the points (i) and (ii). We claim that all the elements of the set $\mathcal{A}(\pi)$ are not larger than b. Suppose that k is a nonnegative integer such that $k \in \mathcal{A}(\pi)$ and $k \geqslant b + 1$. It follows that $k \leqslant \pi_1 - 2 = a - 2$ from the definition of $\mathcal{A}(\pi)$. Moreover, there exists integers l and m with $l < m$ such that $\pi_l = k$ and $\pi_m = k + 1$. It is easy to check that the subsequence $\pi_1\pi_2\pi_l\pi_m$ forms a pattern 4123. This contradicts with the fact that $\pi \in DU_{n+2}(4123)$. Hence, we have completed the proof of the claim.

If $a > b+1$, then we have $b \in \mathcal{A}(\pi)$ since $b+1$ appears right to b in π and $b \leqslant a-2 = \pi_1-2$. If $a = b+1$, then $b \notin \mathcal{A}(\pi)$ since $b+1$ appears left to b in π. Moreover, when $b > 1$, since the entry b appears left to the entry $b-1$ in π, we have $b-1 \notin \mathcal{A}(\pi)$. Finally, it's easy to check from the definitions of $\mathcal{A}(\pi)$ and $\mathcal{A}(\sigma)$ that for any integer k with $k < b-1$, we have $k \in \mathcal{A}(\pi)$ if and only if $k \in \mathcal{A}(\sigma)$. This completes the proof. □

2.2 The bijections

Denote by $\mathcal{P}_n$ and $\mathcal{Q}_n$ the set of Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type (n, n, n) and the set of skew Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type $(n - 1, n, n + 1)$, respectively. Let $\mathcal{P}$ (resp. $\mathcal{Q}$) be the union of $\mathcal{P}_n$ (resp. $\mathcal{Q}_n$) for all $n \geqslant 1$. Similarly, denote by $\mathcal{DU}(4123)$ the union of $\mathcal{DU}_n(4123)$ for all $n \geqslant 1$.

Now we proceed to construct a map ϕ from the set $\mathcal{DU}(4123)$ to the set $\mathcal{P} \cup \mathcal{Q}$ in terms of a recursive procedure, such that for all $m \geqslant 1$, ϕ is a map from $\mathcal{DU}_m(4123)$ to $\mathcal{Q}_n$ when $m = 2n-1$ and from $\mathcal{DU}_m(4123)$ to $\mathcal{P}_n$ when $m = 2n$. Let $\pi = \pi_1\pi_2 \ldots \pi_m \in \mathcal{DU}_m(4123)$. For $m = 1$, we define $\phi(1) = 233$. For $m = 2$, define $\phi(21) = 123$. For $m \geqslant 2$, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_{m-2} \in \mathcal{DU}_{m-2}(4123)$ such that $\pi = (a, b) \to \sigma$ where $a = \pi_1$ and $b = \pi_2$.

Assume that we have obtained the word $v = \phi(\sigma) = v_1v_2 \dots v_{3n-3}$. Then we construct a word $w = \phi(\pi) = w_1w_2 \dots w_{3n}$ from v by inserting two consecutive letters 12 immediately left to v_b and one letter 3 immediately left to v_{a-1}. Namely, w is a word with $w_b = 1$, $w_{b+1} = 2$ and $w_{a+1} = 3$ such that we can recover the word v by removing the entries w_b, w_{b+1} and w_{a+1} from w.

For example, consider the 4123-avoiding down-up alternating permutation $\pi = 63758142$. Let $\sigma = 546132$, $\sigma' = 4132$ and $\sigma'' = 21$. Clearly, we have $\pi = (6,3) \to \sigma$, $\sigma = (5,4) \to \sigma'$ and $\sigma' = (4,1) \to \sigma''$. Applying the map ϕ recursively, we can obtain $\phi(\sigma'') = 123$, $\phi(\sigma') = 121233$, $\phi(\sigma) = 121123233$, and $\phi(\pi) = 121211323233$.

Conversely, we define a map ψ from the set $\mathcal{P} \cup \mathcal{Q}$ to the set $\mathcal{DU}(4123)$, such that for all $n \geqslant 1$, ψ is a map from $\mathcal{P}_n$ to $\mathcal{DU}_{2n}(4123)$ and from $\mathcal{Q}_n$ to $\mathcal{DU}_{2n-1}(4123)$. Given a word $w = w_1w_2 \dots w_{3n} \in \mathcal{P} \cup \mathcal{Q}$, we wish to recover a 4123-avoiding down-up alternating permutation $\psi(w)$ in terms of a recursive procedure. If $w = 233$, we define $\psi(w) = 1$. If $w = 123$, then define $\psi(w) = 21$. For $n \geqslant 2$, let $a = \alpha(w)$. Now we proceed to construct an ordered pair (v, b) from w by the following procedure.

- If $w_{a+2} = 3$, then let b be the largest element of $\mathcal{B}(w)$ and v be a word obtained from w by removing w_b, w_{b+1} and w_{a+1} from w. For example, let $w = 121233$. It is easy to check that $a = \alpha(w) = 4$, $w_6 = 3$ and $\mathcal{B}(w) = \{0, 1\}$. Thus we have $b = 1$ and $v = 123$.

- If $w_{a+2} \neq 3$, then find the largest integer q such that $q \leqslant a - 1$ and $w_qw_{q+1} = 12$. Let $b = q$ and v be a word obtained from w by removing w_b, w_{b+1} and w_{a+1} from w. For example, let $w = 121211323233$. It is easy to check that $a = \alpha(w) = 6$ and $w_8 = 2 \neq 3$. Thus we have $b = 3$ and $v = 121123233$.

Finally, we define $\psi(w) = (a, b) \to \psi(v)$.

For instance, let $w = 121211323233$. By applying the map ψ recursively, we construct a permutation π as follows:

$$\begin{array}{rlclclcl} w = & 12\mathbf{12}11\mathbf{3}23233 & \Longrightarrow & 121\mathbf{12}\mathbf{3}233 & \Longrightarrow & \mathbf{12}12\mathbf{3}3 & \Longrightarrow & 123, \\ \pi = & \mathbf{63}758142 & \Longleftarrow & \mathbf{54}6132 & \Longleftarrow & \mathbf{41}32 & \Longleftarrow & 21. \end{array}$$

Our next goal is to show that the map ϕ is a bijection between the set $\mathcal{DU}_{2n}(4123)$ and the set $\mathcal{P}_n$. Analogously, the map ϕ induces a bijection between the set $\mathcal{DU}_{2n-1}(4123)$ and the set $\mathcal{Q}_n$.

Proposition 4. *For any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n}(4123)$, the word $\phi(\pi)$ is a Yamanouchi word on the alphabet $\{1, 2, 3\}$ of type (n, n, n) satisfying $\pi_1 = \alpha(\phi(\pi))$ and $\mathcal{A}(\pi) = \mathcal{B}(\phi(\pi))$.*

Proof. We proceed by induction on n. For $n = 1$, we have $\phi(21) = 123$, which is a Yamanouchi word of type $(1, 1, 1)$ with the property that $\alpha(\phi(21)) = 2$ and $\mathcal{A}(21) = \mathcal{B}(123) = \{0\}$. For $n \geqslant 2$, choose $\sigma = \sigma_1\sigma_2 \dots \sigma_{2n-2} \in \mathcal{DU}_{2n-2}(4123)$ such that $\pi = (a, b) \to \sigma$ where $a = \pi_1$ and $b = \pi_2$. Suppose that $\mathcal{A}(\sigma) = \{a_0, a_1, \dots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 < a_2 < \dots < a_p < a_{p+1} = \sigma_1$. Assume that $v = v_1v_2 \dots v_{3n-3} = \phi(\sigma)$

is a Yamanouchi word on the alphabet $\{1,2,3\}$ of type $(n-1,n-1,n-1)$ such that $\alpha(v)=\sigma_1$ and $\mathcal{B}(v)=\mathcal{A}(\sigma)$. We aim to show that $w=\phi(\pi)$ is a Yamanouchi word on the alphabet $\{1,2,3\}$ of type (n,n,n) satisfying $\alpha(w)=\pi_1$ and $\mathcal{B}(w)=\mathcal{A}(\pi)$. By the construction of the word w, it is easy to check that each left subword of w does not contain more occurrences of the letter $i+1$ than that of i for $i=1,2$. This implies that w is a Yamanouchi word of length $3n$. Let $w=w_1w_2\ldots w_{3n}$.

By Lemma 2, since $\pi=(a,b)\to\sigma$, we can choose the integer j such that $a_{j+1}+2\geqslant a>b\geqslant a_j+1$. Since $a\leqslant a_{j+1}+2\leqslant a_{p+1}+2=\sigma_1+2=\alpha(v)+2$, there is no occurrence of the letter 3 in the subword $v_1v_2\ldots v_{a-2}$ according to the definition of $\alpha(v)$. This implies that there is no occurrence of the letter 3 in the subword $w_1w_2\ldots w_a$ according to the construction of the word w. Meanwhile, we have $w_{a+1}=3$. So the obtained word w has the initial run of length a, that is, $\alpha(w)=a=\pi_1$. It remains to show that $\mathcal{A}(\pi)=\mathcal{B}(w)$. Recall that $\pi=(a,b)\to\sigma$ and $\mathcal{A}(\sigma)=\mathcal{B}(v)=\{a_0,a_1,\ldots,a_p\}$. By Lemma 3, in order to verify $\mathcal{B}(w)=\mathcal{A}(\pi)$, it suffices to show that

(i)$'$ if $b=a_j+1$ and $j\geqslant 1$, then we have $\mathcal{B}(w)=\{a_0,a_1,\ldots,a_{j-1},b\}$ when $a>b+1$ and $\mathcal{B}(w)=\{a_0,a_1,\ldots,a_{j-1}\}$ when $a=b+1$;

(ii)$'$ otherwise, we have $\mathcal{B}(w)=\{a_0,a_1,\ldots,a_j,b\}$ when $a>b+1$ and $\mathcal{B}(w)=\{a_0,a_1,\ldots,a_j\}$ when $a=b+1$.

We claim that all the elements of the set $\mathcal{B}(w)$ are not larger than b. Suppose that k is a nonnegative integer such that $k\in\mathcal{B}(w)$ and $k\geqslant b+1$. According to the definition of the set $\mathcal{B}(w)$, we have $w_kw_{k+1}=12$ and $k\leqslant\alpha(w)-2=a-2$. Since $w_{b+1}=2$, we have $k\neq b+1$. Now suppose that $b+2\leqslant k\leqslant a-2$. Clearly, we have $v_{k-2}v_{k-1}=w_kw_{k+1}=12$ by the construction of w. Since $k-2\leqslant a-4\leqslant a_{j+1}-2\leqslant\alpha(v)-2$, we have $k-2\in\mathcal{B}(v)$. However, we have $a_j+1\leqslant b\leqslant k-2\leqslant a_{j+1}-2$, which implies that $k-2\notin\mathcal{B}(v)$. Hence, the claim is proved.

If $a=b+1$, then we have $b\notin\mathcal{B}(w)$ since $b=a-1=\alpha(w)-1$. If $a>b+1$, then we have $b\in\mathcal{B}(w)$ since $w_bw_{b+1}=12$ and $b<a-1\leqslant\alpha(w)-1$. Since $w_b=1$, we have $b-1\notin\mathcal{B}(w)$ when $b>1$.

In order to verify (i)$'$ and (ii)$'$, it remains to show that for any nonnegative integer k with $k<b-1$, we have $k\in\mathcal{B}(w)$ if and only if $k\in\mathcal{B}(v)$. According to the construction of the word w, we have $w_kw_{k+1}=v_kv_{k+1}$. By Lemma 2, we have $b\leqslant\sigma_1$. Note that $k<b-1\leqslant\sigma_1-1=\alpha(v)-1$ and $k<b-1<a-1\leqslant\alpha(w)-1$. Thus, we have $k\in\mathcal{B}(w)$ if and only if $k\in\mathcal{B}(v)$ by the definitions of $\mathcal{B}(w)$ and $\mathcal{B}(v)$. This completes the proof. □

Proposition 5. *For any Yamanouchi word $w=w_1w_2\ldots w_{3n}$ on the alphabet $\{1,2,3\}$ of type (n,n,n), the permutation $\psi(w)$ is in $\mathcal{DU}_{2n}(4123)$ such that the first entry of the permutation $\psi(w)$ is equal to $\alpha(w)$ and $\mathcal{A}(\psi(w))=\mathcal{B}(w)$.*

Proof. We proceed by induction on n. For the case $n=1$, we have $\psi(123)=21$. For $n\geqslant 2$, let $a=\alpha(w)$. According to the definition of ψ, we can construct an ordered pair (v,b) from w.

First, we shall show that the word v is a Yamanouchi word on the alphabet $\{1,2,3\}$ of type $(n-1,n-1,n-1)$. Suppose that $\mathcal{B}(w)=\{a_0,a_1,\ldots,a_p\}$, where $p\geqslant 0$ and

$0 = a_0 < a_1 < \ldots < a_p < a_{p+1} = \alpha(w)$. For the case $w_{a+2} = 3$, we have $b = a_p$ by the definition of ψ. Since w is a Yamanouchi word with $\alpha(w) = a$, $w_{a+1} = 3$ and $w_{a+2} = 3$, there are at least two occurrences of 2's left to w_{a+1} and the first occurrence of 2 is preceded immediately by 1. This guarantees that there exists at least one subword $w_k w_{k+1} = 12$ with $k \leqslant a - 2 = \alpha(w) - 2$. This yields that $\mathcal{B}(w) \neq \{0\}$ according to the definition of $\mathcal{B}(w)$. Hence we have $b = a_p > 0$. Moreover, by definition we have $b = a_p \leqslant \alpha(w) - 2 = a - 2$. For the case $w_{a+2} \neq 3$, the property of the Yamanouchi word guarantees that there exists at least one subword $w_k w_{k+1} = 12$ with $k \leqslant a - 1$. Thus, in either case, we have $a > b > 0$. Note that $w_b = 1, w_{b+1} = 2$ and $w_{a+1} = 3$. According to the definition of ψ, v is obtained from w by removing w_b, w_{b+1} and w_{a+1}. It is easy to check that any left subword of v does not contain more occurrences of the letter $i + 1$ than that of i for $i = 1, 2$. Thus the obtained word v is a Yamanouchi word of type $(n-1, n-1, n-1)$. Let $v = v_1 v_2 \ldots v_{3n-3}$. Suppose that $\mathcal{B}(v) = \{c_0, c_1, \ldots, c_m\}$, where $m \geqslant 0$ and $0 = c_0 < c_1 < \ldots < c_m < c_{m+1} = \alpha(v)$.

We claim that $\alpha(v) \geqslant a-2$. According to the definition of $\alpha(w)$, there is no occurrence of the letter 3 in the subword $w_1 w_2 \ldots w_a$. Thus, by the construction of v, there is no occurrence of the letter 3 in the subword $v_1 v_2 \ldots v_{a-2}$. This implies that the word v has the initial run of length at least $a - 2$, that is, $\alpha(v) \geqslant a - 2$.

Assume that the permutation $\psi(v)$ is in $\mathcal{DU}_{2n-2}(4123)$ such that the first element of $\psi(v)$ equals $\alpha(v)$ and $\mathcal{A}(\psi(v)) = \mathcal{B}(v)$. Now we proceed to show that $\psi(w)$ is a down-up alternating permutation in $\mathcal{DU}_{2n}(4123)$ such that the first element of $\psi(w)$ is equal to $\alpha(w)$ and $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$. Recall that $\psi(w) = (a, b) \to \psi(v)$ and $a = \alpha(w)$. This yields that the first element of $\psi(w)$ is equal to $\alpha(w)$. Now we proceed to show that $\psi(w) \in \mathcal{DU}_{2n}(4123)$ such that $\mathcal{B}(w) = \mathcal{A}(\psi(w))$. We have two cases: $w_{a+2} = 3$ or $w_{a+2} \neq 3$.

If $w_{a+2} = 3$, then we have $\alpha(v) = a-2$ since $v_{a-1} = 3$. By the definition of ψ, we have $b = a_p$. By the definition of $\mathcal{B}(w)$, this ensures that there is no subword $w_k w_{k+1} = 12$ in the subword $w_{b+2} \ldots w_{a-1}$. Thus there is no subword $v_k v_{k+1} = 12$ in the subword $v_b \ldots v_{a-3}$ by the construction of v. Thus we have $c_m \leqslant b - 1$ according to the definition of $\mathcal{B}(v)$. Since $c_{m+1}+2 = \alpha(v)+2 = a > b \geqslant c_m+1$ and $b = a_p \leqslant \alpha(w)-2 = a-2 = \alpha(v)$, by Lemma 3, we can verify that $\psi(w) = (a, b) \to \psi(v)$ is in $\mathcal{DU}_{2n}(4123)$.

We next prove that $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$ for the case $w_{a+2} = 3$. Let k be a nonnegative integer with $k < b-1$. From the construction of v, we have $v_k v_{k+1} = w_k w_{k+1}$. Since $a_p \leqslant \alpha(w)-2$ and $b = a_p$, we have $k < b-1 = a_p-1 \leqslant \alpha(w)-3$ and $k \leqslant \alpha(w)-4 = \alpha(v)-2$. By the definitions of $\mathcal{B}(w)$ and $\mathcal{B}(v)$, it follows that $k \in \mathcal{B}(w)$ if and only if $k \in \mathcal{B}(v)$. Observe that $w_b = 1$. This ensures that $b - 1 \notin \mathcal{B}(w)$ when $b \geqslant 2$ by the definition of $\mathcal{B}(w)$. Hence we deduce that if $c_m = b - 1$ and $m \geqslant 1$ then we have $\{a_0, a_1, \ldots, a_{p-1}\} = \{c_0, c_1, \ldots, c_{m-1}\}$; otherwise, we have $\{a_0, a_1, \ldots, a_{p-1}\} = \{c_0, c_1, \ldots, c_m\}$. Note that $\psi(w) = (a, b) \to \psi(v)$, $\mathcal{A}(\psi(v)) = \mathcal{B}(v) = \{c_0, c_1, \ldots, c_m\}$, and $a - 2 \geqslant b \geqslant c_m + 1$. By Lemma 3, we can verify that if $c_m = b-1$ and $m \geqslant 1$, then $\mathcal{A}(\psi(w)) = \{c_0, c_1, \ldots, c_{m-1}, b\}$; otherwise, $\mathcal{A}(\psi(w)) = \{c_0, c_1, \ldots, c_m, b\}$. Since $b = a_p$, we deduce that $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$.

If $w_{a+2} \neq 3$, then we have $\alpha(v) \geqslant a - 1$ since $v_{a-1} \neq 3$. By the definition of ψ, we have $b \leqslant a - 1$. This yields that $b \leqslant \alpha(v) = c_{m+1}$. Choose the integer j such that $c_{j+1} \geqslant$

$b \geqslant c_j + 1$. According to the definition of $\mathcal{B}(w)$, there is no subword $w_k w_{k+1} = 12$ in the subword $w_{b+2} \dots w_{a-1}$. This implies that there is no subword $v_k v_{k+1} = 12$ in the subword $v_b \dots v_{a-3}$. Thus we have $c_{j+1} \geqslant a-2$, that is, $a \leqslant c_{j+1}+2$. Since $c_{j+1}+2 \geqslant a > b \geqslant c_j+1$ and $b \leqslant c_{j+1} \leqslant c_{m+1} = \alpha(v)$, it follows that $\psi(w) = (a, b) \to \psi(v)$ is in $\mathcal{DU}_{2n}(4123)$ from Lemma 3.

Now we turn to the proof of $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$ for the case $w_{a+3} \neq 3$. Let k be a nonnegative integer k with $k < b - 1$. From the construction of v, we have $v_k v_{k+1} = w_k w_{k+1}$. Since $k < b-1 \leqslant a-2 = \alpha(w) - 2$ and $k < b-1 \leqslant \alpha(v) - 1$, we derive that $k \in \mathcal{B}(w)$ if and only if $k \in \mathcal{B}(v)$. Observe that $w_b = 1$. This ensures that $b - 1 \notin \mathcal{B}(w)$ when $b \geqslant 2$ by the definition of the set $\mathcal{B}(w)$. Recall that $b \geqslant c_j + 1$. According to the definitions of ψ and $\mathcal{B}(w)$, we have $b \geqslant a_p$. So, we derive that

- if $b = a_p$, then $\{a_0, a_1, \dots, a_{p-1}\} = \{c_0, c_1, \dots, c_{j-1}\}$ when $b = c_j + 1$ with $j \geqslant 1$; otherwise, $\{a_0, a_1, \dots, a_{p-1}\} = \{c_0, c_1, \dots, c_j\}$;
- if $b > a_p$, then $\{a_0, a_1, \dots, a_p\} = \{c_0, c_1, \dots, c_{j-1}\}$ when $b = c_j + 1$ and $j \geqslant 1$; otherwise, $\{a_0, a_1, \dots, a_p\} = \{c_0, c_1, \dots, c_j\}$.

According to the definition of $\mathcal{B}(w)$ and the construction of (v, b), it is easy to check that if $b = a_p$, then we have $b \leqslant \alpha(w) - 2 = a - 2$; otherwise, we have $b = a - 1$. Note that $\psi(w) = (a, b) \to \psi(v)$, $\mathcal{A}(\psi(v)) = \mathcal{B}(v)$ and $c_{j+1} \geqslant b \geqslant c_j+1$. By Lemma 3, we can verify that

- if $b = a_p$, then $\mathcal{A}(\psi(w)) = \{c_0, c_1, \dots, c_{j-1}, b\}$ when $b = c_j+1$ with $j \geqslant 1$; otherwise, $\mathcal{A}(\psi(w)) = \{c_0, c_1, \dots, c_j, b\}$;
- if $b > a_p$, then $\mathcal{A}(\psi(w)) = \{c_0, c_1, \dots, c_{j-1}\}$ when $b = c_j + 1$ and $j \geqslant 1$; otherwise, $\mathcal{A}(\psi(w)) = \{c_0, c_1, \dots, c_j\}$.

Thus, we derive that $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$. This completes the proof. $\square$

Now we aim to prove that the map ϕ is a bijection by showing that the maps ϕ and ψ are inverses of each other. First, we need to introduce the following definition. Let $w = w_1 w_2 \dots w_n$ be a word. The entry w_i is said to be *positioned* at i.

Theorem 6. *The map ϕ is a bijection between the set $\mathcal{DU}_{2n}(4123)$ and the set of Yamanouchi words on the alphabet $\{1, 2, 3\}$ of type (n, n, n) such that for any permutation $\pi = \pi_1 \pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n}(4123)$, we have $\pi_1 = \alpha(\phi(\pi))$ and $\mathcal{A}(\pi) = \mathcal{B}(\phi(\pi))$*

Proof. It is sufficient to show that the maps ϕ and ψ are inverses of each other. First we need to show that $\psi(\phi(\pi)) = \pi$ for any permutation $\pi \in \mathcal{DU}_{2n}(4123)$. We proceed by induction on n. Obviously, for $n = 1$, we have $\psi(\phi(21)) = \psi(123) = 21$. For $n \geqslant 2$, choose $\sigma = \sigma_1 \sigma_2 \dots \sigma_{2n-2}$ such that $\pi = (a, b) \to \sigma$ where $a = \pi_1$ and $b = \pi_2$. Assume that $\psi(\phi(\sigma)) = \sigma$.

For convenience, let $u = u_1 u_2 \dots u_{3n-3} = \phi(\sigma)$ and $w = w_1 w_2 \dots w_{3n} = \phi(\pi)$. By Proposition 4, u and w are Yamanouchi words on the alphabet $\{1, 2, 3\}$ satisfying $\alpha(u) =$

σ_1, $\alpha(w) = \pi_1 = a$, $\mathcal{B}(u) = \mathcal{A}(\sigma)$ and $\mathcal{B}(w) = \mathcal{A}(\pi)$. Suppose that $\mathcal{A}(\sigma) = \{a_0, a_1, \ldots, a_p\}$, where $p \geqslant 0$ and $0 = a_0 < a_1 < a_2 < \ldots < a_p < a_{p+1} = \sigma_1$. By Lemma 2, we have $a_{j+1} + 2 \geqslant a > b \geqslant a_j + 1$ for some integer j and $b \leqslant \sigma_1$.

When we apply the map ψ to the word $w = \phi(\pi)$, we construct an ordered pair from w. Denote by (v, t) the obtained ordered pair. By the definition of ψ, we have $\psi(\phi(\pi)) = \psi(w) = (a, t) \to \psi(v)$. Since $\pi = (a, b) \to \sigma$ and $\psi(\phi(\sigma)) = \sigma$, in order to prove $\psi(\phi(\pi)) = \pi$, it is sufficient to show that $t = b$ and $v = \phi(\sigma)$. According to the definition of ϕ, the word u can be obtained from w by removing w_b, w_{b+1} and w_{a+1}. Meanwhile, by the construction of the ordered pair (v, t), the word v is obtained from w by removing w_t, w_{t+1} and w_{a+1}. Thus, it is sufficient to show that $t = b$ to prove $\psi(\phi(\pi)) = \pi$. We shall consider two cases.

If $a = a_{p+1} + 2$, then we have $b \geqslant a_p + 1$. Recall that $b \leqslant \sigma_1 = a_{p+1}$. This yields that $a > b + 1$. By Lemma 3, we deduce that if $b = a_p + 1$ and $p \geqslant 1$, then $\mathcal{A}(\pi) = \{a_0, \ldots, a_{p-1}, b\}$; otherwise, $\mathcal{A}(\pi) = \{a_0, \ldots, a_p, b\}$. Since $a_{p+1} = \sigma_1 = \alpha(u)$, we have $u_{a-1} = u_{\alpha(u)+1} = 3$ according to the definition of $\alpha(u)$. According to the definition of ϕ, the word u can be obtained from w by removing w_b, w_{b+1} and w_{a+1}. This implies that $w_{a+2} = u_{a-1} = 3$. By the construction of the ordered pair (v, t), it is easily seen that $t = b$.

If $a < a_{p+1} + 2$, then we have $a - 1 < a_{p+1} + 1 = \alpha(u) + 1$. According to the definition of $\alpha(u)$, we have $u_{a-1} \neq 3$. This implies that $w_{a+2} = u_{a-1} \neq 3$. Since $a_j + 1 \leqslant b < a \leqslant a_{j+1} + 2$, there is no subword $u_k u_{k+1} = 12$ for $b \leqslant k \leqslant a - 3$ according to the definition of $\mathcal{B}(u)$. By the construction of w, we see that there is no subword $w_k w_{k+1} = 12$ for $b + 2 \leqslant k \leqslant a - 1$. On the other hand, according to the definition of ϕ, we have $w_b w_{b+1} = 12$ and $b \leqslant a - 1$. From the construction of the ordered pair (v, t), it follows that $t = b$. Hence, we have $\psi(\phi(\pi)) = \pi$.

Next we turn to the proof of the equality $\phi(\psi(w')) = w'$ for any Yamanouchi word w' on the alphabet $\{1, 2, 3\}$ of type (n, n, n). We proceed by induction on n. Obviously, for $n = 1$, we have $\phi(\psi(123)) = \phi(21) = 123$. For $n \geqslant 2$, let $a' = \alpha(w')$. According to the definition of ψ, we obtain an ordered pair (v', b') from w'. By the construction of the ordered pair (v', b'), the word v' is obtained from w' by removing the entries positioned at b', $b' + 1$ and $a' + 1$. From the proof of Proposition 5, it follows that v' is a Yamanouchi word of type $(n-1, n-1, n-1)$. By Proposition 5, the permutations $\psi(w')$ and $\psi(v')$ are in $\mathcal{DU}_{2n}(4123)$ and $\mathcal{DU}_{2n-2}(4123)$, respectively. Assume that $\phi(\psi(v')) = v'$. According to the definition of ψ, we have $\psi(w') = (a', b') \to \psi(v')$. When we apply the map ϕ to the permutation $\psi(w')$, we get a a Yamanouchi word $\phi(\psi(w'))$ of type (n, n, n). By the definition of ϕ, we can recover the word $\phi(\psi(v'))$ form $\phi(\psi(w'))$ by removing the entries positioned at b', $b' + 1$ and $a' + 1$. Recall that $\phi(\psi(v')) = v'$ and v' is obtained from w' by removing the entries positioned at b', $b' + 1$ and $a' + 1$. Thus we have $\phi(\psi(w')) = w'$. This completes the proof. □

By the same reasoning as in the proofs of Propositions 4 and 5 and Theorem 6, we can obtain the following analogous results for 4123-avoiding down-up alternating permutations of odd length.

Proposition 7. *For any permutation* $\pi = \pi_1\pi_2\ldots\pi_{2n-1} \in \mathcal{DU}_{2n-1}(4123)$*, the word* $\phi(\pi)$ *is a skew Yamanouchi word on the alphabet* $\{1,2,3\}$ *of type* $(n-1,n,n+1)$ *satisfying* $\pi_1 = \alpha(\phi(\pi))$ *and* $\mathcal{A}(\pi) = \mathcal{B}(\phi(\pi))$*.*

Proposition 8. *For any skew Yamanouchi word* $w = w_1w_2\ldots w_{3n}$ *on the alphabet* $\{1,2,3\}$ *of type* $(n-1,n,n+1)$*, the permutation* $\psi(w)$ *is in* $\mathcal{DU}_{2n-1}(4123)$ *such that the first entry of the permutation* $\psi(w)$ *is equal to* $\alpha(w)$ *and* $\mathcal{A}(\psi(w)) = \mathcal{B}(w)$*.*

Theorem 9. *The map* ϕ *is a bijection between the set* $\mathcal{DU}_{2n-1}(4123)$ *and the set of skew Yamanouchi words on the alphabet* $\{1,2,3\}$ *of type* $(n-1,n,n+1)$ *such that for any permutation* $\pi = \pi_1\pi_2\ldots\pi_{2n-1} \in \mathcal{DU}_{2n-1}(4123)$*, we have* $\pi_1 = \alpha(\phi(\pi))$ *and* $\mathcal{A}(\pi) = \mathcal{B}(\phi(\pi))$*.*

So far, we have established bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of Yamanouchi words on the alphabet $\{1,2,3\}$ of type (n,n,n), and between the set of $\mathcal{DU}_{2n-1}(4123)$ and the set of skew Yamanouchi words on the alphabet $\{1,2,3\}$ of type $(n-1,n,n+1)$. Now we proceed to present the desired bijections between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n,n,n), and between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1,n,n-1)$.

Denote by $\mathcal{W}_n$ the set of words $w = w_1w_2\ldots w_n$ on the alphabet $\{1,2,3\}$. Now we define a map β: $\mathcal{W}_n \to \mathcal{W}_n$ as follows. Let $w = w_1w_2\ldots w_{3n}$ be a word on the alphabet $\{1,2,3\}$. Define $\beta(w) = (4-w_n)(4-w_{n-1})\ldots(4-w_1)$. Obviously, the map β is essentially an involution on the set $\mathcal{W}_n$, that is, for any word $w \in \mathcal{W}_n$, we have $\beta(\beta(w)) = w$. Note that the map β can also be called the *reverse-complement* operation.

According to the definitions of skew Yamanouchi words and shifted Yamanouchi words, it is easy to verify that the map β induces a bijection between the set of skew Yamanouchi words of type $(n-1,n,n+1)$ and the set of shifted Yamanouchi words of type $(n+1,n,n-1)$. Similarly, the map β is an involution on the set of Yamanouchi words of type (n,n,n). Moreover, the map β transforms the initial run of a word to the final run. Recall that the map χ is a bijection between the set of Yamanouchi words of type (n,n,n) and standard Young tableaux of shape (n,n,n). Moreover, the map χ is a bijection between the set of shifted Yamanouchi words of type $(n+1,n,n-1)$ and shifted standard Young tableaux of shape $(n+1,n,n-1)$. Observe that given any ordinary or shifted standard Young tableau T of shape (a,b,c) with the $(1,a)$-entry equal to k, its corresponding word $\chi(T)$ has the final run of length $a+b+c-k$. Therefore, we derive the following results.

Proposition 10. *The map* $\chi^{-1}\circ\beta$ *is a bijection between the set of Yamanouchi words of type* (n,n,n) *with the initial run of length* k *and the set of standard Young tableaux of shape* (n,n,n) *with the* $(1,n)$*-entry equal to* $3n-k$*.*

Proposition 11. *The map* $\chi^{-1}\circ\beta$ *induces a bijection between the set of skew Yamanouchi words of type* $(n-1,n,n+1)$ *with the initial run of length* k *and the set of shifted standard Young tableaux of shape* $(n+1,n,n-1)$ *with the* $(1,n+1)$*-entry equal to* $3n-k$*.*

For example, consider a skew Yamanouchi word $w = 112123231323233$ of type $(4,5,6)$ with the initial run of length 5. By applying the map β, we obtain a shifted Yamanouchi word $\beta(w) = 112121312123233$ of type $(6, 5, 4)$ with the final run of length 5. Applying the inverse map χ^{-1} to $\beta(w)$ gives a shifted standard Young tableaux $\chi^{-1}(\beta(w))$ with the (1,6)-entry equal to 10, as illustrated in Figure 2.

1	2	4	6	8	10
	3	5	9	11	13
		7	12	14	15

Figure 2: The shifted standard Young tableau $\chi^{-1}(\beta(w))$.

Combining Theorems 6 and 9 and Propositions 10 and 11, we deduce the following theorems.

Theorem 12. *The map $\Phi = \chi^{-1} \circ \beta \circ \phi$ is a bijection between the set $\mathcal{DU}_{2n}(4123)$ and the set of standard Young tableaux of shape (n, n, n) such that for any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n}(4123)$, the $(1, n)$-entry of the corresponding tableau is equal to $3n - \pi_1$.*

Theorem 13. *The map $\Phi = \chi^{-1} \circ \beta \circ \phi$ is a bijection between the set $\mathcal{DU}_{2n-1}(4123)$ and the set of shifted standard Young tableaux of shape $(n+1, n, n-1)$ such that for any permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{DU}_{2n-1}(4123)$, the $(1, n+1)$-entry of the corresponding tableau is equal to $3n - \pi_1$.*

Recall that there are bijections between the set $\mathcal{UD}_{2n}(1234)$ and the standard Young tableaux of shape (n, n, n), and between the set $\mathcal{UD}_{2n+1}(2143)$ and shifted standard Young tableaux of shape $(n + 2, n + 1, n)$. By the operation of complement, the set $\mathcal{DU}_n(4123)$ is in bijection with the set $\mathcal{UD}_n(1432)$. Thus, by Theorems 12 and 13, we derive that $|\mathcal{UD}_{2n}(1432)| = |\mathcal{UD}_{2n}(1234)|$ and $|\mathcal{UD}_{2n+1}(1432)| = |\mathcal{UD}_{2n+1}(2143)|$, as conjectured by Lewis [7].

3 4123-avoiding up-down alternating permutations

In this section, we show that 4123-avoiding up-down alternating permutations of length $2n + 1$ are in one-to-one correspondence with standard Young tableaux of shape $(n + 1, n, n-1)$. Moreover, for $n \geqslant 2$, there is a bijection between the set of 4123-avoiding up-down permutations of length $2n$ and the set of shifted standard Young tableaux of shape $(n + 2, n, n - 2)$. The following Lemma will be essential in establishing the bijections.

Lemma 14. *Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ be a permutation in $\mathcal{DU}_n(4123)$ and let a be a positive integer. If $a \leqslant \sigma_1$, then $\pi = a \to \sigma$ is in $\mathcal{UD}_{n+1}(4123)$.*

Proof. Let $\pi = \pi_1\pi_2 \dots \pi_{n+1}$. In order to prove $\pi \in \mathcal{UD}_{n+1}(4123)$, it is sufficient to prove that there exists no subsequence $\pi_1\pi_i\pi_j\pi_k$ with $i < j < k$ in π. Assume that $\pi_1\pi_i\pi_j\pi_k$ is a subsequence order-isomorphic to 4123. Since $\pi_1 < \pi_2$, we deduce that $\pi_2\pi_i\pi_j\pi_k$ is also a subsequence order-isomorphic to 4123, which implies that $\sigma_1\sigma_{i-1}\sigma_{j-1}\sigma_{k-1}$ is a subsequence order-isomorphic to 4123. This contradicts with the fact that σ is a 4123-avoiding down-up alternating permutation. This completes the proof. □

Now we proceed to construct a map γ from the set $\mathcal{UD}_{2n+1}(4123)$ to the set of standard Young tableaux of shape $(n+1, n, n-1)$. Given a permutation $\pi = \pi_1\pi_2 \ldots \pi_{2n+1} \in \mathcal{UD}_{2n+1}(4123)$, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_{2n}$ be the permutation such that $\pi = \pi_1 \to \sigma$. Obviously, the permutation σ is in $\mathcal{DU}_{2n}(4123)$. By Theorem 12, the tableau $\Phi(\sigma)$ is a standard Young tableau of shape (n, n, n) with the $(1, n)$-entry equal to $3n - \sigma_1$. Delete the $(3, n)$-entry of $\Phi(\sigma)$, insert a $(1, n+1)$-entry equal to $3n+1-\pi_1$, and increase each entry that is larger than $3n - \pi_1$ by one. Define $T = \gamma(\pi)$ to be the resulting tableau. Since $\pi_1 \leqslant \sigma_1$, the obtained tableau T is a standard Young tableau of shape $(n+1, n, n-1)$. Therefore, the map γ is well defined.

For example, consider a 4123-avoiding up-down alternating permutation $\pi = 4657132$. We get $\sigma = 546132$ such that $\pi = 4 \to \sigma$. By applying the bijection Φ, we get a standard Young tableau $\Phi(\sigma)$:

1	2	4
3	5	8
6	7	9

Removing the $(3, 3)$-entry of $\Phi(\sigma)$ and inserting a $(1, 4)$-entry equal to 6, we get the tableau $\gamma(\pi)$:

1	2	4	6
3	5	9	
7	8		

Theorem 15. *For $n \geqslant 1$, the map γ is a bijection between the set $\mathcal{UD}_{2n+1}(4123)$ and the set of standard Young tableaux of shape $(n+1, n, n-1)$.*

Proof. We proceed to construct a map $\bar{\gamma}$ from the set of standard Young tableaux of shape $(n+1, n, n-1)$ to the set $\mathcal{UD}_{2n+1}(4123)$. Given a standard Young tableau T of shape $(n+1, n, n-1)$, we wish to recover a permutation $\bar{\gamma}(T) \in \mathcal{UD}_{2n+1}(4123)$. Suppose that the $(1, n+1)$-entry and $(1, n)$-entry of T are equal to $3n+1-a$ and $3n-b$, respectively. Then we construct a permutation $\bar{\gamma}(T)$ as follows.

- Remove the $(1, n+1)$-entry from the tableau T and decrease each entry that is larger than $3n+1-a$ by one;
- Insert a $(3, n)$-entry which is equal to $3n$. Denote by T' the obtained standard Young tableaux;
- Finally, set $\bar{\gamma}(T) = a \to \Phi^{-1}(T')$.

Note that T' is a standard Young tableau of shape (n, n, n) such that the $(1, n)$-entry equals $3n - b$. Let $\sigma = \Phi^{-1}(T') = \sigma_1\sigma_2 \ldots \sigma_{2n}$. By Theorem 12, we deduce that σ is a down-up alternating permutation in $\mathcal{DU}_{2n}(4123)$ with $\sigma_1 = b$. Since T is a standard Young tableau, we have $a \leqslant b$. By Lemma 14, the obtained permutation $\bar{\gamma}(T)$ is in $\mathcal{UD}_{2n+1}(4123)$. It is straightforward to check that the construction of the map $\bar{\gamma}$ reverses each step of the construction of the map δ. Thus the maps γ and $\bar{\gamma}$ are inverses of each other. This completes the proof. $\square$

Recall that there is a bijection between the set $\mathcal{UD}_{2n+1}(1234)$ and the set of standard Young tableaux of shape $(n+1, n, n-1)$ [6]. From Theorem 15, we deduce the following result.

Theorem 16. *For $n \geqslant 0$, we have*

$$|\mathcal{UD}_{2n+1}(4123)| = |\mathcal{UD}_{2n+1}(1234)|.$$

Our next goal is to establish an analogous bijection between the set $\mathcal{UD}_{2n}(4123)$ and the set of shifted standard Young tableaux of shape $(n+2, n, n-2)$. We define a map δ from the set of the set of 4123-avoiding up-down alternating permutations of length $2n$ to the set of shifted standard Young tableaux of shape $(n+2, n, n-2)$. For $n \geqslant 2$, given a permutation $\pi = \pi_1\pi_2 \ldots \pi_{2n} \in \mathcal{UD}_{2n}(4123)$, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_{2n-1}$ be the permutation satisfying $\pi = \pi_1 \to \sigma$. Clearly, the permutation σ is in $\mathcal{DU}_{2n-1}(4123)$. By Theorem 13, the tableau $\Phi(\sigma)$ is a shifted standard Young tableau of shape $(n+1, n, n-1)$ with the $(1, n+1)$-entry equal to $3n - \sigma_1$. Finally we obtain a tableau from $\Phi(\sigma)$ by deleting the $(3, n-1)$-entry, inserting a $(1, n+2)$-entry equal to $(3n+1-\pi_1)$ and increasing each entry larger than $3n - \pi_1$ by one. Since $\pi_1 \leqslant \sigma_1$, the obtained tableau is a shifted standard Young tableau of shape $(n+2, n, n-2)$. As in the case for the map γ, we can define the inverse map of δ by reversing each step of the map δ. By Lemma 14 and Theorem 13, we can verify that δ is a bijection.

Theorem 17. *For $n \geqslant 2$, the map δ described above is a bijection between the set $\mathcal{UD}_{2n}(4123)$ and the set of shifted standard Young tableaux of shape $(n+2, n, n-2)$.*

As in the case for standard Young tableaux, there is a simple hook length formula for shifted standard Young tableaux [4]. By simple computation, we derive that the number of shifted standard Young tableaux of shape $(n+2, n, n-2)$ is equal to $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. Recall that the number of 1234-avoiding up-down alternating permutations of length $2n$ is given by $\frac{2(3n)!}{n!(n+1)!(n+2)!}$. Hence, we obtain the following result.

Theorem 18. *For $n \geqslant 0$, we have*

$$|\mathcal{UD}_{2n}(4123)| = |\mathcal{UD}_{2n}(1234)| = \frac{2(3n)!}{n!(n+1)!(n+2)!}.$$

Acknowledgments. The authors would like to thank the referee for helpful suggestions. This work was supported by the National Natural Science Foundation of China (No.10901141) and Zhejiang Innovation Project(Grant No. T200905).

References

[1] M. Bóna, Combinatorics of Permutations. CRC Press, 2004.

[2] M. Bóna, On a family of conjectures of Joel Lewis on alternating Permutations, arXiv: math.CO 1205.1778v1.

[3] S.P. Eu, Skew-standard tableaux with three rows, *Adv. Appl. Math.* **45** (2010), 463–469.

[4] C. Krattenthaler, Bijective proofs of the hook formulas for the number of standard Young tableaux, ordinary and shifted, *Electronic J. Combin.* **2** (1995), #R13.

[5] S. Kitaev, Patterns in permutations and words. Springer Verlag (EATCS monographs in Theoretical Computer Science book series), 2011.

[6] J. B. Lewis, Pattern avoidance for alternating permutations and Young tableaux, *J. Combin. Theory Ser. A* **118** (2011), 1436–1450.

[7] J. B. Lewis, Generating trees and pattern avoidance in alternating permutations, *Electronic J. Combin.* **19** (2012), P21.

[8] T. Mansour, Restricted 132-alternating permutations and Chebyshev polynomials, *Ann. Combin.* **7** (2003), 201–227.

[9] R. P. Stanley, Enumerative Combinatorics, Volume 2. Cambridge University Press, 2001.

[10] R. P. Stanley, Catalan addendum. http://www-math.mit.edu/~rstan/ec/catadd.pdf, 2012.