On (K_q, k) stable graphs with small k

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Submitted: Feb 7, 2011; Accepted: Jun 14, 2012; Published: Jun 28, 2012
Mathematics Subject Classification: 05C35, 05C75, 05C99

Abstract

A graph G is (K_q, k) stable if it contains a copy of K_q after deleting any subset of k vertices. In a previous paper we have characterized the (K_q, k) stable graphs with minimum size for $3 \leqslant q \leqslant 5$ and we have proved that the only (K_q, k) stable graph with minimum size is K_{q+k} for $q \geqslant 5$ and $k \leqslant 3$. We show that for $q \geqslant 6$ and $k \leqslant \frac{q}{2} + 1$ the only (K_q, k) stable graph with minimum size is isomorphic to K_{q+k}.

1 Introduction

For terms not defined here we refer to [1]. As usually, the *order* of a graph G is the number of its vertices (it is denoted by $|G|$) and the *size* of G is the number of its edges (it is denoted by $e(G)$). The degree of a vertex v in a graph G is denoted by $d_G(v)$, or simply by $d(v)$ if no confusion is possible. For any set S of vertices, we denote by $G - S$ the subgraph induced by $V(G) - S$. If $S = \{v\}$ we write $G - v$ for $G - \{v\}$. When e is an edge of G we denote by $G - e$ the spanning subgraph $(V(G), E - \{e\})$. The disjoint union of two graphs G_1 and G_2 is denoted by $G_1 + G_2$. The union of p mutually disjoint copies of a graph G is denoted by pG. A complete subgraph of order q of G is called a *q-clique* of G. The complete graph of order q is denoted by K_q. When a graph G contains a q-clique as subgraph, we say "G contains a K_q".

*The research of APW was partially sponsored by polish Ministry of Science and Higher Education.

In [5] Horváth and Katona consider the notion of (H, k) *stable* graph: given a simple graph H, an integer k and a graph G containing H as subgraph, G is a *a* (H, k) *stable graph* whenever the deletion of any set of k edges does not lead to a H-free graph. These authors consider (P_n, k) stable graphs and prove a conjecture stated in [4] on the minimum size of a (P_4, k) stable graph. In [2], Dudek, Szymański and Zwonek are interested in a vertex version of this notion and introduce the (H, k) vertex stable graphs.

Definition 1.1 *Let H be a graph and k be a natural number. A graph G of order at least k is said to be a (H, k)* vertex stable *graph if for any set S of k vertices the subgraph $G - S$ contains a graph isomorphic to H.*

In this paper, since no confusion will be possible, a (H, k) vertex stable shall be simply called a (H, k) *stable graph.* By $Q(H, k)$ we denote the size of a minimum (H, k) stable graph. It is clear that if G is a (H, k) stable graph with minimum size then the graph obtained from G by addition or deletion of some isolated vertices is also minimum (H, k) stable. Hence we shall assume that all the graphs considered in the paper have no isolated vertices. A (H, k) stable graph with minimum size shall be called a *minimum* (H, k) *stable graph.*

Lemma 1.2 *[2] Let q and k be integers, $q \geqslant 2, k \geqslant 1$. If G is (H, k) stable then, for every vertex v of G, the graph $G - v$ is $(H, k - 1)$ stable.*

Proposition 1.3 *[2] If G is a (H, k) stable graph with minimum size then every vertex as well as every edge is contained in a subgraph isomorphic to H.*

Proof: Let e be an edge of G which is not contained in any subgraph of G isomorphic to H, then $G - e$ would be a (H, k) stable graph with less edges than G, a contradiction. Let x be a vertex of G and e be an edge of G incident with x, since e is an edge of some subgraph isomorphic to H, say H_0, the vertex x is a vertex of H_0. □

2 Preliminary results

We are interested in minimum (K_q, k) stable graphs (where q and k are integers such that $q \geqslant 2$ and $k \geqslant 0$). As a corollary to Proposition 1.3, every edge and every vertex of a minimum (K_q, k) stable graph is contained in a K_q (thus the minimum degree is at least $q - 1$). Note that, for $q \geqslant 2$ and $k \geqslant 0$, the graph K_{q+k} is (K_q, k) stable, hence $Q(K_q, k) \leqslant \binom{q+k}{2}$.

Definition 2.1 *Let H be a non complete graph on $q + t$ vertices ($t \geqslant 1$). We shall say that H is a* near complete graph *when it has a vertex v such that*

- *$H - v$ is complete.*
- *$d_H(v) = q + r$ with $-1 \leqslant r \leqslant t - 2$.*

The previous definition generalizes Definition 1.5 in [3] initially given for $r \in \{-1, 0, 1\}$ and the following lemma generalizes Proposition 2.1 in [3].

Lemma 2.2 *Every minimum (K_q, k) stable graph G, where $q \geqslant 3$ and $k \geqslant 1$, has no component H isomorphic to a near complete graph.*

Proof: Suppose, contrary to our claim, that G has such a component H and let v be the vertex of H such that $H - v$ is a clique of G. Then $|H| = q + t$, with $t \geqslant 1$, and $q - 1 \leqslant d(v) = q + r \leqslant q + t - 2$. Since G is minimum (K_q, k) stable, $G - v$ is $(K_q, k-1)$ stable and is not (K_q, k) stable. Then $G - v$ contains a set S with at most k vertices intersecting every subgraph of $G - v$ isomorphic to a K_q. The graph $G - S$ contains some K_q (at least one) and clearly every subgraph of $G - S$ isomorphic to a K_q contains v. Since $N(v)$ is a K_{q+r} and $N(v) - S$ contains no K_q, $|N(v) - S| \leqslant q - 1$. Since there exists a K_q containing v in $H - S$, $|N(v) - S| = q - 1$ (and hence $|S \cap N(v)| = r + 1$). Since $H - v - S$ contains no K_q, $H - v - S = N(v) - S$. Let a be a vertex of $H - v$ not adjacent to v and let b be a vertex in $N(v) - S$, and consider $S' = S - \{a\} + \{b\}$. We have $| S' | \leqslant k$ and $G - S'$ contains no K_q, a contradiction. □

It is clear that $Q(K_q, 0) = \binom{q}{2}$ and the only minimum $(K_q, 0)$ stable graph is K_q. It is an easy exercise to see that $Q(K_2, k) = k + 1$ and that the matching $(k + 1)K_2$ is the unique minimum (K_2, k) stable graph.

Theorem 2.3 *[3] Let G be a minimum (K_q, k) stable graph, with $k \geqslant 0$ and $3 \leqslant q \leqslant 5$. Then G is isomorphic to $sK_{2q-2} + tK_{2q-3}$, for any choice of s and t such that $s(q - 1) + t(q - 2) = k + 1$.*

In [3] it was proved that if $q \geqslant 4$ and $k \in \{1, 2\}$ then $Q(K_q, k) = \binom{q+k}{2}$ and the only minimum (K_q, k) stable graph is K_{q+k}. We have proved also that if $q \geqslant 5$ then $Q(K_q, 3) = \binom{q+3}{2}$ and the only minimum $(K_q, 3)$ stable graph is K_{q+3}. Dudek, Szymański and Zwonek proved the following result.

Theorem 2.4 *[2] For every $q \geqslant 4$, there exists an integer $k(q)$ such that $Q(K_q, k) \leqslant (2q - 3)(k + 1)$ for $k \geqslant k(q)$.*

As a consequence of this last result, they have deduced that for every $k \geqslant k(q)$ K_{q+k} is not minimum (K_q, k) stable.

Remark 2.5 *From now on, throughout this section we assume that q and k are integers such that $q \geqslant 4$, $k \geqslant 1$ and for every r such that $0 \leqslant r < k$ we have $Q(K_q, r) = \binom{q+r}{2}$ and the only minimum (K_q, r) stable graph is K_{q+r}.*

In view of Theorem 2.4, k is bounded from above and we are interested in obtaining the greatest possible value of k.

Lemma 2.6 *Let G be a (K_q, k) stable graph such that $e(G) \leqslant \binom{q+k}{2}$. Then either for every vertex v we have $d(v) \leqslant q + k - 2$ or G is isomorphic to K_{q+k}.*

Proof: Suppose that some vertex v has degree at least $q+k-1$. By Lemma 1.2 the graph $G-v$ is $(K_q, k-1)$ stable, hence $Q(K_q, k-1) \leqslant e(G-v) = e(G) - d(v)$. Since $Q(K_q, k-1) = \binom{q+k-1}{2}$, we have

$\binom{q+k-1}{2} \leqslant e(G) - d(v) \leqslant \binom{q+k}{2} - (q+k-1) = \binom{q+k-1}{2}$.

It follows that $e(G-v) = \binom{q+k-1}{2}$, $G-v$ is isomorphic to K_{q+k-1} and $d(v) = q+k-1$. Hence, G is isomorphic to K_{q+k}. □

Lemma 2.7 *Let G be a minimum (K_q, k) stable graph. Then one of the following statements is true*

- *G has no component isomorphic to K_q ,*
- *$Q(K_q, k-1) + \binom{q}{2} \leqslant Q(K_q, k)$.*

Proof: Suppose that some component H of G is isomorphic to a K_q. If $G-H$ is not $(K_q, k-1)$ stable, then there is a set S with at most $k-1$ vertices intersecting each K_q of $G-H$. Then, for any vertex a of H, $S+a$ intersects each K_q of G while S has at most $k-1$ vertices, a contradiction. Hence $G-H$ is $(K_q, k-1)$ stable and we have $Q(K_q, k-1) \leqslant e(G-H) = Q(K_q, k) - \binom{q}{2}$. □

Lemma 2.8 *[3] Let G be a minimum (K_q, k) stable graph and let u be a vertex of degree $q-1$. Then one of the following statements is true*

- *$\forall v \in N(u) \quad d(v) \geqslant q+1$,*
- *$Q(K_q, k-1) + 3(q-2) \leqslant Q(K_q, k)$.*

Proof: By Proposition 1.3, since $d(u) = q-1$, $\{u\} \cup N(u)$ induces a complete graph on q vertices. Assume that some vertex $w \in N(u)$ has degree $q+r$ where $r=-1$ or $r=0$, and let v be a neighbour of u distinct from w. Since the degree of u in $G-v$ is $q-2$, no edge incident with u can be contained in a K_q of $G-v$. Since $G-v$ is $(K_q, k-1)$ stable, we can delete the $q-2$ edges incident with u in $G-v$ and the resulting graph G' is still $(K_q, k-1)$ stable. By deleting v, we have $e(G-v) \leqslant e(G) - (q-1)$ and hence

$$e(G^{'}) \leqslant e(G) - (q-1) - (q-2).$$

In G', the degree of w is now $q+r-2$. Hence, no edge incident with w in G' can be contained in a K_q. Deleting these $q+r-2$ edges from G' leads to a graph G" which remains to be $(K_q, k-1)$ stable. We get thus

$$Q(K_q, k-1) \leqslant e(G^{''}) \leqslant e(G) - (q-1) - (q-2) - (q+r-2).$$

Since $e(G) \leqslant Q(K_q, k)$, the result follows. □

Lemma 2.9 *Let G be a minimum (K_q, k) stable graph, where $1 \leqslant k \leqslant 2q-6$, and let v be a vertex of degree $q-1$. Then for every vertex $w \in N(v)$ we have $d(w) \geqslant q+1$.*

Proof: Suppose, contrary to the assertion of the lemma, that $d(w) \leqslant q$ for some vertex $w \in N(v)$. By Lemma 2.8, we have $Q(K_q, k-1) + 3(q-2) \leqslant Q(K_q, k)$. Since $Q(K_q, k-1) = \binom{q+k-1}{2}$ and $Q(K_q, k) \leqslant \binom{q+k}{2}$ we have $\binom{q+k-1}{2} + 3q - 6 \leqslant \binom{q+k}{2}$. Then we obtain $k \geqslant 2q-5$, a contradiction. □

Lemma 2.10 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 5$ and $1 \leqslant k \leqslant q-1$. Then the minimum degree of G is at least q.*

Proof: Suppose that there is a vertex v of degree $q-1$ and let w be a neighbour of v. Since $q-1 \leqslant 2q-6$, by Lemma 2.9, w has degree at least $q+1$. By Lemma 1.2 the graph $G-w$ is $(K_q, k-1)$ stable. In that graph v is not contained in any K_q since its degree is $q-2$. Hence $G - \{w, v\}$ is still $(K_q, k-1)$ stable. We have $e(G - \{w, v\}) = e(G) - (d(v) + d(w) - 1) \leqslant e(G) - 2q + 1$. Since $Q(K_q, k-1) = \binom{q+k-1}{2}$ and $Q(K_q, k) \leqslant \binom{q+k}{2}$ we have $\binom{q+k-1}{2} \leqslant e(G) - 2q + 1 \leqslant \binom{q+k}{2} - 2q + 1$. It follows that $k \geqslant q$, a contradiction. □

Lemma 2.11 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 5$ and $1 \leqslant k \leqslant q-1$, and let v be a vertex of degree q. Then the subgraph induced by $N(v)$ is complete.*

Proof: Suppose not, and assume that $N(v)$ contains two nonadjacent vertices a and b. Let $w \in N(v)$ distinct from a and b (w must exist since $q > 3$). By Lemma 1.2 the graph $G - w$ is $(K_q, k-1)$ stable. In that graph v is not contained in a K_q since its two neighbours a and b are not adjacent. Hence $G - \{w, v\}$ is still $(K_q, k-1)$ stable. By Lemma 2.10, $d(w) \geqslant q$ and hence $e(G - \{w, v\}) = e(G) - (d(v) + d(w) - 1) \leqslant e(G) - 2q + 1$. We have, as in the proof of Lemma 2.10, $\binom{q+k-1}{2} \leqslant e(G) - 2q + 1 \leqslant \binom{q+k}{2} - 2q + 1$, and we obtain $k \geqslant q$, a contradiction. □

Lemma 2.12 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 5$ and $2 \leqslant k \leqslant \frac{q}{2} + 1$, and let v be a vertex of degree at least $q+1$. Then either $N(v)$ induces a complete graph or there exists an ordering $v_1, \ldots, v_{d(v)}$ of the vertices of $N(v)$ such that $\{v_1, \ldots, v_{q-1}\}$ induces a complete graph and $v_{d(v)-1}v_{d(v)}$ is not in $E(G)$. Moreover, there exists a vertex w in $\{v_1, \ldots, v_{q-1}\}$ adjacent to $v_{d(v)-1}$ and $v_{d(v)}$.*

Proof: Suppose that the subgraph induced by $N(v)$ is not complete and let a and b be two nonadjacent neighbours of v.

Claim 2.12.1 *$N(v) - \{a, b\}$ contains a K_{q-1}.*

Proof of Claim: Let us suppose first that $d(a) = q$ (or $d(b) = q$). Hence, by Lemma 2.11, $N(a)$ induces a K_{q+1}. Since $v \in N(a)$ and $b \notin N(a)$, $N(v) - \{a, b\}$ contains a K_{q-1} as claimed. Hence we can assume now that $d(a) \geqslant q + 1$ and $d(b) \geqslant q + 1$. Suppose for contradiction that every K_{q-1} in $N(v)$ intersects $\{a, b\}$, that is, there is no K_q containing v in $G - \{a, b\}$. Since the graph $G - \{a, b\}$ is $(K_q, k-2)$ stable, the graph $G - \{a, b, v\}$ is still $(K_q, k-2)$ stable. Then $e(G - \{a, b, v\}) = e(G) - (d(v) + d(a) + d(b) - 2) \leqslant e(G) - 3q - 1$ and hence $Q(K_q, k-2) \leqslant e(G - \{a, b, v\}) \leqslant e(G) - 3q - 1 = Q(K_q, k) - 3q - 1$. Since $Q(K_q, k-2) = \binom{q+k-2}{2}$ and $Q(K_q, k) \leqslant \binom{q+k}{2}$, we have $\binom{q+k-2}{2} \leqslant \binom{q+k}{2} - 3q - 1$ and hence $\frac{q}{2} + 2 \leqslant k$, a contradiction to $k \leqslant \frac{q}{2} + 1$. □

Thus, we can order the vertices of $N(v)$ in such a way that the $q-1$ first ones $v_1, \ldots, v_{q-1}$ induce a complete graph and the two last vertices $v_{d(v)-1}$ and $v_{d(v)}$ are not adjacent, as claimed.
Set $d(v) = q + r$ with $r \geqslant 1$. By Proposition 1.3, the edges vv_{q+r-1} and vv_{q+r} are contained in two distinct q-cliques, say Q_1 and Q_2. Since v_{q+r-1} and v_{q+r} are not adjacent, each Q_i contains at most r vertices in $N(v) - \{v_1, \ldots, v_{q-1}\}$ and at least $q - r + 1$ vertices in $\{v_1, \ldots, v_{q-1}\}$. Since $N(v)$ is not complete and $e(G) \leqslant \binom{q+k}{2}$, by Lemma 2.6 we have $d(v) \leqslant q + k - 2$, and hence $r \leqslant k - 2$. Since $k \leqslant \frac{q}{2} + 1$, Q_1 (as well as Q_2) has at least $q - r + 1 \geqslant q - k + 3 > \frac{q}{2}$ vertices in $\{v_1, \ldots, v_{q-1}\}$. Hence Q_1 and Q_2 have at least one common vertex w in $\{v_1, \ldots, v_{q-1}\}$, and the Lemma follows. □

Lemma 2.13 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 5$ and $2 \leqslant k \leqslant \frac{q}{2} + 1$, and let H be a component of G. Then either H is complete or for every vertex v of maximum degree in H the subgraph induced by $N(v)$ contains no complete subgraph on $d(v) - 1$ vertices.*

Proof: Assume that H is not complete.

Claim 2.13.1 *The maximum degree in H is at least $q + 1$.*

Proof of Claim: If the minimum degree in H is at least $q + 1$, we are done. If there exists a vertex u of degree $q - 1$ in H then, by Lemma 2.9, the degree of any vertex of $N(u)$ is at least $q + 1$. If there exists a vertex u of degree q then, by Lemma 2.11, $N(u) \cup \{u\}$ induces a K_{q+1}. Since H is connected, there exists a vertex in $H - (N(u) \cup \{u\})$ having at least one neighbour w in $N(u)$, and clearly $d(w) \geqslant q + 1$. □

Let v be a vertex of maximum degree in H and set $d(v) = q + r$, with $k \geqslant 1$. Since H is not complete, the subgraph induced on $N(v)$ is not complete. By Lemma 2.12, there exists an ordering $\{v_1 \ldots v_{q+r}\}$ of the vertices of $N(v)$ such that $\{v_1, \ldots, v_{q-1}\}$ induces a complete graph and $v_{q+r-1}v_{q+r}$ is not an edge of G. Suppose that the subgraph induced by $N(v)$ contains a complete subgraph on $q + r - 1$ vertices. Then, without loss of generality we may suppose that $\{v_1, \ldots, v_{q+r-2}, v_{q+r-1}\}$ induces a complete graph. Let us denote by A the set of neighbours of v_{q+r} in $N(v)$.

Claim 2.13.2 *$|A| \geqslant q-2$, every vertex in A has degree $q+r$ and has no neighbour outside $N(v) \cup \{v\}$.*

Proof of Claim: Since G is a minimum (K_q, k) stable graph, by Proposition 1.3, the edge vv_{q+r} must be contained in a K_q. Hence v_{q+r} has at least $q-2$ neighbours in $\{v_1, \ldots, v_{q+r-2}\}$. Since the subgraph induced by $(N(v)-\{v_{q+r}\})$ is complete, every vertex a in A is adjacent to every vertex in $(N(v)-\{a\}) \cup \{v\}$. Then $d(a) = q+r$, i.e. a has maximum degree in H. Hence, no vertex in A has a neighbour outside $N(v) \cup \{v\}$, and the Claim follows. □

By Lemma 2.2, the $(q+r)$-clique $(N(v)-\{v_{q+r}\}) \cup \{v\}$ is a proper subgraph of $H-\{v_{q+r}\}$. Since H is connected, there exists a vertex w outside $N(v) \cup \{v\}$ adjacent to a vertex u in $N(v)$. Let us denote by B the set of neighbours of w in $N(v)$. Since the edge uw is contained in a K_q by Proposition 1.3, w must have at least $q-2$ common neighbours with u in $N(v)$, and hence $|B| \geqslant q-1$. Since by Claim 2.13.2 A has no neighbour outside $N(v) \cup \{v\}$, A and B are disjoint. Then we have $2q-3 \leqslant |A \cup B| \leqslant |N(v)| = q+r$, and hence $q \leqslant r+3$. Since $r \leqslant k-2$ by Lemma 2.6, we obtain $q \leqslant k+1 \leqslant \frac{q}{2}+2$, that is $q \leqslant 4$, a contradiction. Hence, the subgraph induced by the vertices $\{v_1, \ldots, v_{q+r-2}, v_{q+r-1}\}$ is not complete, and the Lemma follows. □

Proposition 2.14 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 5$ and $2 \leqslant k \leqslant \frac{q}{2}+1$. Then every component of G is a complete graph.*

Proof: Let H be a component of G and v be a vertex of maximum degree in H. If the subgraph induced on $N(v)$ is complete then H is obviously complete. We can thus assume that $N(v)$ is not a clique. By Lemmas 2.10 and 2.11, the minimum degree is at least $q+1$, and hence $d(v) = q+r$ with $r \geqslant 1$.

Claim 2.14.1 *The graph $G-(N(v) \cup \{v\})$ is $(K_q, k-r)$ stable.*

Proof of Claim 2.14.1: By Lemma 2.12, we can consider an ordering $v_1, \ldots, v_{q+r}$ of $N(v)$ such that the set $\{v_1, \ldots, v_{q-1}\}$ induces a K_{q-1}, $v_{q+r-1}v_{q+r} \notin E(G)$ and there is a vertex $w \in \{v_1, \ldots, v_{q-1}\}$ adjacent to v_{q+r-1} and v_{q+r}. By Lemma 2.13, we can find two nonadjacent vertices a and b in $N(v)-\{v_{q+r}\}$ and two nonadjacent vertices c and d in $N(v)-\{v_{q+r-1}\}$. Let us note that since the set $\{v_1, \ldots, v_{q-1}\}$ induces a complete graph, it contains at most one vertex of the set $\{a, b\}$ and at most one vertex of $\{c, d\}$. Then, $|\{v_1, \ldots, v_{q-1}\} \cap \{w, a, b, c, d\}| \leqslant 3$.
Since H is not complete, the graph G is not complete and by Lemma 2.6 we have $r \leqslant k-2$. Since $k \leqslant \frac{q}{2}+1$ and $q \geqslant 6$, there exists a subset $A \subseteq \{v_1 \ldots v_{q-1}\}$ such that

- $|A| = r$,
- $w \notin A$,
- $A \cap \{a, b, c, d\} = \emptyset$.

By repeated applications of Lemma 1.2, the graph G_1 obtained from G by deleting A is $(K_q, k-r)$ stable. In G_1, the degree of v is equal to q.
Without loss of generality, suppose that a is distinct from v_{q+r-1} and c is distinct from v_{q+r}. If there exists a q-clique in G_1 containing the edge vv_{q+r-1} then $\{v_1, \dots, v_{q+r-2}\} - A$ is a $(q-2)$-clique containing a. Since ab is not an edge, we must have $b = v_{q+r-1}$, a contradiction to the fact that av_{q+r-1} is an edge. Thus, there is no q-clique in G_1 containing vv_{q+r-1}. Analogously, we prove that there is no q-clique in G_1 containing vv_{q+r}.
Hence, the graph G_2 obtained from G_1 by deletion of the edges vv_{q+r-1} and vv_{q+r} is still $(K_q, k-r)$ stable. In G_2, v has degree $q-2$, so it is not contained in any K_q. We can thus delete v and we get a $(K_q, k-r)$ stable graph G_3.
Since the maximum degree in G is $q+r$, the degree of w in G_3 is at most $q-1$. Recall that w is adjacent to the two nonadjacent vertices v_{q+r-1} and v_{q+r}. Hence w is not contained in any K_q of G_3, which means that $G_4 = G_3 - w$ is still $(K_q, k-r)$ stable. Since the degree of each vertex in $\{v_1, \dots, v_{q-1}\} - (A \cup \{w\})$ is at most $q-2$ in G_4, none of these vertices can be contained in any K_q of G_4. Hence by deletion of these vertices we get again a $(K_q, k-r)$ stable graph G_5. We shall prove that none of the $r+1$ vertices $v_q, \dots, v_{q+r}$ is contained in a K_q of G_5.
Note that $G_5 = G - \{v, v_1, \dots, v_{q-1}\}$. For $q \leqslant j \leqslant q+r$, denote by d_j the degree of the vertex v_j in the subgraph induced by $\{v_q, \dots, v_{q+r}\}$. Clearly we have $0 \leqslant d_j \leqslant r$. In G, by Proposition 1.3, the edge vv_j is contained in a K_q. Hence v_j is adjacent (in G) to at least $q-2-d_j$ vertices in $\{v_1, \dots, v_{q-1}\}$. Since we have deleted the vertex v and the vertices $v_1, \dots, v_{q-1}$, we have thus $d_{G_5}(v_j) \leqslant q+r-(q-2-d_j)-1 = r+1+d_j$. If $d_j \leqslant r-1$ then $d_{G_5}(v_j) \leqslant 2r \leqslant 2(k-2) \leqslant q-2$ and there is no K_q in G_5 containing v_j. The equality $d_{G_5}(v_j) = q-1$ can only be obtained when $d_j = r$, that is v_j has r neighbours in $v_q \dots v_{q+r}$. Since v_{q+r-1} and v_{q+r} are not adjacent, v_j is not contained in any K_q of G_5.

Hence, the graph $G_6 = G - (N(v) \cup \{v\})$ obtained from G_5 by deletion of all the vertices $v_q, v_{q+1} \dots, v_{q+r}$ is still $(K_q, k-r)$ stable, and the Claim follows. □

Claim 2.14.2

$$\binom{q+k-r}{2} + q + r + \binom{q-1}{2} + \frac{1}{2}(r+1)(2q-r-2) + 1 \leqslant \binom{q+k}{2} \tag{1}$$

Proof of Claim: 2.14.2 To get back G from $G - (N(v) \cup \{v\})$ we add, at least

- the $q+r$ edges incident with v,
- the $\binom{q-1}{2}$ edges of the $(q-1)$-clique induced by the set $\{v_1, \dots, v_{q-1}\}$,
- the edges incident with $\{v_q, \dots, v_{q+r}\}$ and not incident with v.

Let l be the number of edges incident with $v_q, \dots, v_{q+r}$, and not incident with v.

We have

$$e(G - (N(v) \cup \{v\})) + q + r + \binom{q-1}{2} + l \leqslant e(G) \tag{2}$$

In order to find a lower bound of the number of edges incident with the vertices $v_q, \ldots, v_{q+r}$, for each $i \in \{q, \ldots, q+r\}$ let us denote by d_i the degree of the vertex v_i in the subgraph induced by the set $\{v_q, \ldots, v_{q+r}\}$. Then,

$$l = \frac{1}{2}\Sigma_{i=q}^{q+r} d_i + \Sigma_{i=q}^{q+r}(d_G(v_i) - 1 - d_i) = \Sigma_{i=q}^{q+r} d_G(v_i) - (r+1) - \frac{1}{2}\Sigma_{i=q}^{q+r} d_i.$$

Since by Lemma 2.10 the minimum degree in G is at least q, we have

$$l \geqslant q(r+1) - (r+1) - \frac{1}{2}\Sigma_{i=q}^{q+r} d_i.$$

Since for every i in $\{q, \ldots, q+r-2\}$ $d_i \leqslant r$, $d_{q+r-1} \leqslant r-1$ and $d_{q+r} \leqslant r-1$, we obtain

$$l \geqslant q(r+1) - (r+1) - \frac{1}{2}r(r-1) - (r-1),$$

and hence

$$l \geqslant \frac{1}{2}(r+1)(2q-r-2) + 1.$$

By the assumption made at the beginning of the section (see Remark 2.5), a minimum $(K_q, k-r)$ stable graph has $\binom{q+k-r}{2}$ edges. Since $e(G) \leqslant \binom{q+k}{2}$, the inequality (1) follows from Claim 2.14.1 and the inequality (2). This proves the Claim. □

A simple calculation shows that the inequality

$$q^2 + q + 2 \leqslant 2kr$$

can be obtained by starting from the inequality (1).
Since $r \leqslant k-2$ and $k \leqslant \frac{q}{2} + 1$, we have $q^2 + q + 2 \leqslant 2k(k-2) \leqslant (q+2)(\frac{q}{2} - 1)$, hence $\frac{q^2}{2} + q + 4 \leqslant 0$, a contradiction. Thus, $N(v)$ is a clique and the Proposition follows. □

3 Result

In [3], it is shown that if G is minimum (K_q, k) stable and the numbers k and q satisfy one of the following conditions:

- $k = 1$ and $q \geqslant 4$
- $k = 2$ and $q \geqslant 4$
- $k = 3$ and $q \geqslant 5$

then G is isomorphic to K_{q+k}.

Theorem 3.1 *Let G be a minimum (K_q, k) stable graph, where $q \geqslant 6$ and $k \leqslant \frac{q}{2}+1$. Then G is isomorphic to K_{q+k}.*

Proof: For $0 \leqslant k \leqslant 3$ the graph G is isomorphic to K_{q+k}. Let k be such that $4 \leqslant k \leqslant \frac{q}{2}+1$ and suppose that for every r with $0 \leqslant r < k$ the only minimum (K_q, r) stable graph is K_{q+r}. By Proposition 2.14, the graph G is the disjoint union of p complete graphs $H_1 \equiv K_{q+k_1}$, $H_2 \equiv K_{q+k_2}$, $\cdots$, $H_p \equiv K_{q+k_p}$. Suppose, without loss of generality, that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_p \geqslant 0$ and that there exist two components H_i and H_j with $i < j$ such that $k_i - k_j \geqslant 2$. By substituting $H_i' \equiv K_{q+k_i-1}$ for H_i and $H_j' \equiv K_{q+k_j+1}$ for H_j, we obtain a new (K_q, k) stable graph G' such that $e(G') = e(G) - (k_i - k_j - 1) < e(G)$, which is a contradiction. Thus, for any i and any j, $0 \leqslant |k_i - k_j| \leqslant 1$ (cf [2] Proposition 7).
To conclude that G has a unique component, observe the following facts.

- The graphs $2K_{q+l}$ and K_{q+2l+1} are both $(K_q, 2l+1)$ stable, but if $2l+1 \leqslant \frac{q}{2}+1$ then $\binom{q+2l+1}{2} < 2\binom{q+l}{2}$.
- The graphs $K_{q+l} + K_{q+l+1}$ and K_{q+2l+2} are both $(K_q, 2l+2)$ stable but if $2l+2 \leqslant \frac{q}{2}+1$ then $\binom{q+2l+2}{2} < \binom{q+l+1}{2} + \binom{q+l}{2}$.

□

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