The Number of Nilpotent Semigroups of Degree 3

Andreas Distler*
Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa
Portugal
adistler@fc.ul.pt

J. D. Mitchell
University of St Andrews,
Mathematical Institute,
North Haugh,
St Andrews,
Fife, KY16 9SS,
Scotland, UK
jdm3@st-and.ac.uk

Submitted: Jan 16, 2012; Accepted: Jun 15, 2012; Published: Jun 28, 2012
Mathematics Subject Classifications: 05A15, 20M99

Abstract

A semigroup is *nilpotent of degree* 3 if it has a zero, every product of 3 elements equals the zero, and some product of 2 elements is non-zero. It is part of the folklore of semigroup theory that almost all finite semigroups are nilpotent of degree 3.

We give formulae for the number of nilpotent semigroups of degree 3 on a set with $n \in \mathbb{N}$ elements up to equality, isomorphism, and isomorphism or anti-isomorphism. Likewise, we give formulae for the number of nilpotent commutative semigroups on a set with n elements up to equality and up to isomorphism.

Keywords: nilpotent semigroups; power group enumeration; nilpotency degree

1 Introduction

The topic of enumerating finite algebraic or combinatorial objects of a particular type is classical. Many theoretical enumeration results were obtained thanks to the advanced orbit counting methods developed by Redfield [Red27], Polya [Pol37], and de Bruijn [dB59]. Numerous applications of the method known as power group enumeration can be found in [HP73]. Of particular interest for this paper is the usage to count universal algebras in [Har66].

The enumeration of finite semigroups has mainly been performed by exhaustive search and the results are therefore restricted to very small orders. The most recent numbers

*Supported by the University of St Andrews and the project PTDC/MAT/101993/2008 of Centro de Álgebra da Universidade de Lisboa, financed by FCT and FEDER.

are of semigroups of order 9 [Dis10], of semigroups with identity of order 10 [DK09], commutative semigroups of order 10 [Gri03], and linearly ordered semigroups of order 7 [Sla95].

In this paper we use power group enumeration to develop formulae for the number of semigroups of a particular type, which we now define.

A semigroup S is *nilpotent* if there exists a $r \in \mathbb{N}$ such that the set

$$S^r = \{ s_1 s_2 \cdots s_r \mid s_1, s_2, \ldots, s_r \in S \}$$

has size 1. If r is the least number such that $|S^r| = 1$, then we say that S has *(nilpotency) degree* r.

As usual, the number of 'structural types' of objects is of greater interest than the number of distinct objects. Let S and T be semigroups. Then a function $f : S \to T$ is an *isomorphism* if it is a bijection and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in S$. The *dual* S^* of S is the semigroup with multiplication $*$ defined by $x * y = y \cdot x$ on the set S. A bijection $f : S \to T$ is an *anti-isomorphism* if f is an isomorphism from S^* to T. Throughout this article we distinguish between the number of distinct semigroups on a set, the number up to isomorphism, and the number up to isomorphism or anti-isomorphism. We shall refer to the number of distinct semigroups that can be defined on a set as the number *up to equality*.

For $n \in \mathbb{N}$ we let $z(n)$ denote the number of nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$. The particular interest in nilpotent semigroups of degree 3 stems from the observation that almost all finite semigroups are of this type. More precisely, Kleitman, Rothschild, and Spencer identified $z(n)$ in [KRS76] as an asymptotic lower bound for the number of all semigroups on that set. Furthermore, Jürgensen, Migliorini, and Szép suspected in [JMS91] that $z(n)/2n!$ was a good lower bound for the number of semigroups with n elements up to isomorphism or anti-isomorphism based on the comparison of these two numbers for $n = 1, 2, \ldots, 7$. This belief was later supported by Satoh, Yama, and Tokizawa [SYT94, Section 8] and the first author [Dis10] in their analyses of the semigroups with orders 8 and 9, respectively.

This paper is structured as follows: in the next section we present and discuss our results, delaying certain technical details for later sections; in Section 3 we describe a way to construct semigroups of degree 2 or 3; in Section 4 nilpotent semigroups of degree 3 are considered up to equality; in Section 5 we present the relevant background material from power group enumeration and a number of technical results in preparation for Section 6 where we give the proofs for our main theorems. Tables containing the first few terms of the sequences defined by the various formulae in the paper can be found at the appropriate points. The implementation used to obtain these numbers is provided as the function `Nr3NilpotentSemigroups` in the computer algebra system GAP [GAP08] by the package Smallsemi [DM11].

2 Formulae for the number of nilpotent semigroups of degree 3

2.1 Up to equality

The number of nilpotent and commutative nilpotent semigroups of degree 3 on a finite set can be computed using formulae given in [JMS91, Theorems 15.3 and 15.8]. We summarise the relevant results in the following theorem. As the theorems in [JMS91] are stated incorrectly we shall give a proof for Theorem 1 in Section 4.

Theorem 1. *For $n \in \mathbb{N}$ the following hold:*

(i) *the number of distinct nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$ is*

$$\sum_{m=2}^{a(n)} \binom{n}{m} m \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (m-i)^{\binom{(n-m)^2}{}}$$

where $a(n) = \left\lfloor n + 1/2 - \sqrt{n - 3/4} \right\rfloor$;

(ii) *the number of distinct commutative nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$ is*

$$\sum_{m=2}^{c(n)} \binom{n}{m} m \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (m-i)^{(n-m)(n-m+1)/2}$$

where $c(n) = \left\lfloor n + 3/2 - \sqrt{2n + 1/4} \right\rfloor$.

Note that there are no nilpotent semigroups of degree 3 with fewer than 3 elements. Accordingly, the formulae in Theorem 1 yield that the number of nilpotent and commutative nilpotent semigroups of degree 3 with 1 or 2 elements is 0. The first few non-zero terms of the sequences given by Theorem 1 are shown in Tables 1 and 2.

2.2 Up to isomorphism and up to isomorphism or anti-isomorphism

Our main results are explicit formulae for the number of nilpotent and commutative nilpotent semigroups of degree 3 on any finite set up to isomorphism and up to isomorphism or anti-isomorphism. As every commutative semigroup is equal to its dual we obtain three different formulae.

If j is a partition of $n \in \mathbb{N}$, written as $j \vdash n$, then we denote by j_i the number of summands equalling i. The first of our main theorems, dealing with nilpotent semigroups of degree 3 up to isomorphism, can then be stated as follows:

Table 1: Numbers of nilpotent semigroups of degree 3 up to equality

n	number of nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$
3	6
4	180
5	11 720
6	3 089 250
7	5 944 080 072
8	147 348 275 209 800
9	38 430 603 831 264 883 632
10	90 116 197 775 746 464 859 791 750
11	2 118 031 078 806 486 819 496 589 635 743 440
12	966 490 887 282 837 500 134 221 233 339 527 160 717 340
13	17 165 261 053 166 610 940 029 331 024 343 115 375 665 769 316 911 576
14	6 444 206 974 822 296 283 920 298 148 689 544 172 139 277 283 018 112 679 406 098 010

Table 2: Numbers of commutative nilpotent semigroups of degree 3 up to equality

n	number of commutative nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$
3	6
4	84
5	1 620
6	67 170
7	7 655 424
8	2 762 847 752
9	3 177 531 099 864
10	11 942 816 968 513 350
11	170 387 990 514 807 763 280
12	11 445 734 473 992 302 207 677 404
13	3 783 741 947 416 133 941 828 688 621 484
14	5 515 869 594 360 617 154 295 309 604 962 217 274
15	33 920 023 793 863 706 955 629 537 246 610 157 737 736 800
16	961 315 883 918 211 839 933 605 601 923 922 425 713 635 603 848 080
17	160 898 868 329 022 121 111 520 489 011 089 643 697 943 356 922 368 997 915 120

Table 3: Numbers of nilpotent semigroups of degree 3 up to isomorphism

n	number of non-isomorphic nilpotent semigroups of degree 3 of order n
3	1
4	9
5	118
6	4 671
7	1 199 989
8	3 661 522 792
9	105 931 872 028 455
10	24 834 563 582 168 716 305
11	53 061 406 576 514 239 124 327 751
12	2 017 720 196 187 069 550 262 596 208 732 035
13	2 756 576 827 989 210 680 367 439 732 667 802 738 773 384
14	73 919 858 836 708 511 517 426 763 179 873 538 289 329 852 786 253 510
15	29 599 937 964 452 484 359 589 007 277 447 538 854 227 891 149 791 717 673 581 110 642

Theorem 2. *Let $n, p, q \in \mathbb{N}$. For $1 \leqslant q < p$ denote*

$$N(p,q) = \sum_{j \vdash q-1} \sum_{k \vdash p-q} \left(\prod_{i=1}^{q-1} j_i!\, i^{j_i} \prod_{i=1}^{p-q} k_i!\, i^{k_i} \right)^{-1} \prod_{a,b=1}^{p-q} \left(1 + \sum_{d \mid \operatorname{lcm}(a,b)} d j_d \right)^{k_a k_b \gcd(a,b)} . \qquad (1)$$

Then the number of nilpotent semigroups of degree 3 and order n up to isomorphism equals

$$\sum_{m=2}^{a(n)} \left(N(n,n) - N(n-1,m-1) \right) \quad \textit{where} \quad a(n) = \left\lfloor n + 1/2 - \sqrt{n - 3/4} \right\rfloor ,$$

The second of our main theorems gives the number of nilpotent semigroups of degree 3 up to isomorphism or anti-isomorphism.

Theorem 3. *Let $n, p, q \in \mathbb{N}$. For $1 \leqslant q < p$ let $N(p,q)$ as in (1) and denote*

$$L(p,q) = \frac{1}{2} N(p,q) + \frac{1}{2} \sum_{j \vdash q-1} \sum_{k \vdash p-q} \left(\prod_{i=1}^{q-1} j_i!\, i^{j_i} \prod_{i=1}^{p-q} k_i!\, i^{k_i} \right)^{-1} \prod_{a=1}^{p-q} \left(q_a^{k_a} p_{a,a}^{k_a^2 - k_a} \prod_{b=1}^{a-1} p_{a,b}^{2 k_a k_b} \right), \quad (2)$$

where

$$p_{a,b} = \left(1 + \sum_{d \mid \operatorname{lcm}(2,a,b)} d j_d \right)^{ab/\operatorname{lcm}(2,a,b)}$$

and

$$q_a = \begin{cases} (1 + \sum_{d \mid a} d j_d)(1 + \sum_{d \mid 2a} d j_d)^{(a-1)/2} & \text{if } a \equiv 1 \mod 2 \\ (1 + \sum_{d \mid a} d j_d)^a & \text{if } a \equiv 0 \mod 4 \\ (1 + \sum_{d \mid a/2} d j_d)^2 (1 + \sum_{d \mid a} d j_d)^{a-1} & \text{if } a \equiv 2 \mod 4. \end{cases}$$

Table 4: Numbers of nilpotent semigroups of degree 3 up to isomorphism or anti-isomorphism

n	number of non-(anti-)isomorphic nilpotent semigroups of degree 3 of order n
3	1
4	8
5	84
6	2 660
7	609 797
8	1 831 687 022
9	52 966 239 062 973
10	12 417 282 095 522 918 811
11	26 530 703 289 252 298 687 053 072
12	1 008 860 098 093 547 692 911 901 804 990 610
13	1 378 288 413 994 605 341 053 354 105 969 660 808 031 163
14	36 959 929 418 354 255 758 713 676 933 402 538 920 157 765 946 956 889
15	14 799 968 982 226 242 179 794 503 639 146 983 952 853 044 950 740 907 666 303 436 922

Then the number of nilpotent semigroups of degree 3 *and order* n *up to isomorphism or anti-isomorphism equals*

$$\sum_{m=2}^{a(n)} (L(n,m) - L(n-1,m-1)) \quad \textit{where} \quad a(n) = \left\lfloor n + 1/2 - \sqrt{n - 3/4} \right\rfloor.$$

A semigroup is *self-dual* if it is isomorphic to its dual. The concept of anti-isomorphism has no relevance for self-dual semigroups. Combining Theorems 2 and 3, it is possible to deduce a formula for the number of self-dual, nilpotent semigroups of degree 3 up to isomorphism. More generally, considering semigroups of a certain type the number of self-dual semigroups up to isomorphism is equal to twice the number of semigroups up to isomorphism and anti-isomorphism minus the number of semigroups up to isomorphism.

Corollary 4. *Let* $n \in \mathbb{N}$ *and let* $N(p,q)$ *and* $L(p,q)$ *be as defined in* (1) *and* (2)*, respectively. Then the number of self-dual, nilpotent semigroups of degree* 3 *and order* n *up to isomorphism equals*

$$\sum_{m=2}^{a(n)} (2L(n,m) - N(n,m) - 2L(n-1,m-1) + N(n-1,m-1))$$
$$\textit{where} \quad a(n) = \left\lfloor n + 1/2 - \sqrt{n - 3/4} \right\rfloor.$$

Substituting in the previous corollary the actual formula for $2L(p,q)$ we notice that $N(p,q)/2$ appears as a term in $L(p,q)$ and cancels. The resulting simplified formula is implemented as part of the function `Nr3NilpotentSemigroups` in Smallsemi [DM11].

Since commutative semigroups are self-dual, we obtain just one formula up to isomorphism for commutative nilpotent semigroups of degree 3.

Table 5: Numbers of self-dual nilpotent semigroups of degree 3 up to isomorphism

n	number of non-isomorphic self-dual nilpotent semigroups of degree 3 of order n
3	1
4	7
5	50
6	649
7	19 605
8	1 851 252
9	606 097 491
10	608 877 121 317
11	1 990 358 249 778 393
12	25 835 561 207 401 249 185
13	1 739 268 479 271 518 877 288 942
14	590 686 931 539 550 985 679 107 660 268
15	846 429 051 478 198 751 690 097 659 025 763 202

Theorem 5. *Let $n, p, q \in \mathbb{N}$. For $1 \leqslant q < p$ denote*

$$K(p,q) = \sum_{j \vdash q-1} \sum_{k \vdash p-q} \left[\left(\prod_{i=1}^{q-1} j_i!\, i^{j_i} \prod_{i=1}^{p-q} k_i!\, i^{k_i} \right)^{-1} \prod_{a=1}^{\lfloor \frac{n}{2} \rfloor} \left(1 + \sum_{d|a} d j_d \right)^{k_{2a}} \left(1 + \sum_{d|2a} d j_d \right)^{a k_{2a}} \cdot \prod_{a=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(1 + \sum_{d|2a-1} d j_d \right)^{a k_{2a-1}} \prod_{a<b} \left(1 + \sum_{d|\mathrm{lcm}(a,b)} d j_d \right)^{k_a k_b \gcd(a,b)} \right].$$

Then the number of nilpotent, commutative semigroups of degree 3 and order n up to isomorphism equals

$$\sum_{m=2}^{c(n)} \left(K(n,m) - K(n-1,m-1) \right) \quad \text{where} \quad c(n) = \left\lfloor n + 3/2 - \sqrt{2n + 1/4} \right\rfloor.$$

To determine the number of nilpotent semigroups of degree 3 up to isomorphism or up to isomorphism or anti-isomorphism, we use the technique of power group enumeration in a similar way as Harrison did for universal algebras [Har66]. In Section 5 we present the relevant background material and a number of technical results in preparation for Section 6 where we give the proofs for Theorems 2, 3, and 5.

2.3 Bounds and asymptotics

The formula for the number of nilpotent semigroups of degree 3 up to isomorphism or anti-isomorphism in Theorem 3 provides a new lower bound for the number of semigroups up to isomorphism or anti-isomorphism of a given size. Presumably this bound is asymptotic,

Table 6: Numbers of commutative nilpotent semigroups of degree 3 up to isomorphism

n	number of non-isomorphic commutative nilpotent semigroups of degree 3 of order n
3	1
4	5
5	23
6	155
7	2 106
8	79 997
9	9 350 240
10	3 377 274 621
11	4 305 807 399 354
12	23 951 673 822 318 901
13	608 006 617 857 847 433 462
14	63 282 042 551 031 180 915 403 659
15	25 940 470 166 038 603 666 194 391 357 972
16	45 946 454 978 824 286 601 551 283 052 739 171 318
17	452 361 442 895 926 947 438 998 019 240 982 893 517 749 169
18	30 258 046 596 218 438 115 657 059 107 812 634 405 962 381 166 457 711
19	12 094 270 656 160 403 920 767 935 604 624 748 908 993 169 949 317 454 767 617 795

that is, the ratio tends to 1 while the order tends to infinity, although this is not a consequence of the result for semigroups up to equality in [KRS76]. The comparison in Table 7 shows also that the lower bound $z(n)/2n!$ from [JMS91] seems to converge rapidly towards our new bound. Analogous observations can be made considering only commutative semigroups though the convergence appears slower as mentioned by Grillet in the analysis in [Gri03].

Our formulae also yield a large qualitative improvement over the old lower bound since they give exact numbers of nilpotent semigroups of degree 3. In particular, the provided numbers can be used to cut down the effort required in an exhaustive search to determine the number of semigroups of a given order, as already done for semigroups of order 9 in [Dis10].

The conjectured asymptotic behaviour of the lower bound of $z(n)/2n!$ for the number of semigroups of order n would imply that almost all sets of isomorphic semigroups on $\{1, 2, \ldots, n\}$ are of size $n!$. In other words, most semigroups have trivial automorphism group; a property that is known for various types of algebraic and combinatorial objects, for example graphs [ER63]. Our formulae could help to prove this conjecture at least for nilpotent semigroups of degree 3. In each summand in (1) those semigroups of degree 3 are counted for which a bijection with cycle structure corresponding to the partitions j and k is an automorphism. It remains to estimate the contribution of all summands that do not correspond to the identity map.

Table 7: Numbers of semigroups and nilpotent semigroups of degree 3

n	number of semigroups up to isomorphism or anti-isomorphism	number of semigroups of degree 3 up to isomorphism or anti-isomorphism	lower bound $\lceil z(n)/2n! \rceil$
3	18	1	1
4	126	8	4
5	1 160	84	49
6	15 973	2 660	2 146
7	836 021	609 797	589 691
8	1 843 120 128	1 831 687 022	1 827 235 556
9	52 989 400 714 478	52 966 239 062 973	52 952 220 887 436
10	*unknown*	12 417 282 095 522 918 811	12 416 804 146 790 463 082

3 Construction of nilpotent semigroups of degree 2 or 3

In this section we describe how to construct nilpotent semigroups of degree 2 or 3 on an n-element set. A similar construction is given in [KRS76]. For the sake of brevity we will denote by $[n]$ the set $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$.

Definition 6. Let $n \geqslant 2$, let A be a non-empty proper subset of $[n]$, and let B denote the complement of A in $[n]$. If $z \in B$ is arbitrary and $\psi : A \times A \to B$ is any function, then we can define multiplication on $[n]$ by

$$xy = \begin{cases} \psi(x, y) & \text{if } x, y \in A \\ z & \text{otherwise.} \end{cases} \tag{3}$$

We will denote the set $[n]$ with the operation given above by $H(A, \psi, z)$.

Any product abc in $H(A, \psi, z)$ equals z, and so the multiplication defined in (3) is associative. It follows that $H(A, \psi, z)$ is a nilpotent semigroup of degree 2 or 3. The semigroup $H(A, \psi, z)$ has degree 2 if and only if $H(A, \psi, z)$ is a zero semigroup if and only if ψ is the constant function with value z. Conversely, if T is a nilpotent semigroup of degree 3 with elements $[n]$, then setting $A = T \setminus T^2$, letting $\psi : A \times A \to T^2$ be defined by $\psi(x, y) = xy$ for all $x, y \in T$, and setting z to be the zero element of T, we see that $T = H(A, \psi, z)$. Therefore when enumerating nilpotent semigroups of degree 3 it suffices to consider the semigroups $H(A, \psi, z)$.

4 Semigroups and commutative semigroups of degree 3 up to equality

Denote by Z_n the set of nilpotent semigroups of degree 3 on $\{1, 2, \ldots, n\}$. A formula for the cardinality of a proper subset of Z_n is stated in Theorem 15.3 of [JMS91]. However, the

formula given in [JMS91] actually yields $|Z_n|$ and this is what the proof of the theorem in [JMS91] shows. Similarly, the formula in Theorem 15.8 of [JMS91] can be used to determine the number of all commutative semigroups in Z_n even though the statement says otherwise. For the sake of completeness and to avoid confusion we prove that the formulae as given in Theorem 1 are correct.

Proof of Theorem 1. In both parts of the proof, we let A be a fixed non-empty proper subset of $[n] = \{1, 2, \ldots, n\}$, let B denote the complement of A in $[n]$, let $m = |B|$, and let $z \in B$ be fixed. We consider semigroups of the form $H(A, \psi, z)$ where $\psi : A \times A \to B$ as given in Definition 6.

(i). The number of functions from $A \times A$ to B is $m^{(n-m)^2}$. To avoid counting semigroups twice for different m, we will only consider those functions ψ where every element in $B \setminus \{z\}$ appears in the image of ψ. For a subset X of $B \setminus \{z\}$ of size i, there are $(m-i)^{(n-m)^2}$ functions with no element from X in their image. Using the Inclusion-Exclusion Principle, the number of functions from $A \times A$ to B with image containing $B \setminus \{z\}$ is

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (m-i)^{(n-m)^2}. \tag{4}$$

The function ψ is defined on a set with $(n-m)^2$ elements. Hence the condition that $B \setminus \{z\}$ is contained in the image of ψ implies that $m - 1 \leqslant (n-m)^2$. Reformulation yields

$$m \leqslant n + 1/2 - \sqrt{n - 3/4}. \tag{5}$$

If $m = 1$, then every function $\psi : A \times A \to B$ is constant, and so, as mentioned above, $H(A, \psi, z)$ is not nilpotent of degree 3. Summing (4) over all appropriate values of m, the $\binom{n}{m}$ choices for B and the m choices for $z \in B$ concludes the proof of this part.

(ii). If $H(A, \psi, z)$ is a commutative semigroup, then the function $\psi : A \times A \to B$ is defined by its values on pairs (i, j) with $i \leqslant j$. There are $(n-m)(n-m+1)/2$ such pairs and hence there are $m^{(n-m)(n-m+1)/2}$ such functions ψ.

The rest of the proof follows the same steps as the proof of part (i) with $m^{(n-m)(n-m+1)/2}$ replacing $m^{(n-m)^2}$ and where the inequality $m - 1 \leqslant (n-m)(n-m+1)/2$ yields the parameter $c(n)$. □

5 Power group enumeration

In this section, we shall introduce the required background material relating to power group enumeration and determine the cycle indices of certain power groups necessary to prove our main theorems. The presentation in this section is based on [HP73].

Let X be a non-empty set and let S_X denote the symmetric group on X. We again denote the set $\{1, 2, \ldots, n\}$ by $[n]$, and will write S_n instead of S_X if $X = [n]$. For a permutation $\pi \in S_X$, let $\delta(\pi, k)$ denote the number of cycles of length k in the disjoint cycle decomposition of π.

Definition 7. Let G be a subgroup of S_n. Then the polynomial

$$\mathcal{Z}(G; x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} x_k^{\delta(g,k)}$$

is called the *cycle index* of the group G; in short, we write $\mathcal{Z}(G)$.

The cycle structure of a permutation $\pi \in S_n$ corresponds to a partition of n, and all elements with the same cycle structure form a conjugacy class of S_n. Remember that if j is a partition of n, written as $j \vdash n$, then we denote by j_i the number of summands equalling i. This yields $j_i = \delta(\pi, i)$ for all i and for each element π in the conjugacy class corresponding to j. This observation allows us to write the cycle index of the symmetric group in a compact form.

Lemma 8 ([HP73, (2.2.5)])**.** *The cycle index of S_n is*

$$\mathcal{Z}(S_n) = \sum_{j \vdash n} \left(\prod_{i=1}^{n} j_i! i^{j_i} \right)^{-1} \prod_{a=1}^{n} x_a^{j_a}.$$

In what follows we require actions other than the natural action of the symmetric group S_X on X. In particular, we require actions on functions in which two groups act independently on the domains and on the images of the functions. If G is a group acting on a set X, then we denote by x^g the image of $x \in X$ under the action of $g \in G$.

Definition 9. Let A and B be subgroups of S_X and S_Y, respectively, where X and Y are finite disjoint sets. Then we define an action of the group $A \times B$ on the set Y^X of functions from X to Y in the following way: the image of $f \in Y^X$ under $(\alpha, \beta) \in A \times B$ is given by

$$f^{(\alpha,\beta)}(x) = \left(f\left(x^{\alpha}\right)\right)^{\beta}$$

for all $x \in X$. We will refer to $A \times B$ with this action as a *power group*.

The cycle index itself is not required for the power groups used in this paper. Of interest is the constant form of the Power Group Enumeration Theorem given below, which states the number of orbits under the action of a power group. The result goes back to de Bruijn [dB59], but is presented here in the form given in [HP73, Section 6.1].

Theorem 10. *Let $A \times B$ be a power group acting on the functions Y^X as in Definition 9. Then the number of orbits of $A \times B$ on Y^X equals*

$$\frac{1}{|B|} \sum_{\beta \in B} \mathcal{Z}(A; c_1(\beta), c_2(\beta), \ldots, c_{|X|}(\beta)),$$

where

$$c_i(\beta) = \sum_{d \mid i} d\, \delta(\beta, d).$$

To apply Theorem 10 in the enumeration of nilpotent semigroups of degree 3 we require the cycle indices of the specific group actions defined below.

Definition 11. Let A be a group acting on a set X. Then we define:

(i) by $A^{\times 2}$ the group A acting on $X \times X$ componentwise, that is,

$$(x_1, x_2)^\alpha = (x_1^\alpha, x_2^\alpha)$$

for $\alpha \in A$;

(ii) by $2A^{\times 2}$ the group $S_2 \times A$ acting on $X \times X$ by

$$(x_1, x_2)^{(\pi,\alpha)} = (x_{1^\pi}^\alpha, x_{2^\pi}^\alpha)$$

for $\alpha \in A$ and $\pi \in S_2$.

(iii) by $A^{\{2\}}$ the group A acting pointwise on the set $\{\{x_1, x_2\} \mid x_i \in X\}$ of subsets of a set X with 1 or 2 elements, that is,

$$\{x_1, x_2\}^\alpha = \{x_1^\alpha, x_2^\alpha\}$$

for $\alpha \in A$.

We will show in Section 6 that it is possible to distinguish nilpotent semigroups of degree 3 of the form $H(A, \psi, z)$ as defined in Definition 6 up to isomorphism, and up to isomorphism or anti-isomorphism, by determining the orbit the function ψ belongs to under certain power groups derived from the actions in Definition 11.

In the next lemma, we obtain the cycle indices of the groups $S_n^{\times 2}$, $S_n^{\{2\}}$, and $2S_n^{\times 2}$ using the cycle index of S_n given in Lemma 8.

Lemma 12. *For $n \in \mathbb{N}$ the following hold:*

(i) *the cycle index of $S_n^{\times 2}$ is*

$$\mathcal{Z}(S_n^{\times 2}) = \sum_{j \vdash n} \left(\prod_{i=1}^{n} j_i!\, i^{j_i} \right)^{-1} \prod_{a,b=1}^{n} x_{\operatorname{lcm}(a,b)}^{j_a j_b \gcd(a,b)};$$

(ii) *the cycle index of $2S_n^{\times 2}$ is*

$$\mathcal{Z}(2S_n^{\times 2}) = \frac{1}{2}\mathcal{Z}(S_n^{\times 2}) + \frac{1}{2} \sum_{j \vdash n} \left(\prod_{i=1}^{n} j_i!\, i^{j_i} \right)^{-1} \prod_{a=1}^{n} \left(q_a^{j_a} p_{a,a}^{j_a^2 - j_a} \prod_{b=1}^{a-1} p_{a,b}^{2 j_a j_b} \right),$$

where $p_{a,b} = x_{\operatorname{lcm}(2,a,b)}^{ab/\operatorname{lcm}(2,a,b)}$ and

$$q_a = \begin{cases} x_a x_{2a}^{(a-1)/2} & \text{if } a \equiv 1 \mod 2 \\ x_a^a & \text{if } a \equiv 0 \mod 4 \\ x_{a/2}^2 x_a^{a-1} & \text{if } a \equiv 2 \mod 4; \end{cases}$$

(iii) *the cycle index of* $S_n^{\{2\}}$ *is*

$$\mathcal{Z}(S_n^{\{2\}}) = \sum_{j\vdash n} \left(\prod_{i=1}^{n} j_i! i^{j_i}\right)^{-1} \prod_{a=1}^{\lfloor n/2\rfloor} r_a \prod_{a=1}^{\lfloor (n+1)/2\rfloor} s_a \prod_{a=1}^{n} t_a \left(\prod_{b=1}^{a-1} x_{\mathrm{lcm}(a,b)}^{j_a j_b \gcd(a,b)}\right),$$

where the monomials are $r_a = x_a^{j_{2a}} x_{2a}^{a j_{2a}}$, $s_a = x_{2a-1}^{a j_{2a-1}}$, *and* $t_a = x_a^{a(j_a^2 - j_a)/2}$.

Proof. **(i).** By definition each permutation in S_n induces a permutation in $S_n^{\times 2}$. Let $\alpha \in S_n$ and let z_a and z_b be two cycles thereof with length a and b respectively. Consider the action of α on those pairs in $[n] \times [n]$ which have as first component an element in z_a and as second component an element in z_b. Let $(i,j) \in [n] \times [n]$ be one such pair. Since $i^{\alpha^k} = i$ if and only if a divides k, and $j^{\alpha^k} = j$ if and only if b divides k, the pair (i,j) is in an orbit of length $\mathrm{lcm}(a,b)$. The total number of pairs with first component in z_a and second component in z_b equals ab. Hence the number of orbits equals $\gcd(a,b)$. Repeating this consideration for every pair of cycles in α leads to

$$\prod_{a,b=1}^{n} x_{\mathrm{lcm}(a,b)}^{\delta(\alpha,a)\delta(\alpha,b)\gcd(a,b)}$$

as the summand corresponding to α in the cycle index $\mathcal{Z}(S_n^{\times 2})$. This yields

$$\mathcal{Z}(S_n^{\times 2}) = \frac{1}{n!} \sum_{\alpha\in S_n} \prod_{a,b=1}^{n} x_{\mathrm{lcm}(a,b)}^{\delta(\alpha,a)\delta(\alpha,b)\gcd(a,b)}.$$

That the contribution of α to $\mathcal{Z}(S_n^{\times 2})$ only depends on its cycle structure allows us to replace the summation over all group elements by a summation over partitions of n; one for each conjugacy class of S_n. The number of elements with cycle structure associated to a partition $j \vdash n$ equals $n!/\prod_{i=1}^{n} j_i!\, i^{j_i}$. Therefore

$$\mathcal{Z}(S_n^{\times 2}) = \frac{1}{n!} \sum_{j\vdash n} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}} \prod_{a,b=1}^{n} x_{\mathrm{lcm}(a,b)}^{j_a j_b \gcd(a,b)},$$

and cancelling the factor $n!$ concludes the proof.

(ii). For elements $(\mathrm{id}_{\{1,2\}}, \alpha) \in 2S_n^{\times 2}$ the contribution to the cycle index of $2S_n^{\times 2}$ equals the contribute of α to $\mathcal{Z}(S_n^{\times 2})$ given in (i). It is rearranged as follows to illustrate which contributions come from identical cycles and which from disjoint cycles:

$$\prod_{a,b=1}^{n} x_{\mathrm{lcm}(a,b)}^{\delta(\alpha,a)\delta(\alpha,b)\gcd(a,b)} = \prod_{a=1}^{n} \left(x_a^{a\delta(\alpha,a)} x_a^{a(\delta(\alpha,a)^2 - \delta(\alpha,a))} \prod_{b<a} x_{\mathrm{lcm}(a,b)}^{2\delta(\alpha,a)\delta(\alpha,b)\gcd(a,b)} \right).$$

For group elements of the form $((1\,2), \alpha)$ the contribution is going to be deduced from the one of α. Let z_a and z_b again be two cycles in α of length a and b respectively, and assume at first, they are disjoint. Then z_a and z_b induce $2\gcd(a,b)$ orbits of length

$\operatorname{lcm}(a,b)$ on the $2ab$ pairs in $[n] \times [n]$ with one component from each of the two cycles. Let

$$\omega = \big\{(i_1, j_1), (i_2, j_2), \ldots, (i_{\operatorname{lcm}(a,b)}, j_{\operatorname{lcm}(a,b)})\big\} \tag{6}$$

be such an orbit. Then

$$\bar{\omega} = \big\{(j_1, i_1), (j_2, i_2), \ldots, (j_{\operatorname{lcm}(a,b)}, i_{\operatorname{lcm}(a,b)})\big\} \tag{7}$$

is another one. The set $\omega \cup \bar{\omega}$ is closed under the action of $((1\,2), \alpha)$. In how many orbits $\omega \cup \bar{\omega}$ splits depends on the parity of a and b. Acting with $((1\,2), \alpha)$ on (i_1, j_1) for $\operatorname{lcm}(a,b)$ times gives (i_1, j_1) if $\operatorname{lcm}(a,b)$ is even and (j_1, i_1) if $\operatorname{lcm}(a,b)$ is odd. Hence the two orbits ω and $\bar{\omega}$ merge to one orbit in the latter case and give two new orbits of the original length otherwise. This yields the monomial

$$x_{\operatorname{lcm}(2,a,b)}^{2ab/\operatorname{lcm}(2,a,b)} = \begin{cases} x_{\operatorname{lcm}(a,b)}^{2\gcd(a,b)} & \text{if} \quad \operatorname{lcm}(a,b) \equiv 0 \bmod 2 \\ x_{2\operatorname{lcm}(a,b)}^{\gcd(a,b)} & \text{if} \quad \operatorname{lcm}(a,b) \equiv 1 \bmod 2, \end{cases}$$

which appears $\delta(\alpha, a)\delta(\alpha, b)$ times if $a \neq b$ and $(\delta(\alpha, a)^2 - \delta(\alpha, a))/2$ times if $a = b$.

Let z_a and z_b now be identical and equal to the cycle $(i_1 i_2 \cdots i_a)$. The contribution to the monomial of α is the factor x_a^a. The orbits are of the form

$$\{(i_g, i_h) \mid 1 \leqslant g, h \leqslant a, g \equiv h + s \bmod a\}$$

for $0 \leqslant s \leqslant a-1$. For $s = 0$ the orbit consists of pairs with equal entries, that is, $\{(i_1, i_1), (i_2, i_2) \ldots (i_a, i_a)\}$, and thus stays the same under $((1\,2), \alpha)$. For an orbit $\omega = \{(i_g, i_h) \mid 1 \leqslant g, h \leqslant a, g \equiv h + s \bmod a\}$ with $s \neq 0$ define $\bar{\omega}$ as in (7). If $\omega \neq \bar{\omega}$ one argues like in the case of two disjoint cycles and gets the result depending on the parity of a. Note that $\omega = \bar{\omega}$ if and only if $s = a/2$. In particular this does not occur for a odd in which case

$$x_a x_{2a}^{(a-1)/2}$$

is the factor contributed to the monomial of $((1\,2), \alpha)$. If on the other hand a is even, one more case split is needed to deal with the orbit

$$\omega = \{(i_g, i_h) \mid 1 \leqslant g, h \leqslant a, g \equiv h + a/2 \bmod a\}.$$

Acting with $((1\,2), \alpha)$ on $(i_a, i_{a/2})$ for $a/2$ times gives $(i_a, i_{a/2})$ if $a/2$ is odd and $(i_{a/2}, i_a)$ if $a/2$ is even. Thus ω splits into two orbits of length $a/2$ in the former case and stays one orbit in the latter. The resulting factors contributed to the monomial of $((1\,2), \alpha)$ are therefore

$$\begin{aligned} x_a^a \quad & \text{if} \quad a \equiv 0 \mod 4 \\ x_{a/2}^2 x_a^{a-1} \quad & \text{if} \quad a \equiv 2 \mod 4. \end{aligned}$$

Following the analysis for all pairs of cycles in α leads to the contribution of $((1\,2), \alpha)$ to the cycle index. Summing as before over all partitions of n, which correspond to the different cycle structures, proves the formula for $\mathcal{Z}(2S_n^{\times 2})$.

(iii). To compute $\mathcal{Z}(S_n^{\{2\}})$ let ω and $\bar{\omega}$ as in (6) and (7) be orbits for two cycles z_a and z_b from $\alpha \in S_n$ acting on $[n] \times [n]$. If the two cycles z_a and z_b are disjoint then both ω and $\bar{\omega}$ correspond to the same orbit

$$\big\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_{\mathrm{lcm}(a,b)}, j_{\mathrm{lcm}(a,b)}\}\big\}$$

of α acting on $[n]^{\{2\}}$. The contribution to the monomial of α in $\mathcal{Z}(S_n^{\{2\}})$ is therefore $x_{\mathrm{lcm}(a,b)}^{\gcd(a,b)}$. Let z_a and z_b now be identical and equal to the cycle $(i_1 i_2 \cdots i_a)$. In $S_n^{\times 2}$ this gave rise to the orbits $\{(i_g, i_h) \mid 1 \leqslant g, h \leqslant a, g \equiv h + s \bmod a\}$ for $0 \leqslant s \leqslant a-1$. The corresponding orbit under $S_n^{\{2\}}$ for $s = 0$ becomes $\{\{i_1\}, \{i_2\}, \ldots, \{i_a\}\}$. All other orbits become $\{\{i_g, i_h\} \mid 1 \leqslant g, h \leqslant a, g \equiv h + s \bmod a\}$ in the same way as before, but these are identical for s and $a - s$. This yields one further exception if a is even and $s = a/2$, in which case the orbit collapses to $\{\{i_g, i_{g+a/2}\} \mid 1 \leqslant g \leqslant a/2\}$. In total, identical cycles lead to the monomials

$$\begin{array}{rcl} x_{a/2} x_a^{a/2} & \text{if} & a \equiv 0 \mod 2 \\ x_a^{(a+1)/2} & \text{if} & a \equiv 1 \mod 2. \end{array}$$

Summing once more over conjugacy classes and making the case split depending on the parity proves the formula for $\mathcal{Z}(S_n^{\{2\}})$. □

Formulae like those in the previous lemma for slightly different actions are given in [HP73, (4.1.9)] and [HP73, (5.1.5)]. The proof techniques used here are essentially the same as in [HP73].

6 Proofs of the main theorems

In this section, we prove Theorem 2. The proofs of Theorems 3, and 5 are very similar to the proof of Theorem 2, and so, for the sake of brevity we show how to obtain these proofs from the one presented, rather than giving the proofs in detail.

We consider the following sets of nilpotent semigroups of degree 3: for $m, n \in \mathbb{N}$ with $2 \leqslant m \leqslant n-1$ we define

$$Z_{n,m} = \{\, H([n] \setminus [m], \psi, 1) \mid \psi : [n] \setminus [m] \times [n] \setminus [m] \to [m] \text{ with } [m] \setminus \{1\} \subseteq \mathrm{im}(\psi) \,\},$$

where $H([n] \setminus [m], \psi, 1)$ is as in Definition 6, and $[n]$ is short for $\{1, 2, \ldots, n\}$, as before. From this point on, we will only consider semigroups belonging to $Z_{n,m}$, and so we write $H(\psi)$ instead of $H([n] \setminus [m], \psi, 1)$.

If $H(\psi) \in Z_{n,m}$ is commutative, then we define a function ψ' from the set of subsets of $[n]$ with 1 or 2 elements to $[m]$ by

$$\psi'\{i, j\} = \psi(i, j) \tag{8}$$

for $i \leqslant j$. Since the equality $\psi(i, j) = \psi(j, i)$ holds for all i, j, the function ψ' is well-defined. Moreover, every function from the set of subsets of $[n]$ with 1 or 2 elements to $[m]$ is induced in this way by a function ψ such that $H(\psi) \in Z_{n,m}$ and $H(\psi)$ is commutative.

Lemma 13. *Let S be a nilpotent semigroup of degree 3 with n elements. Then S is isomorphic to a semigroup in $Z_{n,m}$ if and only if $m = |S^2|$.*

Proof. Let z denote the zero element of S, and let $f : S \to [n]$ be any bijection such that $f(z) = 1$ and $f(S^2) = [m]$. Then define $\psi : ([n] \setminus [m]) \times ([n] \setminus [m]) \to [m]$ by

$$\psi(i, j) = f(f^{-1}(i)f^{-1}(j)).$$

Now, since S is nilpotent, if $x \in [m] \setminus \{1\}$, there exist $s, t \in S \setminus S^2$ such that $f(st) = x$. Thus $\psi(f(s), f(t)) = x$ and $[m] \setminus \{1\} \subseteq \mathrm{im}(\psi)$. Hence $H(\psi) \in Z_{n,m}$ and it remains to show that f is an isomorphism. If $x, y \in S \setminus S^2$, then $f(x)f(y) = \psi(f(x), f(y)) = f(xy)$. Otherwise, $x \in S^2$ or $y \in S^2$, in which case $f(x)f(y) = 1 = f(z) = f(xy)$. □

It follows from Lemma 13 that we can determine the number of isomorphism types in each of the sets $Z_{n,m}$ independently. Of course, if S is a nilpotent semigroup of degree 3 and $m = |S^2|$, then it is not true in general that there exists a unique semigroup in $Z_{n,m}$ isomorphic to S. Instead isomorphisms between semigroups in $Z_{n,m}$ induce an equivalence relation on the functions ψ, which define the semigroups in $Z_{n,m}$.

If $H(\psi) \in Z_{n,m}$ and T is a nilpotent semigroup of degree 3 such that $H(\psi) \cong T$, then there exists $\pi \in S_n$ such that $S^\pi = T$. Hence $T \in Z_{n,m}$ if and only if π stabilises $[n] \setminus [m]$ and $\{1\}$ – and hence $[m] \setminus \{1\}$ – setwise. In particular, the action of π on the domain and range of ψ are independent, and so equivalence can be captured using a power group action.

Lemma 14. *For $m, n \in \mathbb{N}$ with $2 \leqslant m \leqslant n-1$ let $H(\psi), H(\chi) \in Z_{n,m}$, and let U_m denote the pointwise stabiliser of 1 in S_m. Then the following hold:*

(i) *the semigroups $H(\psi)$ and $H(\chi)$ are isomorphic if and only if ψ and χ are in the same orbit under the power group $S^{\times 2}_{[n]\setminus[m]} \times U_m$;*

(ii) *the semigroups $H(\psi)$ and $H(\chi)$ are isomorphic or anti-isomorphic if and only if ψ and χ are in the same orbit under the power group $2S^{\times 2}_{[n]\setminus[m]} \times U_m$.*

If in addition $H(\psi)$ and $H(\chi)$ are commutative, then:

(iii) the semigroups $H(\psi)$ and $H(\chi)$ are isomorphic if and only if ψ' and χ' (as defined in (8)) are in the same orbit under the power group $S^{\{2\}}_{[n]\setminus[m]} \times U_m$.

Proof. **(i).** ($\Rightarrow$) By assumption there exists $\pi \in S_n$ such that $\pi : H(\psi) \to H(\chi)$ is an isomorphism. From the comments before the lemma, π stabilises $[n] \setminus [m]$ and 1, and so there exist $\tau \in U_m$ and $\sigma \in S_{[n]\setminus[m]}$ such that $\tau\sigma = \pi$. Then for all $x, y \in [n] \setminus [m]$

$$\psi(x, y) = (\psi(x, y)^\pi)^{\pi^{-1}} = (\chi(x^\pi, y^\pi))^{\pi^{-1}} = (\chi(x^\sigma, y^\sigma))^{\tau^{-1}}.$$

It follows that χ acted on by $(\sigma, \tau^{-1}) \in S^{\times 2}_{[n]\setminus[m]} \times U_m$ equals ψ, as required.

($\Leftarrow$) As ψ and χ lie in the same orbit under the action of the power group $S^{\times 2}_{[n]\setminus[m]} \times U_m$, there exist $\sigma \in S_{[n]\setminus[m]}$ and $\tau \in U_m$ such that $\psi^{(\sigma,\tau)} = \chi$. Let $\pi = \sigma\tau^{-1} \in S_n$. We will

show that π is an isomorphism from $H(\psi)$ to $H(\chi)$. Let $x, y \in [n]$ be arbitrary. If $x, y \in [n] \setminus [m]$, then

$$x^\pi y^\pi = \psi(x^\sigma, y^\sigma) = (\psi(x^\sigma, y^\sigma)^\tau)^{\tau^{-1}} = (\psi^{(\sigma,\tau)}(x, y))^{\tau^{-1}} = (\chi(x, y))^{\tau^{-1}} = (xy)^\pi.$$

If $x \in [n] \setminus [m]$ and $y \in [m]$, then $(xy)^\pi = 1^\pi = 1 = x^\sigma y^{\tau^{-1}} = x^\pi y^\pi$. The case when $x \in [m]$ and $y \in [n] \setminus [m]$ and the case when $x, y \in [m]$ follow by similar arguments.

(ii). In this part of the proof we write (α, β, γ) instead of $((\alpha, \beta), \gamma)$ when referring to elements of $2S^{\times 2}_{[n]\setminus[m]} \times U_m$.

($\Rightarrow$) If $H(\psi)$ and $H(\chi)$ are isomorphic, then, by part (i), the functions ψ and χ are in the same orbit under the action of $S^{\times 2}_{[n]\setminus[m]} \times U_m$. Since $S^{\times 2}_{[n]\setminus[m]} \times U_m$ is contained in $2S^{\times 2}_{[n]\setminus[m]} \times U_m$, it follows that ψ and χ are in the same orbit under the action of $2S^{\times 2}_{[n]\setminus[m]} \times U_m$.

If $H(\psi)$ and $H(\chi)$ are not isomorphic, then there exists $\pi \in S_n$ such that $\pi : H(\psi) \to H(\chi)$ is an anti-isomorphism. As in the proof of part (i), there exist $\tau \in U_m$ and $\sigma \in S_{[n]\setminus[m]}$ such that $\pi = \tau\sigma$. Then, for all $x, y \in [n] \setminus [m]$,

$$\psi(x, y) = (\psi(x, y)^\pi)^{\pi^{-1}} = (\chi(y^\pi, x^\pi))^{\pi^{-1}} = (\chi(y^\sigma, x^\sigma))^{\tau^{-1}} = \chi^{(\sigma,\tau^{-1})}(y, x).$$

Hence χ acted on by $((1\,2), \sigma, \tau^{-1}) \in 2S^{\times 2}_{[n]\setminus[m]} \times U_m$ equals ψ.

($\Leftarrow$) If $\psi = \chi^{(\mathrm{id}_{\{1,2\}},\sigma,\tau)}$ for some $(\mathrm{id}_{\{1,2\}}, \sigma, \tau) \in 2S^{\times 2}_{[n]\setminus[m]} \times U_m$, then $H(\psi)$ and $H(\chi)$ are isomorphic by part (i). So, we may assume that $\psi = \chi^{((1\,2),\sigma,\tau)}$. Let $\pi = \sigma\tau^{-1} \in S_n$. We show that π is an anti-isomorphism from $H(\psi)$ to $H(\chi)$. Let $x, y \in [n]$ be arbitrary. If $x, y \in [n] \setminus [m]$, then

$$x^\pi y^\pi = \psi(x^\sigma, y^\sigma) = (\psi(x^\sigma, y^\sigma)^\tau)^{\tau^{-1}} = (\psi^{((1\,2),\sigma,\tau)}(y, x))^{\tau^{-1}} = (\chi(y, x))^{\tau^{-1}} = (yx)^\pi.$$

If $x \in [n] \setminus [m]$ and $y \in [m]$, then $(xy)^\pi = 1^\pi = 1 = y^{\tau^{-1}} x^\sigma = y^\pi x^\pi$. The case when $x \in [m]$ and $y \in [n] \setminus [m]$ and the case when $x, y \in [m]$ follow by similar arguments.

(iii). The proof follows from (i) and the observation that ψ' and χ' are in the same orbit under $S^{\{2\}}_{[n]\setminus[m]} \times U_m$ if and only if ψ and χ are in the same orbit under $S^{\times 2}_{[n]\setminus[m]} \times U_m$. □

Lemma 14(i) shows that the number of non-isomorphic semigroups in $Z_{n,m}$ equals the number of orbits of functions defining semigroups in $Z_{n,m}$ under the appropriate power group action. Together with Theorem 10 this provides the essential information required to prove the formula given in Theorem 2 for the number of nilpotent semigroups of degree 3 of order n up to isomorphism.

Proof of Theorem 2. Denote by U_q the stabiliser of 1 in S_q. We shall first show that $N(p, q)$ is the number of orbits of the power group $S^{\times 2}_{[p]\setminus[q]} \times U_q$ on functions from $([p] \setminus [q]) \times ([p] \setminus [q])$ to $[q]$. By Theorem 10 the latter equals

$$\frac{1}{(q-1)!} \sum_{\beta \in H} \mathcal{Z}(S^{\times 2}_{[p]\setminus[q]}; c_1(\beta), \ldots, c_{(p-q)^2}(\beta)), \tag{9}$$

where

$$c_i(\beta) = \sum_{d|i} d\delta(\beta, d).$$

If $\beta \in U_q$, then $\mathcal{Z}(S^{\times 2}_{[p]\setminus[q]}; c_1(\beta), \ldots, c_{(p-q)^2}(\beta))$ only depends on the cycle structure of β and is therefore an invariant of the conjugacy classes of U_q. These conjugacy classes are in 1-1 correspondence with the partitions of $q-1$. If j is a partition of $q-1$ corresponding to the conjugacy class of β, then $\delta(\beta, 1) = j_1 + 1$ and $\delta(\beta, i) = j_i$ for $i = 2, \ldots, q-1$ (where j_i denotes, as before, the number of summands in j equalling i). This yields that $c_i(\beta) = 1 + \sum_{d|i} d\, j_d$. The size of the conjugacy class in U_q corresponding to the partition j is $(q-1)!/\prod_{i=1}^{q-1} j_i!\, i^{j_i}$. Hence summing over conjugacy classes in (9) gives:

$$\sum_{j \vdash q-1} \left(\prod_{i=1}^{q-1} j_i!\, i^{j_i} \right)^{-1} \mathcal{Z}\left(S^{\times 2}_{[p]\setminus[q]}; 1 + \sum_{d|1} dj_d, \ldots, 1 + \sum_{d|(p-q)^2} dj_d \right). \tag{10}$$

Substituting the cycle index of $S^{\times 2}_{[p]\setminus[q]}$ from Lemma 12(i) into (10) yields the formula given in the statement of the Theorem for $N(p, q)$.

By Lemma 14(i), the number of non-isomorphic semigroups in $Z_{n,m}$ for $m \in \mathbb{N}$ with $2 \leqslant m \leqslant n-1$ equals the number of orbits under the power group $S^{\times 2}_{[n]\setminus[m]} \times U_m$ of functions from $([n] \setminus [m]) \times ([n] \setminus [m])$ to $[m]$ having $[m] \setminus \{1\}$ in their image. The orbits counted in $N(n, m)$ include those of functions which do not contain $[m] \setminus \{1\}$ in their image. The number of such orbits equals $N(n-1, m-1)$, the number of orbits of functions with one fewer element in the image set. Hence the number of non-isomorphic semigroups in $Z_{n,m}$ equals $N(n, m) - N(n-1, m-1)$. With Lemma 13, it follows that the number of non-isomorphic nilpotent semigroups of degree 3 with n elements is

$$\sum_{m=2}^{a(n)} (N(m, n) - N(m-1, n-1)) \quad \text{where} \quad a(n) = \left\lfloor n + 1/2 - \sqrt{n - 3/4} \right\rfloor. \qquad \square$$

Replacing the cycle index in (9) by that of $2S^{\times 2}_{[p]\setminus[q]}$ and $S^{\{2\}}_{[p]\setminus[q]}$ proves Theorems 3 and 5, respectively, using the same argument as above.

References

[dB59] N. G. de Bruijn. Generalization of Polya's fundamental theorem in enumerative combinatorial analysis. *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.*, 21:59–69, 1959.

[Dis10] Andreas Distler. *Classification and Enumeration of Finite Semigroups*. Shaker Verlag, Aachen, 2010. *also* PhD thesis, University of St Andrews, 2010, http://hdl.handle.net/10023/945.

[DK09] Andreas Distler and Tom Kelsey. The monoids of orders eight, nine & ten. *Ann. Math. Artif. Intell.*, 56(1):3–21, 2009.

[DM11] A. Distler and J. D. Mitchell. *Smallsemi — a GAP package, version 0.6.4*, 2011. http://tinyurl.com/jdmitchell/smallsemi/.

[ER63] P. Erdős and A. Rényi. Asymmetric graphs. *Acta Math. Acad. Sci. Hungar*, 14:295–315, 1963.

[GAP08] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008. http://www.gap-system.org.

[Gri03] Pierre Antoine Grillet. Computing finite commutative semigroups. II, III. *Semigroup Forum*, 67(2):159–184, 185–204, 2003.

[Har66] Michael A. Harrison. The number of isomorphism types of finite algebras. *Proc. Amer. Math. Soc.*, 17:731–737, 1966.

[HP73] Frank Harary and Edgar M. Palmer. *Graphical enumeration*. Academic Press, New York, 1973.

[JMS91] H. Jürgensen, F. Migliorini, and J. Szép. *Semigroups*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1991.

[KRS76] Daniel J. Kleitman, Bruce R. Rothschild, and Joel H. Spencer. The number of semigroups of order n. *Proc. Amer. Math. Soc.*, 55(1):227–232, 1976.

[Pol37] G. Polya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.*, 68(1):145–254, 1937.

[Red27] J. Howard Redfield. The Theory of Group-Reduced Distributions. *Amer. J. Math.*, 49(3):433–455, 1927.

[Sla95] John Slaney. *FINDER – Finite Domain Enumerator, Version 3.0*, 1995. ftp://arp.anu.edu.au/pub/papers/slaney/finder/finder.ps.gz.

[SYT94] S. Satoh, K. Yama, and M. Tokizawa. Semigroups of order 8. *Semigroup Forum*, 49(1):7–29, 1994.