# Expansions of $k$-Schur functions in the affine nilCoxeter algebra

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**Abstract**

We give a type free formula for the expansion of $k$-Schur functions indexed by fundamental coweights within the affine nilCoxeter algebra. Explicit combinatorics are developed in affine type $C$.

## 1 Introduction

In [1], Berg, Bergeron, Thomas and Zabrocki gave several formulas for the expansion of certain $k$-Schur functions (indexed by fundamental weights) inside the affine nilCoxeter algebra of type A. In particular, they gave an explicit combinatorial description for the reduced words which appear in the expansion.

These coefficients have been studied extensively; they are the coefficients which appear in the product of two $k$-Schur functions. These functions have been identified with representing the homology of the affine Grassmannian in type $A$.

They verified their formula by identifying terms in the expansion of a $k$-Schur function with pseudo-translations (elements of the nilCoxeter algebra which act by translating alcoves in prescribed directions). This generalized Proposition 4.5 of Lam [4], where he gave formulas for $k$-Schur functions indexed by root translations.

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Since then, Lam and Shimozono [6] have discovered a type free analogue of this fact for $k$-Schur functions indexed by coweights. The main goal of this paper is to combine the new result of Lam and Shimozono with the techniques of [1] to give descriptions of the corresponding reduced words appearing in the decomposition of these $k$-Schur functions, with an emphasis on combinatorics.

Section 2 develops a type free formula for $k$-Schur functions indexed by special Grassmannian permutations, Section 3 focuses on the specific combinatorics of affine type $C$, and Section 4 discusses a few examples of the combinatorics in affine types $B$ and $D$.

## 1.1 A brief introduction to root systems

### 1.1.1 Root systems

Let $(I, A)$ be a Cartan datum, i.e., a finite index set $I$ and a generalized Cartan matrix $A = (a_{ij} \mid i, j \in I)$ such that $a_{ii} = 2$ for all $i \in I$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$ if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. If the corank of $A$ is 1, then $A$ is of *affine type*; in this case, we write $(I_{\text{af}}, A_{\text{af}})$, and let $I_{\text{af}} = \{0, 1, \ldots, n\}$. From a Cartan datum of affine type we may recover a corresponding Cartan datum $(I_{\text{fin}}, A_{\text{fin}})$ of *finite type* by considering $I_{\text{fin}} = I_{\text{af}} \setminus \{0\}$. In general, we denote affine root system data with an "af" subscript, and finite root system data with a "fin" subscript. Root system data of arbitrary type has no subscript.

Also associated to a Cartan datum we have a root datum, which consists of a free $\mathbb{Z}$-module $\mathfrak{h}$, its dual lattice $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{Z})$, a pairing $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{Z}$ given by $\langle \mu, \lambda \rangle = \lambda(\mu)$, and sets of linearly independent elements $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. The $\alpha_i$ are known as *simple roots*, and the $\alpha_i^\vee$ are *simple coroots*. The spaces $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \otimes \mathbb{R}$ and $\mathfrak{h}^*_\mathbb{R} = \mathfrak{h}^* \otimes \mathbb{R}$ are the coroot and root spaces, respectively.

### 1.1.2 The affine Weyl group

Associated to a Cartan datum we have the *Weyl group* $W$, with generators $s_i$ for $i \in I$, and relations $s_i^2 = 1$ and

$$\underbrace{s_i s_j s_i s_j \cdots}_{m(i,j)} = \underbrace{s_j s_i s_j s_i \cdots}_{m(i,j)},$$

where $m(i, j) = 2, 3, 4, 6$ or $\infty$ as $a_{ij}a_{ji} = 0, 1, 2, 3$ or $\geqslant 4$, respectively. An element of the Weyl group may be expressed as a word in the generators $s_i$; given the relations above, an element of the Weyl group may have multiple *reduced words*, words of minimal length that express that element. The length of any reduced word of $w$ is the *length* of $w$, denoted $\ell(w)$. The *Bruhat order* on Weyl group elements is a partial order where $v < w$ if there is a reduced word for $v$ that is a subword of a reduced word for $w$. If $v < w$ and $\ell(v) = \ell(w) - 1$, we write $v \lessdot w$.

If $j$ is in $I_{\text{af}}$, we denote by $W_j$ the subgroup of $W$ generated by the elements $s_i$ with $i \neq j$. We denote by $W^j$ a set of minimal length representatives of the cosets $W/W_j$. The elements of $W^0$ will be referred to as *Grassmannian* elements.

### 1.1.3 Weyl group actions

Given a simple root $\alpha_i$, there is an action $\star$ of $W$ on $\mathfrak{h}_{\mathbb{R}}$ or $\mathfrak{h}^*_{\mathbb{R}}$, defined by the action of the generators of $W$ as

$$s_i \star \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad \text{for } i \in I, \lambda \in \mathfrak{h}^*_{\mathbb{R}} \tag{1}$$

$$s_i \star \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee \quad \text{for } i \in I, \mu \in \mathfrak{h}_{\mathbb{R}}. \tag{2}$$

This action by $W$ satisfies $\langle w \star \mu, w \star \lambda \rangle = \langle \mu, \lambda \rangle$.

The set of *real roots* is $\Phi_{\mathrm{re}} = W \star \{\alpha_i \mid i \in I\}$. Given a real root $\alpha = w \star \alpha_i$, we have an *associated coroot* $\alpha^\vee = w \star \alpha_i^\vee$ and an *associated reflection* $s_\alpha = w s_i w^{-1}$ (these are well-defined, and independent of choice of $w$ and $i$).

The action by $W$ preserves the *root lattice* $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and *coroot lattice* $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$. The *fundamental weights* are $\{\Lambda_i \in \mathfrak{h}^*_{\mathbb{R}} \mid \Lambda_i(\alpha_j^\vee) = \delta_{ij} \text{ for } i, j \in I\}$, and the *fundamental coweights* are $\{\Lambda_i^\vee \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(\Lambda_j^\vee) = \delta_{ij} \text{ for } i, j \in I\}$. These generate the *weight lattice* $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ and *coweight lattice* $P^\vee = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i^\vee$.

We let $\mathfrak{h}_{\mathrm{fin}}$ denote the linear span of $\{\alpha_i^\vee \mid i \neq 0\}$ and $\mathfrak{h}^*_{\mathrm{fin}}$ denote the span of $\{\alpha_i \mid i \neq 0\}$. Then there is another action $\diamond$ of $W$ on $\mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$, called the level one action in [16], which is defined by:

$$s_i \diamond \mu = \begin{cases} s_i \star \mu & \text{if } i \neq 0 \\ s_0 \star \mu - \alpha_0^\vee & \text{if } i = 0 \end{cases}$$

where $\alpha_0^\vee$ is interpreted as $\alpha_0^\vee = -\sum_{i \in I_{\mathrm{fin}}} \alpha_i^\vee$.

In addition to reflections $s_\alpha$, we have the translation endomorphisms of $\mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$ given by

$$t_\gamma \diamond \mu = \mu + \gamma \tag{3}$$

for $\gamma \in \mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$. One can show that $t_\mu t_\gamma = t_{\mu+\gamma}$ and that $t_{w(\mu)} = w t_\mu w^{-1}$ for $w \in W_{\mathrm{fin}}$, $\gamma, \mu \in \mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$.

If by abuse of notation we let $Q^\vee_{\mathrm{fin}} = \{t_{\alpha^\vee} \mid \alpha^\vee \in Q^\vee_{\mathrm{fin}}\}$, then the affine Weyl group has an alternate presentation as

$$W_{\mathrm{af}} = W_{\mathrm{fin}} \ltimes Q^\vee_{\mathrm{fin}}.$$

**Remark 1.** *Elements of $W_{af}$ corresponding to translations act trivially via the $\star$ action, i.e. $t_\gamma \star \mu = \mu$.*

### 1.1.4 The extended affine Weyl group

We can define the *extended affine Weyl group* $W_{\mathrm{ext}}$ by

$$W_{\mathrm{ext}} = W_{\mathrm{fin}} \ltimes P^\vee_{\mathrm{fin}}.$$

$W_{\mathrm{ext}}$ also has an action on $\mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$ and $\mathfrak{h}^*_{\mathrm{fin}} \otimes \mathbb{R}$ via the translation formula (3). Translations in the extended affine Weyl group also act trivially under the $\star$ action on $\mathfrak{h}_{\mathrm{fin}} \otimes \mathbb{R}$.

### 1.1.5 Affine hyperplanes and alcoves

In $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$, let $H_{\alpha,k} = \{\mu \mid \langle \mu, \alpha \rangle = k\}$, where $\alpha$ is a finite root and $k \in \mathbb{Z}$. Reflection over the hyperplane $H_{\alpha,k}$ is equivalent to $t_{k\alpha^\vee} s_\alpha$ acting by the $\diamond$ action. Each hyperplane $H_{\alpha,k}$ is stabilized by the action of $W_{\text{af}}$ and the set of hyperplanes $\mathcal{H} = \cup_{\alpha,k} H_{\alpha,k}$ is stabilized by the action of $W_{\text{ext}}$.

The *fundamental alcove* is the polytope bounded by $H_{\alpha_i,0}$ for $i \in I_{\text{fin}}$ and $H_{\theta,1}$, where $\theta$ is the *highest root*. It is a fundamental domain for the $\diamond$ action of $W$ on $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$. Therefore, we may identify alcoves with affine Weyl group elements; we define $\mathcal{A}_w$ to be the alcove $w^{-1} \diamond \mathcal{A}_\emptyset$, where $\mathcal{A}_\emptyset$ is the fundamental alcove. Additionally, we may identify alcoves with their *centroids*, i.e., the average of the vertices of the alcove.

### 1.1.6 Dynkin diagram automorphisms

The length of an element $w \in W$, defined earlier in terms of reduced words, may equivalently be defined to be the number of hyperplanes $H_{\alpha,k}$ that lie between the alcoves $\mathcal{A}_w$ and $\mathcal{A}_\emptyset$. We can similarly define the length of an element $w \in W_{\text{ext}}$ to be the number of hyperplanes that lie between $\mathcal{A}_w$ and $\mathcal{A}_\emptyset$. This definition of length implies that there are non-trivial elements of $W_{\text{ext}}$ of length 0. In fact, it is known [9] that

$$\Omega := \{u \in W_{\text{ext}} \mid \ell(u) = 0\} \cong \text{Aut}(D) \cong P^\vee_{\text{fin}}/Q^\vee_{\text{fin}},$$

where $\text{Aut}(D)$ is the set of Dynkin diagram automorphisms.

The first of the above isomorphisms can be viewed concretely as follows. We let $\Omega = \{u \in W_{\text{ext}} \mid \ell(u) = 0\}$. Let $J = \{i \in I_{\text{fin}} \mid \tau(0) = i, \tau \in \text{Aut}(D)\}$ be the set of *cominiscule coweights*. Define $x_\lambda$ to be a minimal length representative of the coset $t_\lambda W_{\text{fin}}$ for $\lambda \in P^\vee_{\text{fin}}$. It can be shown that $x_\lambda = t_\lambda v_\lambda^{-1}$, where $v_\lambda \in W_{\text{fin}}$ is shortest element of $W_{\text{fin}}$ such that $v_\lambda(\lambda) = \lambda_-$, and $\lambda_-$ is the unique antidominant element of the $W_{\text{fin}}$-orbit of $\lambda$. Then $\Omega = \{x_{\Lambda_i} \mid i \in J\}$, and the element $x_{\Lambda_i}$ corresponds to the Dynkin diagram automorphism sending the node 0 to the node $i$. Under this map and the action of $W_{\text{ext}}$ on $P^\vee_{\text{af}}$ given above, an element $\tau \in \text{Aut}(D)$ acts on the coweight lattice $P^\vee_{\text{af}}$ via $\tau \star \alpha_i^\vee = \alpha^\vee_{\tau(i)}$. Furthermore,

$$\tau s_i = s_{\tau(i)} \tau \tag{4}$$

for $i \in I_{\text{af}}, \tau \in \Omega$. Finally, for $\tau \in \Omega$, $u = s_{i_1} s_{i_2} \cdots s_{i_k} \in W$, we define $\tau(u) = s_{\tau(i_1)} s_{\tau(i_2)} \cdots s_{\tau(i_k)}$.

The extended affine Weyl group can be realized as a semi-direct product of the affine Weyl group and $\Omega$:

$$W_{ext} = W_{af} \ltimes \Omega.$$

The relation (4) describes how elements commute in this realization of $W_{ext}$.

### 1.2 k-Schur functions for general type

Let $\mathbb{F} = \mathbb{C}((t))$ and $\mathbb{O} = \mathbb{C}[[t]]$. The *affine Grassmannian* is defined as $\mathrm{Gr}_G := G(\mathbb{F})/G(\mathbb{O})$. $\mathrm{Gr}_G$ can be decomposed into *Schubert cells* $\Omega_w = \mathcal{B}wG(\mathbb{O}) \subset G(\mathbb{F})/G(\mathbb{O})$, where $\mathcal{B}$ denotes the Iwahori subgroup and $w \in W^0$, the set of Grassmannian elements in the associated affine Weyl group. The Schubert varieties, denoted $X_w$, are the closures of $\Omega_w$, and we have $\mathrm{Gr}_G = \sqcup \Omega_w = \cup X_w$, for $w \in W^0$. The homology $H_*(\mathrm{Gr}_G)$ and cohomology $H^*(\mathrm{Gr}_G)$ of the affine Grassmannian have corresponding Schubert bases, $\{\xi_w\}$ and $\{\xi^w\}$, respectively, also indexed by Grassmannian elements. It is well-known that $\mathrm{Gr}_G$ is homotopy-equivalent to the space $\Omega K$ of based loops in $K$ (due to Quillen, see [13, §8] or [10]). The group structure of $\Omega K$ gives $H_*(\mathrm{Gr}_G)$ and $H^*(\mathrm{Gr}_G)$ the structure of dual Hopf algebras over $\mathbb{Z}$.

The *nilCoxeter algebra* $\mathbb{A}_0$ may be defined via generators and relations from any Cartan datum, with generators $\mathbf{u}_i$ for $i \in I$, and relations $\mathbf{u}_i^2 = 0$ and

$$\underbrace{\mathbf{u}_i\mathbf{u}_j\mathbf{u}_i\mathbf{u}_j\cdots}_{m(i,j)} = \underbrace{\mathbf{u}_j\mathbf{u}_i\mathbf{u}_j\mathbf{u}_i\cdots}_{m(i,j)},$$

where $m(i,j) = 2, 3, 4, 6$ or $\infty$ as $a_{ij}a_{ji} = 0, 1, 2, 3$ or $\geqslant 4$, respectively.

Since the braid relations are exactly those of the corresponding Weyl group, we may index nilCoxeter elements by elements of the Weyl group, e.g., $\mathbf{u}(w) = \mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_k}$, whenever $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word for $w$.

By work of Peterson [11], there is an injective ring homomorphism $j_0 : H_*(\mathrm{Gr}_G) \hookrightarrow \mathbb{A}_0$. This map is an isomorphism on its image (actually a Hopf algebra isomorphism) $j_0 : H_*(\mathrm{Gr}_G) \to \mathbb{B}$, where $\mathbb{B}$ is known as the *affine Fomin-Stanley subalgebra*.

**Definition 2.** For $W$ of affine type $X$ and $w \in W^0$ we define the non-commutative $k$-Schur function $\mathfrak{s}_w^X$ of affine type $X$ to be the image of the Schubert class $\xi_w$ under the isomorphism $j_0$, so $\mathfrak{s}_w^X = j_0(\xi_w)$. When obvious from context, we will simply write $\mathfrak{s}_w$, omitting the type. This definition comes from the realization of $k$-Schur functions identified with the homology of the affine Grassmannian in [4]. In type $C$ this was first properly developed in [5], and in types $B$ and $D$ this was first developed in [12].

**Example 3.** In type $A_n^{(1)}$, the elements $\mathfrak{s}_w^A$ are the non-commutative $k$-Schur functions defined in [4]. One can define a further isomorphism between the affine Fomin-Stanley subalgebra and the ring of symmetric functions generated by the homogeneous symmetric functions $h_\lambda$ with $\lambda_1 \leqslant n-1$. Under this isomorphism, the non-commutative $k$-Schur functions are conjectured to correspond to the $t = 1$ specializations of the $k$-Schur functions of Lapointe, Lascoux and Morse [7] indexed by a $k$-bounded partition corresponding to the element $w$ and are isomorphic to the $k$-Schur functions of Lapointe and Morse [8].

## 2 A type-free formula

Given an element $t = w\tau \in W_{\mathrm{ext}}$ with $w \in W_{\mathrm{af}}$ and $\tau \in \Omega$, we denote by $\bar{t} = w$ the image of $t$ modulo $\Omega$. For $\lambda \in P_{\mathrm{fin}}^\vee$ recall that $t_\lambda \in W_{\mathrm{ext}}$ is the translation which acts on $\mathfrak{h}_{\mathrm{af}}$

according to (3). We let $z_\lambda = \overline{t_\lambda}^{-1}$. In [1], the $z_\lambda$ were called *pseudo-translations*. For a coweight $\gamma$, we let $\Gamma_\gamma = W_{\text{fin}}\gamma$.

Independently from Lam and Shimozono [6], we have simultaneously discovered a generalization of [1] and [4] which gives a formula for the $k$-Schur functions indexed by coweight translations. Rather than include our long proof, we will rely on their result.

**Proposition 4** (Lam, Shimozono [6]). *For a dominant coweight $\gamma$,*

$$\mathfrak{s}_{z_\gamma} = \sum_{\eta \in \Gamma_\gamma} \mathbf{u}(z_\eta).$$

Proposition 4 is the starting point for a type-free combinatorial formula generalizing the one that appears in [1]. It should be noted though, that this formula does not give reduced words for the terms $z_\eta$; they are defined only as the image of translations in prescribed directions. In Theorem 18, we will give a combinatorial description of the explicit reduced words which appear in this sum.

## 2.1 Commutation for $k$-Schur functions

Theorem 5.1 of [1] gives a nice commutation relation for $k$-Schur functions in type $A$ and a generator of the affine nil-Coxeter algebra. In this section we will deduce a similar commutation property in Theorem 8 which will allow us to provide more explicit formulas for $\mathfrak{s}_w$.

**Definition 5.** We now fix some notation. $\gamma$ will denote the $j^{th}$ fundamental coweight $\Lambda_j^\vee$. If $t_\gamma = z_\gamma^{-1}\tau_\gamma$ then we let $t = t_\gamma, z = z_\gamma$, and $\tau = \tau_\gamma$.

**Lemma 6.** *For a coweight $\gamma$, $z$ is the unique element of $W$ which satisfies $\mathcal{A}_z = \mathcal{A}_\emptyset + \gamma$.*

*Proof.* The alcove $\mathcal{A}_z = z^{-1} \diamond \mathcal{A}_\emptyset$ and $z = \overline{t}^{-1}$, where the action of $t$ corresponds to translation by $\gamma$. The uniqueness follows from the fact that $W$ is in bijection with the set of alcoves. □

**Proposition 7.** *For $\gamma$, $z$, $\tau$ as in Definition 5 and $w \in W$,*

$$z_{w\star\gamma} w = \tau(w) z.$$

*Proof.* Let $w \in W$. In $W_{\text{ext}}$, we have $wt_{w^{-1}\star\gamma} = tw$. Let $t_{w^{-1}\star\gamma} = z_{w^{-1}\star\gamma}^{-1}\tau'$ for some $\tau' \in \Omega$. Then we have

$$wz_{w^{-1}\star\gamma}^{-1}\tau' = z^{-1}\tau w, \tag{5}$$

$$wz_{w^{-1}\star\gamma}^{-1}\tau' = z^{-1}\tau(w)\tau, \tag{6}$$

with the last equality coming from Equation (4). Therefore, we must have $\tau' = \tau$, and

$$wz_{w^{-1}\star\gamma}^{-1} = z^{-1}\tau(w).$$

Inverting both sides and replacing $w^{-1}$ with $w$ gives the desired result. □

The following theorem is a generalization of the commutation property for rectangular $k$-Schur functions found in [1].

**Theorem 8.** *Let $\gamma$ be a fundamental coweight, and let $w \in W$. Then*

$$\mathfrak{s}_z \mathbf{u}(w) = \mathbf{u}(\tau(w))\mathfrak{s}_z.$$

*Proof.* This follows from Proposition 7 and Proposition 4. □

## 2.2 An algebraic formula

We let $W_{0,j}$ denote the subgroup of $W$ generated by the simple reflections $s_i$ with $i \neq 0, j$ and let $W_0^j$ denote the set of minimal length coset representatives of $W_0/W_{0,j}$. This subsection provides another formula for the $k$-Schur functions which correspond to fundamental coweights.

**Remark 9.** *Let $\gamma$ denote the $j^{th}$ fundamental coweight, as in Definition 5. Then $\Gamma_\gamma$ is naturally identified with $W_0^j$. We can construct a bijection between $W_0^j$ and $\Gamma_\gamma$ as follows. First we give a map from $W_0$ to $\Gamma_\gamma$: for $v \in W_0$, we define a map $v \to v(\gamma)$. This map is clearly onto; $\Gamma_\gamma$ is defined to be the image of this map. From equation (2), we see that $s_i \star \gamma = \gamma$ for $i \neq 0, j$. Therefore, $W_0/W_{0,j}$ is in bijection with $\Gamma_\gamma$.*

**Lemma 10.** *Let $w \in W$ and $\mu, \nu \in \mathfrak{h}_{af}$. The two actions $\star$ and $\diamond$ are related by:*

$$w \diamond (\mu + \nu) = w \diamond \mu + w \star \nu.$$

*Proof.* We prove this on the generators $s_i$. If $i$ is not zero, then $s_i$ is linear and the two actions agree, so there is nothing to prove. If $i = 0$, then

$$s_0 \diamond (\mu + \nu) = s_0 \star (\mu + \nu) - \alpha_0^\vee = (s_0 \star \mu - \alpha_0^\vee) + s_0 \star \nu = s_0 \diamond \mu + s_0 \star \nu.$$ □

The following proposition is a stepping stone to proving our main theorem; It is used to connect Proposition 4 to Theorem 18.

**Proposition 11.** *Let $\gamma$ be a fundamental coweight as in Def 5. Then*

$$\mathfrak{s}_z = \sum_{v \in W_0^j} \mathbf{u}(\tau(v) z v^{-1}).$$

*Proof.* We will use Proposition 4; we show that each $\tau(v)zv^{-1}$ is in fact $z_{v\star\gamma}$.

Let $w = \tau(v)zv^{-1}$. We compute

$$\mathcal{A}_{wv} = \mathcal{A}_{\tau(v)z} = \mathcal{A}_{z_{v\star\gamma}v} = v^{-1} z_{v\star\gamma}^{-1} \diamond \mathcal{A}_\emptyset = v^{-1} \diamond (\mathcal{A}_\emptyset + v \star \gamma) = \mathcal{A}_v + \gamma,$$

where the second equivalence comes from Proposition 7 and the last two use Lemma 10. Applying $v$ to the left and right of the equation above yields $\mathcal{A}_w = \mathcal{A}_\emptyset + v \star \gamma$. By Lemma 6, $w = z_{v\star\gamma}$. Combined with Remark 9, this concludes the proof. □

## 2.3 Towards a general combinatorial formula

We will outline in this section how to build a combinatorial formula for the $k$-Schur functions indexed by a fundamental coweight. Section 3 will give more explicit formulas in affine type $C$.

**Definition 12.** A set of combinatorial objects $\mathcal{R}$ will be called a *combinatorial affine Grassmannian set for* $W$ if:

- There is a transitive action of $W$ on $\mathcal{R}$.
- There exists an element $\emptyset \in \mathcal{R}$ which satisfies $W_0\emptyset = \{\emptyset\}$.
- The map $W^0 \to \mathcal{R}$ defined by $w \to w\emptyset$ is a bijection.

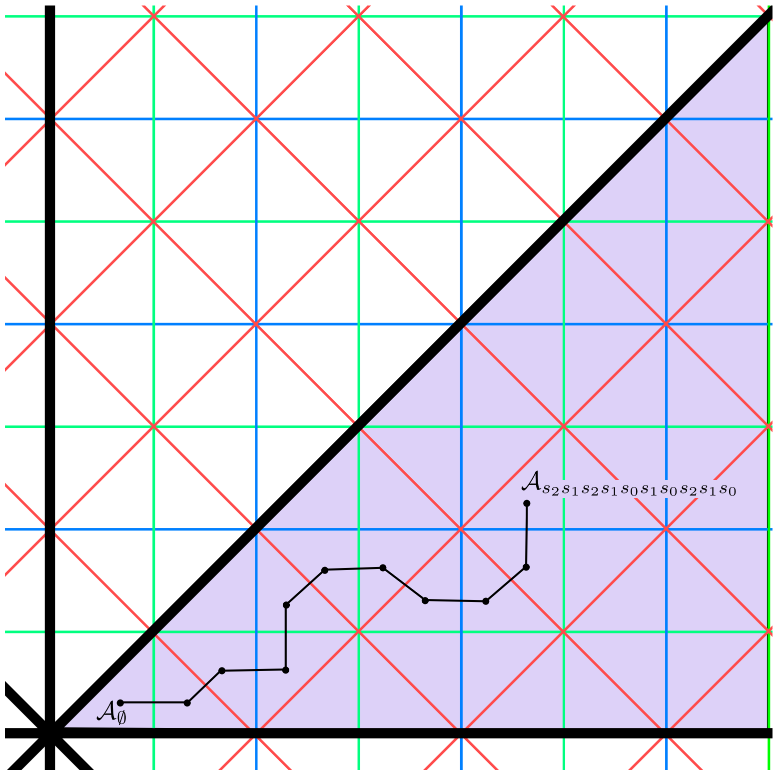
Given a combinatorial affine Grassmannian set $\mathcal{R}$, $\mu \in \mathcal{R}$, and the above bijection, we define $w_\mu \in W^0$ by $w_\mu\emptyset = \mu$.

**Remark 13.** *There is another way of calculating the location of an alcove* $\mathcal{A}_w$ *given a reduced word of the element* $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ *that we picture as an alcove walk. Given a word* $w = s_{i_1}s_{i_2}\cdots s_{i_r}$*, the location of* $\mathbf{u}(w)$ *is calculated by a path starting at* $\mathcal{A}_\emptyset$ *followed by the alcove* $\mathcal{A}_{s_{i_r}}$*, then*

$$\mathcal{A}_{s_{i_{r-1}}s_{i_r}}, \mathcal{A}_{s_{i_{r-2}}s_{i_{r-1}}s_{i_r}}, \ldots, \mathcal{A}_{s_{i_1}s_{i_2}\cdots s_{i_{r-1}}s_{i_r}}.$$

*Each of these alcoves is adjacent (see [1, Proposition 1.1]) and the word for* $w$ *determines a path which travels from the fundamental alcove to* $\mathcal{A}_w$ *traversing a single hyperplane for each simple reflection in the word.*

**Example 14.** A walk that corresponds to the reduced word $s_2s_1s_2s_1s_0s_1s_0s_2s_1s_0$ appears below. Each hyperplane is colored according to the simple reflection that corresponds to a crossing of that hyperplane; e.g., crossing a green hyperplane corresponds to an $s_0$, a red hyperplane corresponds to an $s_1$, and a blue hyperplane corresponds to an $s_2$.




In the diagram above, the path represents a particular reduced word for the element of $W^0$ of type $C_2$. The vertices of this path are in correspondence with the sequence of alcoves: $\mathcal{A}_\emptyset \to \mathcal{A}_{s_0} \to \mathcal{A}_{s_1s_0} \to \mathcal{A}_{s_2s_1s_0} \to \mathcal{A}_{s_0s_2s_1s_0} \to \mathcal{A}_{s_1s_0s_2s_1s_0} \to \mathcal{A}_{s_0s_1s_0s_2s_1s_0} \to \mathcal{A}_{s_1s_0s_1s_0s_2s_1s_0} \to \mathcal{A}_{s_2s_1s_0s_1s_0s_2s_1s_0} \to \mathcal{A}_{s_1s_2s_1s_0s_1s_0s_2s_1s_0} \to \mathcal{A}_{s_2s_1s_2s_1s_0s_1s_0s_2s_1s_0}$.

We can define $x \in \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$ to be on the *positive* or *negative* side of the hyperplane $H_j := H_{\alpha_j,0}$ by $\langle x, \alpha_j \rangle > 0$ or $\langle x, \alpha_j \rangle < 0$, respectively.

**Lemma 15.** *(see for instance [17]) Minimal length expressions of $w \in W$ correspond to alcove walks which do not cross the same affine hyperplane twice.*

**Lemma 16.** *[See for instance [2]] Let $j \in \{1, 2, \ldots, k\}$. Then $w$ has a right $j$ descent ($ws_j < w$) if and only if the alcove $\mathcal{A}_w$ is on the negative side of the hyperplane $H_j$.*

**Lemma 17.** *For all $v \in W_0^j$, $\tau(v)z \in W^0$.*

*Proof.* By Proposition 7, $\tau(v)z = z_{v\star\gamma}v$. Therefore, the alcove

$$\mathcal{A}_{\tau(v)z} = \mathcal{A}_{z_{v\star\gamma}v} = (z_{v\star\gamma}v)^{-1} \diamond \mathcal{A}_\emptyset =$$

$$v^{-1}z_{v\star\gamma}^{-1} \diamond \mathcal{A}_\emptyset = v^{-1} \diamond (\mathcal{A}_\emptyset + v \star \gamma) = \mathcal{A}_v + \gamma,$$

by Lemma 10. Since $v \in W_0^j \subset W^j$, the only right descent of $v$ is a $j$ descent, so for $x \in \mathcal{A}_v$ and $i \neq 0, j$ we have $\langle x, \alpha_i \rangle \geqslant 0$, by Lemma 16. Furthermore, $v \in W_0^j \subset W_0$, so $\langle x, \alpha_j \rangle \geqslant -1$ for $x \in \mathcal{A}_v$ (as every alcove corresponding to $v \in W_0$ has a vertex at the origin). Combining these two facts, we get that $\langle x + \gamma, \alpha_i \rangle \geqslant 0$ for all $i \neq 0$ (since $\langle \gamma, \alpha_j \rangle \geqslant 1$). Therefore, the alcove $\mathcal{A}_v + \gamma$ is dominant, so the corresponding element is Grassmannian, i.e. $\tau(v)z \in W^0$. □

We let $w_0^j$ be the (unique) maximal length element of $W_0^j$. The set $\mathcal{R}$ inherits a partial order from $W^0$; for $\mu, \nu \in \mathcal{R}$ we say $\mu \leqslant \nu$ whenever $w_\mu \leqslant w_\nu$. For $\mu, \nu \in \mathcal{R}$ with $\nu \leqslant \mu$, we let $w_{\mu/\nu} := w_\mu w_\nu^{-1}$.

**Theorem 18.** *Let $R = z\emptyset$ and $S = \tau(w_0^j)z\emptyset$. Then*

$$\mathfrak{s}_z = \sum_{S \leqslant \lambda \leqslant R} \mathbf{u}(w_\lambda \tau^{-1}(w_{R/\lambda})).$$

*Proof.* We construct a map $\Phi : W_0^j \to \mathcal{R}$ by sending $v \in W_0^j$ to $\Phi(v) = \tau(v)z\emptyset$. By Lemma 17, $\Phi$ is injective and hence $\Phi$ is a bijection on its image, which is precisely all $\lambda \in \mathcal{R}$ satisfying $S \leqslant \lambda \leqslant R$. In other words, $w_\lambda = \tau(v)z$ whenever $\Phi(v) = \lambda$.

Now $w_{R/\lambda} = w_R w_\lambda^{-1}$, so $w_{R/\lambda} = z(\tau(v)z)^{-1} = \tau(v^{-1})$. Therefore $w_\lambda \tau^{-1}(w_{R/\lambda}) = \tau(v)z\tau^{-1}(\tau(v^{-1})) = \tau(v)zv^{-1}$. By Proposition 11, the theorem follows. □

**Remark 19.** *It should be noted that Theorem 18 gives the reduced words which appear in the expansion of $\mathfrak{s}_z$; they are precisely the reduced words which correspond to objects from $\mathcal{R}$. Once the bijection between $W^0$ and $\mathcal{R}$ is understood, the terms in Theorem 18 are as well.*

*In this sense the theorem is stronger than Proposition 4, although its proof relies entirely on the proposition. In particular, this theorem generalizes Definition 2.1 of [1] to all affine types.*

# 3 Type C combinatorics

As an application of Theorem 18, we use this section to develop the combinatorics of affine type $C$.

## 3.1 Type C root system background

Fix an integer $k > 1$. We recall some facts about roots and weights in affine type $C$ (see [2] for more details). We let $\epsilon_1, \dots, \epsilon_k$ denote an orthonormal basis for $V := \mathbb{R}^k \equiv \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$. We realize $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \dots, \alpha_{k-1} = \epsilon_{k-1} - \epsilon_k, \alpha_k = 2\epsilon_k$ as the simple roots of finite type $C_k$.

The fundamental weights are realized as $\Lambda_i := \epsilon_1 + \cdots + \epsilon_i$ for $i = 1, \dots, k$. The fundamental coweights are $\Lambda_i^\vee = 2\Lambda_i$ for $i \neq k$ and $\Lambda_k^\vee = \Lambda_k$.

The fundamental coweights $\Lambda_1^\vee, \dots, \Lambda_{k-1}^\vee$ also belong to the coroot lattice $Q_{\text{fin}}^\vee$. The elements $t_{\Lambda_i^\vee}$ actually equal $z_{\Lambda_i^\vee}^{-1}$ (for $i \neq k$) in $W_{\text{ext}}$, i.e. these elements have trivial Dynkin diagram automorphisms (as compared to type $A$, where all fundamental coweights correspond to distinct non-trivial Dynkin diagram automorphisms).

Since $\Lambda_k^\vee$ is not in $Q_{\text{fin}}^\vee$, $t_{\Lambda_k^\vee}$ corresponds to a non-trivial Dynkin diagram automorphism. In affine type $C$ there is only one such automorphism, which we will denote $\tau$. It is defined by $\tau(i) = k - i$ for all $i \in \{0, 1, \dots, k\}$.

We let $W$ denote the affine Coxeter group of type $C$. Recall it is generated by $s_0, s_1, \ldots, s_k$ subject to the relations:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i \in \{0, 1, \ldots, k\}, \\ s_i s_j &= s_j s_i \text{ if } i - j \neq \pm 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in \{1, \ldots, k-2\}, \\ s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i \text{ for } i \in \{0, k-1\}. \end{aligned}$$

## 3.2 Bijection between Grassmannian elements and symmetric $2k$-cores

**Definition 20.** The *hook length* of a cell $x$ in the Young diagram of a partition $\lambda$ is the number of cells of the Young diagram of $\lambda$ to the right of $x$ and above $x$, including the box $x$. A partition $\lambda$ is called an $n$-core if for every cell $x$ in the Young diagram of $\lambda$, $n$ does not divide the hook length of $x$.

In [3], Hanusa and Jones give a construction for a combinatorial affine Grassmannian set for $W$ for all classical affine $W$ (the affine Grassmannians corresponding to $B_k^{(1)}/B_k, C_k^{(1)}/C_k, D_k^{(1)}/D_k, B_k^{(1)}/D_k$).

In affine type $C$, the set $\mathcal{R}$ of combinatorial affine Grassmannian elements they give are symmetric $2k$-core partitions (symmetry is with respect to transposing the partition). We give a short outline of the action of $W$ on $\mathcal{R}$ as follows:

Let the residue of a cell $(i, j)$ of a Young diagram be:

$$res(i,j) = \begin{cases} j - i \mod 2k & \text{if } 0 \leqslant (j-i) \mod 2k \leqslant k \\ 2k - ((j-i) \mod 2k) & \text{if } k < (j-i) \mod 2k < 2k \end{cases}$$

We can then define an action on symmetric 2k-core partitions by letting $s_i\lambda =$

$$\begin{cases} \lambda \cup \{\text{residue } i \text{ cells}\} & \text{if } \lambda \text{ has addable cell of residue } i \\ \lambda \setminus \{\text{residue } i \text{ cells}\} & \text{if } \lambda \text{ has removable cell of residue } i \\ \lambda & \text{else} \end{cases}$$

**Theorem 21** (Hanusa, Jones [3])**.** *The action of $W$ on $\mathcal{R}$ described above makes $\mathcal{R}$ into a combinatorial affine Grassmannian set for $W$.*

**Example 22.** Let $k = 3$ and let $w = s_1 s_2 s_3 s_2 s_0 s_1 s_0 \in W^0$. Then $w$ corresponds to the symmetric 6-core $(6, 3, 2, 1, 1, 1)$.

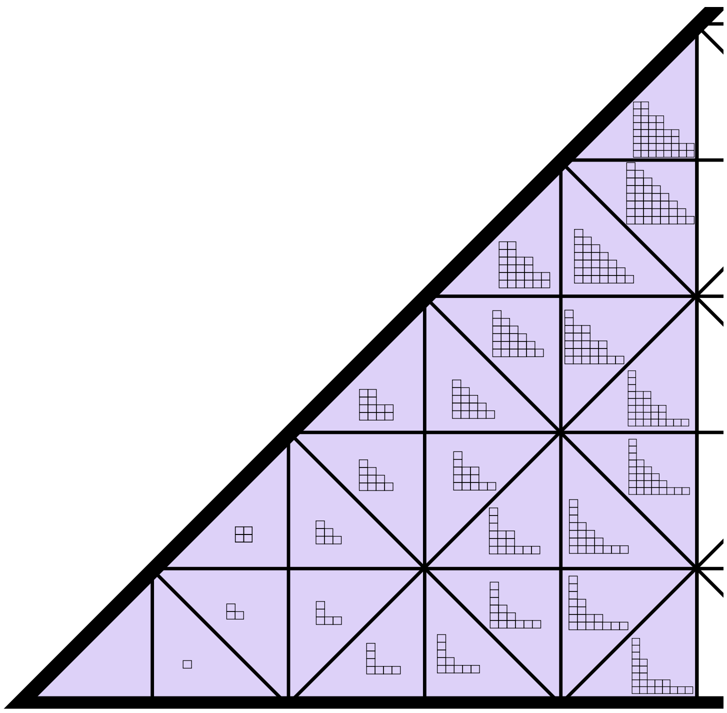
| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 2 | 1 | | | | |
| 1 | 0 | 1 | | | |
| 0 | 1 | 2 | 3 | 2 | 1 |

**Remark 23.** *Symmetric $2k$-core partitions have extraneous data. Half of the partition is determined from the other, so we will sometimes think of a symmetric $2k$-core as a diagram with boxes $(i, j)$ for $j \geqslant i$. We call such a diagram a* shifted diagram.

**Example 24.** Let $k = 3$ and $w = s_1 s_2 s_3 s_2 s_0 s_1 s_0$ as above. Then the shifted diagram for the 6-core is:

| | | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | | | |
| 0 | 1 | 2 | 3 | 2 | 1 |

**Example 25.** A portion of the lattice of symmetric 4-cores coming from the action of the affine Coxeter group of type $C_2^{(1)}$ is pictured below.



The pseudo-translation $z_{\Lambda_1^\vee}$ corresponding to the fundamental coweight $\Lambda_1^\vee = 2\epsilon_1$ takes the fundamental alcove to the alcove indexed by the symmetric 4-core $(4, 1, 1, 1)$ and the pseudo-translation $z_{\Lambda_2^\vee}$ corresponding to the fundamental coweight $\Lambda_2^\vee = \epsilon_1 + \epsilon_2$ takes the fundamental alcove to the alcove indexed by the symmetric 4-core $(2, 2)$.

## 3.3 Words and cores corresponding to fundamental coweights

Each $s_i$ acts on $V$ by reflecting across the hyperplane corresponding to the simple root $\alpha_i$ for $i \neq 0$ and reflecting across the affine hyperplane $H_{\theta,1} = \{v \in V : \langle v, \theta \rangle = 1\}$, where $\theta$ is the highest root, for $i = 0$. Specifically, if we let $(a_1, \ldots, a_k) \in V$ represent $\sum_i a_i \epsilon_i$, then:

$$s_i \diamond (a_1, \ldots, a_k) = \begin{cases} (a_1, \ldots, a_{i+1}, a_i, \ldots, a_k) & \text{for } i = 1, \ldots, k-1; \\ (a_1, \ldots, a_{k-1}, -a_k) & \text{for } i = k; \\ (2 - a_1, \ldots, a_k) & \text{for } i = 0. \end{cases}$$

For $i \leqslant k + 1$ we let $w_i := s_{i-1} s_{i-2} \cdots s_1 s_0 \in W$.

**Lemma 26.** *For $i \leqslant k$, the element $w_i$ acts on $v = (a_1, \ldots, a_k) \in V$ by:*

$$w_i \diamond v = (a_2, a_3, \ldots, a_i, 2 - a_1, a_{i+1}, \ldots, a_k).$$

*Also,*

$$w_{k+1} \diamond v = (a_2, a_3, \ldots, a_k, a_1 - 2)$$

*Proof.* Simple calculation using Weyl group action described above. □

**Lemma 27.** $w_{k+1}^{-1} w_k w_j^{-1} \diamond (a_1, \ldots, a_k) = (a_j - 2, a_1, a_2, \ldots, \widehat{a_j}, \ldots, a_k)$.

*Proof.* Simple calculation using Lemma 26. □

If $G_\emptyset$ is the centroid of $\mathcal{A}_\emptyset$, then

$$G_\emptyset = \frac{1}{k+1} \sum_i \Lambda_i = (\frac{k}{k+1}, \frac{k-1}{k+1}, \ldots, \frac{1}{k+1}).$$

Recall that for a fixed $j$ we let $\gamma$ denote the coweight $\Lambda_j^\vee$.

**Lemma 28.** *For $j \neq k$, $z_\gamma = (w_j w_k^{-1} w_{k+1})^j$.*

*Proof.* Let $w = w_j w_k^{-1} w_{k+1}$. We compute the centroid of the alcove $G_{w^j} = w^{-j} \diamond G_\emptyset = G_\emptyset - (\underbrace{2, 2, \ldots, 2}_{j}, 0, 0, \ldots, 0)$ by Lemma 27. Therefore $w^j = z_\gamma$ by Lemma 6. □

**Corollary 29.** *For $j \neq k$, $z_\gamma$ corresponds to the symmetric $2k$-core $\lambda = ((2k)^j, j^{2k-j})$. Equivalently, $z_\gamma$ corresponds to the shifted partition $(2k, 2k-1, \ldots, 2k-j+1)$.*

*Proof.* Let $w = w_j w_k^{-1} w_{k+1}$. The first application of $w$ will add $2k - j + 1$ boxes to the shifted diagram. Every subsequent application adds $2k - j + 1$ boxes to a new row of the shifted diagram and one box to each previous row. □

The last case, when $\gamma = \Lambda_k^\vee$, is slightly different. We end this section by describing the corresponding symmetric $2k$-core in this case.

**Lemma 30.** *If $\gamma = \Lambda_k^\vee$ then $z_\gamma = w_k^{-1} w_{k-1}^{-1} \cdots w_1^{-1}$.*

*Proof.*

$$G_{w_k^{-1} w_{k-1}^{-1} \cdots w_1^{-1}} = (w_k^{-1} w_{k-1}^{-1} \cdots w_1^{-1})^{-1} \diamond G_\emptyset = w_1 \cdots w_{k-1} w_k \diamond G_\emptyset =$$

$$(2 - \frac{1}{k+1}, 2 - \frac{2}{k+1}, \ldots, 2 - \frac{k}{k+1}) = (1, 1, \ldots, 1) + G_\emptyset = \gamma + G_\emptyset.$$

By Lemma 6, the statement follows. □

**Lemma 31.** *With the action on partitions described above,*

$$w_i^{-1} w_{i-1}^{-1} \cdots w_2^{-1} w_1^{-1} \emptyset = \underbrace{(i, i, \ldots, i)}_{i}.$$

*Proof.* The proof is by induction. $w_1 = s_0$, and $s_0\emptyset = (1)$. If $w_{i-1}^{-1} \cdots w_1^{-1}\emptyset = (i-1, i-1, \ldots, i-1)$, then $w_i^{-1}(i-1, i-1, \cdots, i-1) = s_0 s_1 \cdots s_{i-1} s_i (i-1, \ldots, i-1) = s_0 s_1 \cdots s_{i-1}(i, i-1, \ldots, i-1, 1) = s_0 s_1 \cdots s_{i-2}(i, i, i-1, \ldots, i-1, 2) = \cdots = (i, i, \ldots, i)$. □

**Corollary 32.** *$z_{\Lambda_k^\vee}$ corresponds to the symmetric $2k$-core*

$$\underbrace{(k, k, \ldots, k)}_{k}.$$

*Equivalently, this corresponds to the shifted partition $(k, k-1, \ldots, 2, 1)$.*

*Proof.* Follows from Lemma 30 and Lemma 31. □

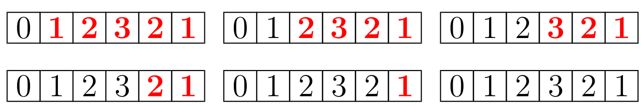
## 3.4 Subcores and a combinatorial formula

We now illustrate our formulas for $k = 3$. We first introduce the shorthand notation $\mathbf{u}(i_1 i_2 \ldots i_m)$ to denote $\mathbf{u}(s_{i_1} s_{i_2} \cdots s_{i_m})$. The simplest example is $j = 1$.

**Example 33.** Let $j = 1$. Then $z = z_{\Lambda_1^\vee} = s_1 s_2 s_3 s_2 s_1 s_0$. The Dynkin automorphism $\tau$ corresponding to $z$ is trivial. $w_0^1$ is the element $s_1 s_2 s_3 s_2 s_1$. Therefore $R = z\emptyset = (6, 1, 1, 1, 1, 1)$ and $S = w_0^1 z\emptyset = (1)$. There are 6 symmetric 6-cores between $S$ and $R$, they are:

$$(1), (2, 1), (3, 1, 1), (4, 1, 1, 1), (5, 1, 1, 1, 1), (6, 1, 1, 1, 1, 1).$$

They correspond respectively to the following shifted diagrams.



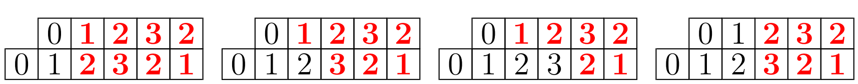

Therefore

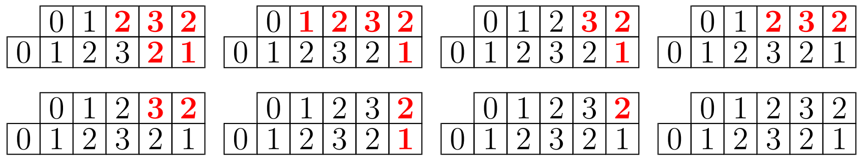
$$\mathfrak{s}^C_{z_{\Lambda_1^\vee}} = \mathbf{u}(012321) + \mathbf{u}(101232) + \mathbf{u}(210123)$$
$$+\mathbf{u}(321012) + \mathbf{u}(232101) + \mathbf{u}(123210).$$

**Example 34.** Let $j = 2$. Then $z = z_{\Lambda_2^\vee} = s_2 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0$. The Dynkin automorphism $\tau$ corresponding to $z$ is trivial. $w_0^2 = s_2 s_1 s_3 s_2 s_1 s_3 s_2$. Therefore $R = z\emptyset = (6, 6, 2, 2, 2, 2)$ and $S = w_0^2 z\emptyset = (2, 2)$. There are 12 symmetric 6-cores between $S$ and $R$, they are:

$$(2, 2), (3, 2, 1), (4, 2, 1, 1), (3, 3, 2),$$
$$(4, 3, 2, 1), (5, 2, 1, 1, 1), (5, 4, 2, 2, 1), (6, 3, 2, 1, 1, 1),$$
$$(6, 4, 2, 2, 1, 1), (5, 5, 2, 2, 2), (6, 5, 2, 2, 2, 1), (6, 6, 2, 2, 2, 2).$$

They correspond respectively to the following shifted diagrams.





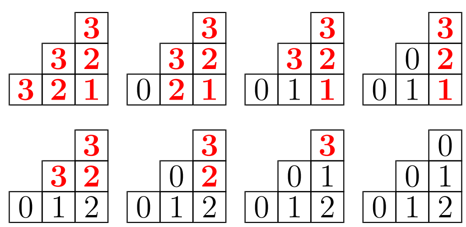
By Theorem 18,

$$\mathfrak{s}^C_{z_{\Lambda_2^\vee}} = \mathbf{u}(0102132132) + \mathbf{u}(0210232123) + \mathbf{u}(0321023212) + \mathbf{u}(1021023123)$$
$$+\mathbf{u}(1032102312) + \mathbf{u}(0232102321) + \mathbf{u}(2103210231) + \mathbf{u}(1023210232)$$
$$+\mathbf{u}(2102321023) + \mathbf{u}(3210321021) + \mathbf{u}(3210232102) + \mathbf{u}(2321023210).$$

**Example 35.** Let $j = 3$. The word $z = z_{\Lambda_3^\vee}$ is $s_0s_1s_2s_0s_1s_0$. Then $z$ corresponds to the unique non-trivial Dynkin automorphism defined by $\tau(i) = 3 - i$. The corresponding shifted diagram is $(3, 2, 1)$. Let $R = (3, 3, 3) = z\emptyset$ and $S = \tau(w_0^3)z\emptyset = \emptyset$. There are 8 symmetric 6 cores between $S$ and $R$. They are

$$\emptyset, (1), (2, 1), (2, 2), (3, 1, 1), (3, 2, 1), (3, 3, 2), (3, 3, 3).$$

These correspond respectively to the following shifted diagrams, where the bold letters correspond to elements not in $\lambda$ which have $\tau^{-1}$ applied to them.



By Theorem 18,

$$\mathfrak{s}^C_{z_{\Lambda_3^\vee}} = \mathbf{u}(321323) + \mathbf{u}(032312) + \mathbf{u}(103231) + \mathbf{u}(010321)$$
$$+\mathbf{u}(210323) + \mathbf{u}(021032) + \mathbf{u}(102103) + \mathbf{u}(010210).$$

# 4 Remaining types

Although Hanusa and Jones [3] did give descriptions of combinatorial affine Grassmannian sets for the type $B$ and $D$ cases, the combinatorics involved are not as nice. It seems plausible that some different collection of elements better suited to describing the terms appearing in expansions of $k$-Schur functions in these types will arise in the future. Rather than spending a good deal of space here to developing these in full generality, we will include the case of affine $B$ of rank 3 and affine $D$ of rank 4 as examples of what the combinatorics would look like; the compelled reader should easily be able to develop a corresponding expansion in full generality from these examples, the concepts of Section 3, and a full understanding of Hanusa and Jones' combinatorics in these types.

## 4.1 Affine type $B$, rank 3

Affine type $B$ has one non-trivial Dynkin diagram automorpism $\tau$, which is defined by permuting the indices 0 and 1, and fixing all other $i$.

**Example 36.** The affine Grassmannian element $z = s_0 s_2 s_3 s_2 s_0$ corresponds to translation by the fundamental coweight $\Lambda_1^\vee$, which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(7, 2, 1, 1, 1, 1, 1)$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 0 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 2 | | | | | | |
| 0 | 0 | | | | | |
| 0 | 0 | 2 | 3 | 2 | 0 | 0 |

This fundamental coweight corresponds to the nontrivial Dynkin automorphism $\tau$. Again, as in type $C$, the objects involved in the bijection of Hanusa and Jones are symmetric cores, so we will remove half of the diagram, and study the skew partition:

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | 2 | 3 | 2 | 0 | 0 |

In this case, $\tau(w_0^1) = z$, so we need to look at all sub-diagrams between $S = \emptyset$ and $R = (7, 1)$. There are six such diagrams:

| | | | | | | |
|---|---|---|---|---|---|---|
| | **1** | | | | | |
| **1** | **1** | **2** | **3** | **2** | **1** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | **2** | **3** | **2** | **1** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | 2 | **3** | **2** | **1** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | 2 | 3 | **2** | **1** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | 2 | 3 | 2 | **1** | **1** |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | | | | | |
| 0 | 0 | 2 | 3 | 2 | 0 | 0 |

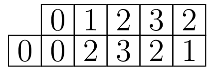
By Theorem 18,

$$\mathfrak{s}^B_{z_{\Lambda_1}^\vee} = \mathbf{u}(12321) + \mathbf{u}(01232) + \mathbf{u}(20123) + \mathbf{u}(32012) + \mathbf{u}(23201) + \mathbf{u}(02320).$$

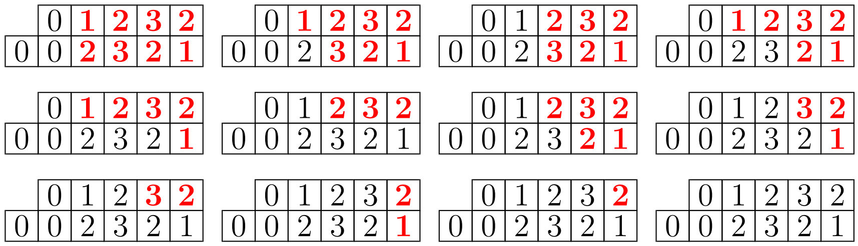
**Example 37.** The affine Grassmannian element $z = s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_0$ corresponds to translation by the Fundamental coweight $\Lambda_2^\vee$, which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(6, 6, 2, 2, 2, 2)$:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | | | | |
| 2 | 3 | | | | |
| 3 | 2 | | | | |
| 2 | 1 | | | | |
| 0 | 0 | 1 | 2 | 3 | 2 |
| 0 | 0 | 2 | 3 | 2 | 1 |

This coweight corresponds to the trivial Dynkin automorphism, and the even symmetric 6-core corresponds to the following skew partition:



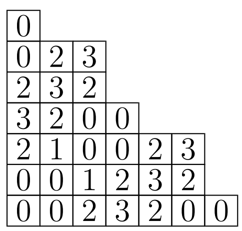
In this case, $w_0^2 = s_2s_3s_1s_2s_3s_1s_2$, so we need to look at all skew sub-diagrams between $S = w_0^2 z\emptyset = (2,1)$ and $R = (7,1)$. There are twelve such diagrams:



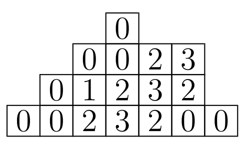
By Theorem 18,

$$\mathfrak{s}^B_{z^\vee_{\Lambda_2}} = \mathbf{u}(02132132) + \mathbf{u}(20213231) + \mathbf{u}(12021323) + \mathbf{u}(32021321)$$
$$\mathbf{u}(23202321) + \mathbf{u}(12320232) + \mathbf{u}(31202132) + \mathbf{u}(23120231)$$
$$+\mathbf{u}(12312023) + \mathbf{u}(32312021) + \mathbf{u}(13231202) + \mathbf{u}(21323120).$$

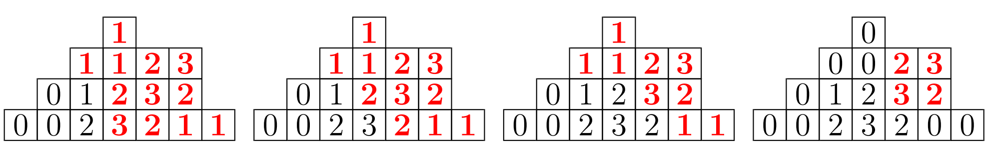
**Example 38.** The affine Grassmannian element $z = s_3s_2s_3s_0s_2s_3s_1s_2s_0$ corresponds to translation by the fundamental coweight $\Lambda_3^\vee$, which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(7,6,6,4,3,3,1)$:

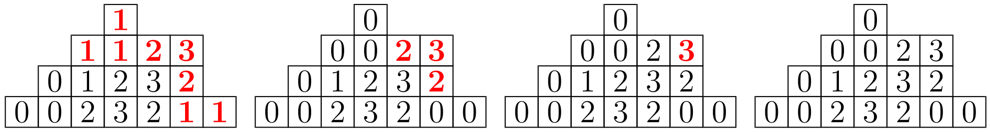


This coweight corresponds to the nontrivial Dynkin automorphism $\tau$, and the even symmetric 6-core corresponds to the following skew partition:



In this case, $w_0^3 = s_3s_2s_1s_3s_2s_3$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^3)z\emptyset = (3,2)$ and $R = z\emptyset = (7,5,4,1)$. There are eight such diagrams:





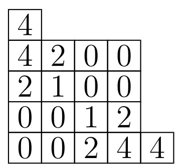
By Theorem 18,

$$\mathfrak{s}^B_{z^\vee_{\Lambda_3}} = \mathbf{u}(120\mathbf{323123}) + \mathbf{u}(3120\mathbf{32312}) + \mathbf{u}(23120\mathbf{3231}) + \mathbf{u}(023120\mathbf{323})$$
$$\mathbf{u}(323120\mathbf{321}) + \mathbf{u}(3023120\mathbf{32}) + \mathbf{u}(23023120\mathbf{3}) + \mathbf{u}(323023120).$$

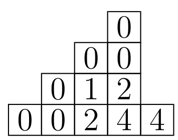
## 4.2 Affine type $D$, rank 4

All Dynkin automorphisms in affine $D$ rank 4 leave the index 2 fixed. Here we give explicit expansions for two of the fundamental coweights.

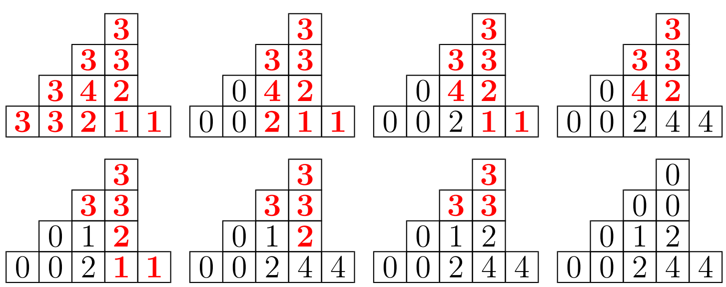
**Example 39.** The affine Grassmannian element $z = s_0 s_2 s_4 s_1 s_2 s_0$ corresponds to translation by the fundamental coweight $\Lambda_3^\vee$, which under the identification of Hanusa and Jones corresponds to the even symmetric 8-core $(5, 4, 4, 4, 1)$:



This coweight corresponds to a nontrivial Dynkin automorphism $\tau$ which swaps 0 with 3 and 1 with 4, and the even symmetric 8-core corresponds to the following skew partition:



In this case, $w_0^3 = s_3 s_2 s_4 s_1 s_2 s_3$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^3) z \emptyset = \emptyset$ and $R = z\emptyset = (5, 3, 2, 1)$. There are eight such diagrams:



By Theorem 18,

$$\mathfrak{s}^D_{z^\vee_{\Lambda_4}} = \mathbf{u}(\mathbf{321423}) + \mathbf{u}(0\mathbf{32142}) + \mathbf{u}(20\mathbf{3241}) + \mathbf{u}(420\mathbf{324})$$
$$\mathbf{u}(120\mathbf{321}) + \mathbf{u}(4120\mathbf{32}) + \mathbf{u}(24120\mathbf{3}) + \mathbf{u}(024120).$$

**Example 40.** The affine Grassmannian element $z = s_0 s_2 s_3 s_1 s_2 s_0$ corresponds to translation by the fundamental coweight $\Lambda_4^\vee$, which under the identification of Hanusa and Jones corresponds to the even symmetric 8-core $(4,4,4,4)$:

| | | | |
|---|---|---|---|
| 3 | 2 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 0 | 0 | 1 | 2 |
| 0 | 0 | 2 | 3 |

This coweight corresponds to a nontrivial Dynkin automorphism $\tau$ which swaps 0 with 4 and 1 with 3, and the even symmetric 8-core corresponds to the following skew partition:

| | | | |
|---|---|---|---|
| | | | 0 |
| | | 0 | 0 |
| | 0 | 1 | 2 |
| 0 | 0 | 2 | 3 |

In this case, $w_0^4 = s_4 s_2 s_3 s_1 s_2 s_4$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^4) z\emptyset = \emptyset$ and $R = z\emptyset = (4,3,2,1)$. There are eight such diagrams:

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | **4** | **3** | **2** |
| **4** | **4** | **2** | **1** |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | **3** | **2** |
| 0 | 0 | **2** | **1** |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | **3** | **2** |
| 0 | 0 | 2 | **1** |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | **3** | **2** |
| 0 | 0 | 2 | 3 |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | 1 | **2** |
| 0 | 0 | 2 | **1** |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | 1 | **2** |
| 0 | 0 | 2 | 3 |

| | | | |
|---|---|---|---|
| | | | **4** |
| | | **4** | **4** |
| | 0 | 1 | 2 |
| 0 | 0 | 2 | 3 |

| | | | |
|---|---|---|---|
| | | | 0 |
| | | 0 | 0 |
| | 0 | 1 | 2 |
| 0 | 0 | 2 | 3 |

By Theorem 18,

$$\mathfrak{s}^D_{z^\vee_{\Lambda_4}} = \mathbf{u}(\mathbf{421324}) + \mathbf{u}(0\mathbf{42132}) + \mathbf{u}(20\mathbf{4231}) + \mathbf{u}(320\mathbf{423})$$

$$\mathbf{u}(120\mathbf{421}) + \mathbf{u}(3120\mathbf{42}) + \mathbf{u}(23120\mathbf{4}) + \mathbf{u}(023120).$$

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