A note on zero-sum 5-flows in regular graphs

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Abstract

Let G be a graph. A *zero-sum flow* of G is an assignment of non-zero real numbers to the edges of G such that the sum of the values of all edges incident with each vertex is zero. Let k be a natural number. A *zero-sum k-flow* is a flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$. It has been conjectured that every r-regular graph, $r \geq 3$, admits a zero-sum 5-flow. In this paper we provide an affirmative answer to this conjecture, except for $r = 5$.

1. Introduction

Nowhere-zero flows on graphs were introduced by Tutte [8] in 1949 and since then have been extensively studied by many authors. A great deal of research in the area has been motivated by Tutte's 5-Flow Conjecture which states that every 2-edge connected graph can have its edges directed and labeled by integers from $\{1, 2, 3, 4\}$ in such a way that Kirchhoffs current law is satisfied at each vertex. In 1983, Bouchet [4] generalized

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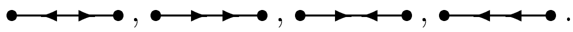
this concept to bidirected graphs. A *bidirected graph* G is a graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities:



Let G be a bidirected graph. For every $v \in V(G)$, the set of all edges with tails (respectively, heads) at v is denoted by $E^+(v)$ (respectively, $E^-(v)$). The function $f : E(G) \longrightarrow \mathbb{R}$ is a *bidirected flow* of G if for every $v \in V(G)$, we have

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).$$

If f takes its values from the set $\{\pm 1, \ldots, \pm(k-1)\}$, then it is called a *nowhere-zero bidirected k-flow.*

Bouchet proposed the following interesting conjecture.

Bouchet's Conjecture. [4, 9] *Every bidirected graph that has a nowhere-zero bidirected flow admits a nowhere-zero bidirected* 6*-flow.*

Bouchet showed that his conjecture is true if 6 is replaced by 216. Then Zyka [10] reduced 216 to 30. Also, DeVos proved Bouchet's Conjecture with 6 replaced by 12, see [5].

Let G be a graph. A *k-regular* graph is a graph where each vertex is of degree k. A *zero-sum flow* of G is an assignment of non-zero real numbers to the edges of G such that the sum of the values of all edges incident with each vertex is zero. Let k be a natural number. A *zero-sum k-flow* is a flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$. The following conjecture was posed on the zero-sum flows in graphs.

Zero-Sum Conjecture (ZSC). [1] *If G is a graph with a zero-sum flow, then G admits a zero-sum* 6*-flow.*

The following theorem shows the relation between Bouchet's Conjecture and ZSC.

Theorem 1. [2] *Bouchet's Conjecture and ZSC are equivalent.*

The following conjecture is a stronger version of ZSC for regular graphs.

Conjecture A. [2] *Every r-regular graph ($r \geq 3$) admits a zero-sum* 5*-flow.*

Motivated by Bouchet's Conjecture and along with Theorem 1 we focused our attention to establish the Conjecture A. In the next section, except for the case $r = 5$, we prove Conjecture A. The following two results show the validity of this conjecture for some special cases.

Theorem 2. [1] *Let r be an even integer with $r \geq 4$. Then every r-regular graph has a zero-sum 3-flow.*

Theorem 3. [2] *Let G be an r-regular graph. If r is divisible by 3, then G has a zero-sum 5-flow.*

Remark. There are some regular graphs with no zero-sum 4-flow. To see this consider the graph given in Figure 1. Assume, to the contrary, that this graph has a zero-sum 4-flow. Since the sum of the values of all three edges incident with a vertex is zero, not all can be odd, so -2 or 2 should appear on (exactly) one edge incident to the vertex. On the other hand two numbers with absolute value 2 cannot appear in the neighborhood of a vertex. So the edges of G with values ± 2 form a perfect matching. But by a celebrated Theorem of Tutte [3, p.76], G has no perfect matching, a contradiction.

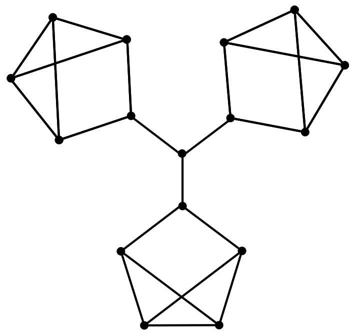


Figure 1. A 3-regular graph with no zero-sum 4-flow

2. The Main Result

In this section we prove that every r-regular graph, $r \geq 3$, $r \neq 5$, admits a zero-sum 5-flow. Before establishing our main result we need some notations and definitions.

A *factor* of a graph is a spanning subgraph. A *k-factor* is a factor which is k-regular. In particular a 2-factor is a disjoint union of cycles that cover all the vertices. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *k-factorization* of G is a partition of the edges of G into disjoint k-factors. For integers a and b, $1 \leq a \leq b$, an *$[a, b]$-factor* of G is defined to be a factor F of G such that $a \leq d_F(v) \leq b$, for every $v \in V(G)$. For any vertex $v \in V(G)$, let $N_G(v) = \{\, u \in V(G) \mid uv \in E(G) \,\}$.

Below we state two known theorems about the factorization of graphs.

Theorem 4. [7] *Every $2k$-regular multigraph admits a 2-factorization.*

Theorem 5. [6] *Let $r \geq 3$ be an odd integer and let k be an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every r-regular graph has a $[k-1, k]$-factor each component of which is regular.*

Lemma 6. *Let G be an r-regular graph. Then for every even integer q, $2r \leq q \leq 4r$, there exists a function $f : E(G) \to \{2, 3, 4\}$ such that for every $u \in V(G)$, $\sum_{v \in N_G(u)} f(uv) = q$.*

Proof. For every edge $e = uv$, we add a new edge $e' = uv$ to the graph G and call the resulting graph G'. Clearly, G' is a $2r$-regular multigraph. By Theorem 4, G' admits a 2-factorization with 2-factors $F_1, \ldots, F_r$. Now, for every $e \in F_i$, $1 \leq i \leq r$, we define a function $g : E(G') \to \{1, 2\}$ as follows:

$$g(e) = \begin{cases} 2, & 1 \leq i \leq \frac{q-2r}{2}; \\ 1, & \frac{q-2r}{2} < i. \end{cases}$$

Therefore, for each $u \in V(G')$, $\sum_{v \in N_{G'}(u)} g(uv) = q$. Now, define a function $f : E(G) \to \{2, 3, 4\}$ such that for every $e = uv \in E(G)$, $f(e) = g(e) + g(e')$, where $e' = uv$ in G'. Then for every $u \in V(G)$, $\sum_{v \in N_G(u)} f(uv) = q$, as desired. □

Now, we are in a position to prove our main theorem.

Theorem 7. *Let $r \geq 3$ and $r \neq 5$. Then every r-regular graph has a zero-sum 5-flow.*

Proof. If $r = 3$, then by Theorem 3, the assertion holds. First we prove the theorem for $r = 7$. Let G be a 7-regular graph. Then, by Theorem 5, G is a disjoint union of a 3-regular graph H_1 and a 4-regular graph H_2. By Theorem 4, H_2 can be decomposed into two 2-factors H_2' and H_2''. Assign 1 and 2 to all edges of H_2' and H_2'', respectively. By Lemma 6, there exists a function $f : E(H_1) \to \{2, 3, 4\}$ such that for every $u \in V(H_1)$, $\sum_{v \in N_{H_1}(u)} f(uv) = 8$. Now, assign -2 to every edge in $E(G) \setminus E(H)$ and we are done.

Now, let $r \geq 9$ be an odd integer. By Theorem 5, for every k, $k \leq \frac{2r}{3}$, G has a $[k-1, k]$-factor whose components are regular. Let $k = \lfloor \frac{2r}{3} \rfloor$, $k' = r - k$, and H be a $[k-1, k]$-factor of G such that H_1 is the union of the $(k-1)$-regular components of H and $H_2 = H \setminus H_1$. It can be easily checked that $k \leq 2k' \leq 2k - 4$. Hence by Lemma 6, there exists a function $f : E(H_1) \longrightarrow \{2, 3, 4\}$ such that for every $u \in V(H_1)$, $\sum_{v \in N_{H_1}(u)} f(uv) = 4k' + 4$. Also by Lemma 6, there exists a function $f : E(H_2) \longrightarrow \{2, 3, 4\}$ such that for every $v \in V(H_2)$, $\sum_{v \in N_{H_2}(u)} f(uv) = 4k'$. Finally assign -4 to every edge of $E(G) \setminus E(H)$. Now, by Theorem 2 the proof is complete. □

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