Multiranks for partitions into Multi-Colors

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Abstract

We generalize Hammond-Lewis birank to multiranks for partitions into colors and give combinatorial interpretations for multipartitions such as $b(n)$ defined by H. Zhao and Z. Zhong and $Q_{p_1,p_2}(n)$ defined by Toh congruences modulo 3, 5, 7.

Keywords: Partition congruences; multirank; Jacobi's triple product identity; Quintuple product identity

1 Introduction and Motivation

For the two indeterminates q and z with $|q| < 1$, the q-shifted factorial of infinite order and the modified Jacobi theta function are defined respectively by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n) \quad \text{and} \quad \langle z;q\rangle_\infty = (z;q)_\infty(q/z;q)_\infty$$

where the multi-parameter expression for the former will be abbreviated as

$$[\alpha, \beta, \cdots, \gamma; q]_\infty = (\alpha; q)_\infty(\beta; q)_\infty \cdots (\gamma; q)_\infty.$$

For brevity we denote $E_m = (q^m; q^m)_\infty$.

Let $p(n)$ be the number of unrestricted partitions of n, where n is nonnegative integer. In 1921 Ramanujan [27] discovered the following congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7). \end{aligned}$$

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There exist many proofs in mathematical literature, for which one can be found, for example, in [6, 7, 12, 17, 26, 30].

In 1944 F. J. Dyson [15] defined the rank of a partition as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m and let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. In 1953 A. O. L. Atkin and H. P. F. Swinnerton-Dyer [3] proved

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{p(5n+4)}{5}$$

and

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{p(7n+5)}{7}.$$

Following from the fact that the operation of conjugation reverses the sign of the rank, the trivial consequences are

$$N(m,n) = N(-m,n) \quad \text{and} \quad N(m,t,n) = N(t-m,t,n).$$

In 2010 Chan [9] introduced the partition function $a(n)$ by $\sum_{n=0}^{\infty} a(n)q^n := \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}$, and obtained the following congruence

$$a(3n+2) \equiv 0 \pmod 3.$$

Another proof has been given by B. Kim [24]. He defined a crank analog $M'(m,N,n)$ for $a(n)$ and proved that

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

for all nonnegative integers n, where $M'(m,N,n)$ is the number of partition of n with crank congruent to m modulo N.

Hammond and Lewis [21] investigated some elementary results for 2-colored partitions mod 5. They defined

$$birank(\pi_1,\pi_2) = \#(\pi_1) - \#(\pi_2),$$

where $\#(\pi)$ is the number of parts in the partition π. They explained that the residue of the birank mod 5 divided the bipattitions of n into 5 equal classes provided $n \equiv 2, 3$ or $4 \pmod 5$. F. G. Garvan [19] found two other analogs the Dyson-birank and the 5-core-birank.

H. Zhao and Z. Zhong [31] have also investigated the arithmetic properties of a certain function $b(n)$ given by $\sum_{n=0}^{\infty} b(n)q^n = (q;q)_\infty^{-2}(q^2;q^2)_\infty^{-2}$. They have found $b(5n+4) \equiv 0 \pmod 5$ and $b(7n+2) \equiv b(7n+3) \equiv b(7n+4) \equiv b(7n+6) \equiv 0 \pmod 7$ for any $n \geqslant 0$.

Toh [28] has also given lots of partition congruences such as $Q_{(p_1,p_2)}(n) \equiv 0 \pmod \ell$ for prime number ℓ, which is defined in section 3.

The main purpose of this paper is to define multirank for partition into colors and prove multipartitions congruences applying the method of [18], which uses roots of unity. It also has used the modified Jacobi triple product identity and quintuple product identity as follows:

- Jacobi triple product identity [23]:

$$\sum_{n=-\infty}^{+\infty} (-1)^n \, q^{\binom{n}{2}} \, x^n \;=\; [q, x, q/x; q]_\infty \,. \tag{1}$$

(See also [1, 4, 5, 11, 20, 22, 16, 25].)

- Modified Jacobi triple product identity [21]:

$$[q, zq, q/z; q]_\infty = \sum_{n\geqslant 0} (-1)^n (z^n + z^{n-1} + \cdots + z^{-n}) q^{\binom{n+1}{2}}. \tag{2}$$

- Quintuple product identity [8, 13, 29]:

$$[q, z, q/z; q]_\infty \times \left[qz^2, q/z^2; q^2\right]_\infty = \sum_{n=-\infty}^{+\infty} \left\{1 - z^{1+6n}\right\} q^{3\binom{n}{2}} \left(q^2/z^3\right)^n \tag{3a}$$

$$= \sum_{n=-\infty}^{+\infty} \left\{1 - (q/z^2)^{1+3n}\right\} q^{3\binom{n}{2}} \left(qz^3\right)^n. \tag{3b}$$

- Modified quintuple product identity

$$[q, zq, q/z; q]_\infty \times \left[qz^2, q/z^2; q^2\right]_\infty = \sum_{n=-\infty}^{+\infty} \left\{z^{3n} + z^{3n-1} + \cdots + z^{-3n}\right\} q^{3\binom{n}{2}+2n} \tag{4a}$$

$$\left[q^2, zq, q/z, q^2\right]_\infty \times \left[q^4z^2, q^4/z^2; q^4\right]_\infty = \sum_{n=-\infty}^{+\infty} \left\{z^{3n} + z^{3n-2} + \cdots + z^{-3n}\right\} q^{6\binom{n}{2}+5n} \,. \tag{4b}$$

2 Multiranks and multipartition congruences of $b(n)$

In this section, we give combinatorial interpretations for congruences properties of partition into 4-colors $b(n)$ given by H. Zhao and Z. Zhong [31].

After Andrews and Garvan [2], for a partition λ, we define $\#(\lambda)$ is the number of parts in λ and $\sigma(\lambda)$ is the sum of the parts of λ with the convention $\#(\lambda) = \sigma(\lambda) = 0$ for the empty partition λ. Let $\mathcal{P}$ be the set of all ordinary partitions, $\mathcal{D}$ be the set of all partitions into distinct parts, $\mathcal{O}$ be the set of all partitions into odd parts, $\mathcal{DO}$ be the set of all partitions into distinct odd parts.

We denote

$$C_{1^22^2} = \{(\lambda_1, \lambda_2, 2\lambda_3, 2\lambda_4) | \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathcal{P}\}.$$

For $\lambda \in C_{1^22^2}$, we define the sum of parts s, two multiranks $R_{1^22^2}(\lambda)$ and $R^2_{1^22^2}(\lambda)$ by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + 2\sigma(\lambda_3) + 2\sigma(\lambda_4) \\ R_{1^22^2}(\lambda) &= \#(\lambda_1) - \#(\lambda_2) + \#(\lambda_3) - \#(\lambda_4) \\ R^2_{1^22^2}(\lambda) &= \#(\lambda_1) - \#(\lambda_2) + 3\#(\lambda_3) - 3\#(\lambda_4). \end{aligned}$$

The number of 4-colored partitions of n if $s(\lambda)=n$ having $R_{1^22^2}(\lambda)=m$ will be written as $N_{C_{1^22^2}}(m,n)$, and $N_{C_{1^22^2}}(m,t,n)$ is the number of such 4-colored partitions of n having multirank $R_{1^22^2}$ congruent to $m \pmod t$. The number of 4-colored partitions of n if $s(\lambda)=n$ having $R^2_{1^22^2}(\lambda)=m$ is denoted by $N^2_{C_{1^22^2}}(m,n)$, and $N^2_{C_{1^22^2}}(m,t,n)$ is the number of such 4-colored partitions of n having multirank $R^2_{1^22^2}(\lambda)\equiv m \pmod t$. Now, summing over all 4-colored partitions $\lambda\in C_{1^22^2}$ gives

$$N_{C_{1^22^2}}(m,n)=\sum_{\substack{\lambda\in C_{1^22^2},s(\lambda)=n,\\ R_{1^22^2}(\lambda)=m}}1,\quad N^2_{C_{1^22^2}}(m,n)=\sum_{\substack{\lambda\in C_{1^22^2},s(\lambda)=n,\\ R^2_{1^22^2}(\lambda)=m}}1.$$

Let $T(z,q)$ and $T^2(z,q)$ denote the two variable generating functions for $N_{C_{1^22^2}}(m,n)$ and $N^2_{C_{1^22^2}}(m,n)$,

$$T(z,q):=\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}N_{C_{1^22^2}}(m,n)z^mq^n = \frac{1}{[zq,q/z;q]_\infty\,[zq^2,q^2/z;q^2]_\infty} \tag{5a}$$

$$T^2(z,q):=\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}N^2_{C_{1^22^2}}(m,n)z^mq^n = \frac{1}{[zq,q/z;q]_\infty\,[z^3q^2,q^2/z^3;q^2]_\infty}. \tag{5b}$$

By setting $z=1$ in the identity (5a) and (5b) we get

$$\sum_{m=-\infty}^{\infty}N_{C_{1^22^2}}(m,n)=b(n),\quad \sum_{m=-\infty}^{\infty}N^2_{C_{1^22^2}}(m,n)=b(n).$$

By interchanging λ_3 and λ_4, we can easily obtain

$$\begin{aligned} N_{C_{1^22^2}}(m,n)&=N_{C_{1^22^2}}(-m,n) \quad\text{and}\quad N_{C_{1^22^2}}(m,t,n)=N_{C_{1^22^2}}(t-m,t,n);\\ N^2_{C_{1^22^2}}(m,n)&=N^2_{C_{1^22^2}}(-m,n) \quad\text{and}\quad N^2_{C_{1^22^2}}(m,t,n)=N^2_{C_{1^22^2}}(t-m,t,n).\end{aligned}$$

Denote $N_{C_{1^22^2}}[\cdot]$ by $N_{C_{1^22^2}}[\cdot]:=\sum_{n\geqslant 0}N_{C_{1^22^2}}(\cdot,5,n)q^n$.

Theorem 1. *For $n\geqslant 0$,*

$$N_{C_{1^22^2}}(0,5,5n+4)=N_{C_{1^22^2}}(1,5,5n+4)=N_{C_{1^22^2}}(2,5,5n+4)=\frac{b(5n+4)}{5}.$$

Proof. Putting $z=\zeta$ in (5a), where $\zeta=\exp\frac{2\pi i}{5}$, gives

$$\begin{aligned} T(\zeta,q)&=\frac{1}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty}\\ &=\frac{[q,\zeta^2q,\zeta^{-2}q;q]_\infty\,[q^2,\zeta^2q^2,\zeta^{-2}q^2;q^2]_\infty}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty}.\end{aligned}$$

Using the modified Jacobi triple product identity (2), and the method of Hammond and Lewis [21], we have

$$
\begin{aligned}
T(\zeta,q) &= \left\{\frac{1}{\langle q^5;q^{25}\rangle_\infty}+\frac{(\zeta+\zeta^{-1})q}{\langle q^{10};q^{25}\rangle_\infty}\right\}\left\{\frac{1}{\langle q^{10};q^{50}\rangle_\infty}+\frac{(\zeta+\zeta^{-1})q^2}{\langle q^{20};q^{50}\rangle_\infty}\right\}\\
&= \frac{1}{\langle q^5;q^{25}\rangle_\infty\langle q^{10};q^{50}\rangle}+\frac{q^3}{\langle q^{10};q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}\\
&\quad+(\zeta+\zeta^{-1})\left\{\frac{q}{\langle q^{10};q^{25}\rangle_\infty\langle q^{10};q^{50}\rangle}+\frac{q^2}{\langle q^5;q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}-\frac{q^3}{\langle q^{10};q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}\right\}.
\end{aligned}
$$

While the 5-dissection of $T(\zeta,q)$ is

$$T(\zeta,q)=N_{C_{1^22^2}}[0]-N_{C_{1^22^2}}[2]+(\zeta+\zeta^{-1})(N_{C_{1^22^2}}[1]-N_{C_{1^22^2}}[2]).$$

It follows that

$$
\begin{aligned}
\sum_{n\geqslant 0}\{N_{C_{1^22^2}}(0,5,n)-N_{C_{1^22^2}}(2,5,n)\}q^n &= \frac{1}{\langle q^5;q^{25}\rangle_\infty\langle q^{10};q^{50}\rangle}+\frac{q^3}{\langle q^{10};q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}\\
\sum_{n\geqslant 0}\{N_{C_{1^22^2}}(1,5,n)-N_{C_{1^22^2}}(2,5,n)\}q^n &= \frac{q}{\langle q^{10};q^{25}\rangle_\infty\langle q^{10};q^{50}\rangle}+\frac{q^2}{\langle q^5;q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}\\
&\quad-\frac{q^3}{\langle q^{10};q^{25}\rangle_\infty\langle q^{20};q^{50}\rangle}.
\end{aligned}
$$

No term involving q^{5n+4} appears on the right side of the last identity, we finish the proof of Theorem 1. $\square$

Theorem 2. *For $n\geqslant 0$,*

$$
\begin{aligned}
&N^2_{C_{1^22^2}}(0,7,7n+k)=N^2_{C_{1^22^2}}(1,7,7n+k)=N^2_{C_{1^22^2}}(2,7,7n+k)\\
&=N^2_{C_{1^22^2}}(3,7,7n+k)=\frac{b(7n+k)}{7}\quad \textit{for}\quad k=2,3,4,6.
\end{aligned}
$$

Proof. Replacing z by ϖ in (5b), where $\varpi=\exp\frac{2\pi i}{7}$, gives

$$
\begin{aligned}
T^2(\varpi,q) &= \frac{1}{(\varpi q;q)_\infty(\varpi^{-1}q;q)_\infty(\varpi^3q^2;q^2)_\infty(\varpi^{-3}q^2;q^2)_\infty}\\
&= \frac{[q,\varpi^2q,\varpi^{-2}q;q]_\infty\,[\varpi^4q,\varpi^{-4}q;q^2]_\infty}{(q^7;q^7)_\infty}.
\end{aligned}
$$

Using the modified quintuple product identity (4a), we have

$$T^2(\varpi,q)=\frac{\sum_{-\infty}^{\infty}(\varpi^{6n}+\varpi^{6n-2}+\cdots+\varpi^{-6n})q^{\frac{3n^2+n}{2}}}{(q^7;q^7)_\infty}.$$

Since $\frac{3n^2+n}{2}\equiv 0,1,2,5\pmod 7$, and $\varpi^{6n}+\varpi^{6n-2}+\cdots+\varpi^{-6n}=0$ only when $n\equiv_7 1$, hence the coefficient of q^n on the right side of the last identity is zero when $n\equiv 2\pmod 7$, $n\equiv 3\pmod 7$, $n\equiv 4\pmod 7$, and $n\equiv 6\pmod 7$. The Theorem 2 is completed. $\square$

We illustrate Theorem 1 and Theorem 2 for the case $n = 4$.

λ	4_1	3_2+1_2	2_2+2_4	$2_2+1_1+1_1$	$2_4+1_1+1_2$
$R_{1^22^2}(\lambda)$	1	-2	-2	1	-1
$R^2_{1^22^2}(\lambda)$	1	-2	-4	1	-3
λ	4_2	2_1+2_1	2_3+2_3	$2_2+1_1+1_2$	$2_4+1_2+1_2$
$R_{1^22^2}(\lambda)$	-1	2	2	-1	-3
$R^2_{1^22^2}(\lambda)$	-1	2	6	-1	-5
λ	4_3	2_1+2_2	2_3+2_4	$2_2+1_2+1_2$	$1_1+1_1+1_1+1_1$
$R_{1^22^2}(\lambda)$	1	0	0	-3	4
$R^2_{1^22^2}(\lambda)$	3	0	0	-3	4
λ	4_4	2_1+2_3	2_4+2_4	$2_3+1_1+1_1$	$1_1+1_1+1_1+1_2$
$R_{1^22^2}(\lambda)$	-1	2	-2	3	2
$R^2_{1^22^2}(\lambda)$	-3	4	-6	5	2
λ	3_1+1_1	2_1+2_4	$2_1+1_1+1_1$	$2_3+1_1+1_2$	$1_1+1_1+1_2+1_2$
$R_{1^22^2}(\lambda)$	2	0	3	1	0
$R^2_{1^22^2}(\lambda)$	2	-2	3	3	0
λ	3_1+1_2	2_2+2_2	$2_1+1_1+1_2$	$2_3+1_2+1_2$	$1_1+1_2+1_2+1_2$
$R_{1^22^2}(\lambda)$	0	-2	1	-1	-2
$R^2_{1^22^2}(\lambda)$	0	-2	1	1	-2
λ	3_2+1_1	2_2+2_3	$2_1+1_2+1_2$	$2_4+1_1+1_1$	$1_2+1_2+1_2+1_2$
$R_{1^22^2}(\lambda)$	0	0	-1	1	-4
$R^2_{1^22^2}(\lambda)$	0	2	-1	-1	-4

We have

$$N_{C_{1^22^2}}(0,5,4) = N_{C_{1^22^2}}(1,5,4) = N_{C_{1^22^2}}(2,5,4) = 7$$

$$N^2_{C_{1^22^2}}(0,7,4) = N^2_{C_{1^22^2}}(1,7,4) = N^2_{C_{1^22^2}}(2,7,4) = N^2_{C_{1^22^2}}(3,7,4) = 5.$$

3 Multiranks and multipartitions congruences modulo 3

In this section, we give some statistics that divide the relevant partitions into equinumerous classes and present the combinatorial interpretation for multipartition congruences modulo 3.

Before defining the multiranks, we need to introduce some natation. In Corteel and Lovejoy [14], an overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. Just as Toh [28]: let

- $po(n)$ denote the number of partitions of n into odd parts;
- $pe(n)$ denote the number of partitions of n into even parts;

- $\overline{p}(n)$ denote the number of overpartitions of n;
- $\overline{po}(n)$ denote the number of overpartitions of n into odd parts;
- $\overline{pe}(n)$ denote the number of overpartitions of n into even parts;
- $pod(n)$ denote the number of partitions of n where the odd parts are distinct;
- $ped(n)$ denote the number of partitions of n where the even parts are distinct.

The corresponding generating functions are

$$\sum_{n=0}^{\infty} po(n)q^n = \frac{E_2}{E_1}; \sum_{n=0}^{\infty} pe(n)q^n = \frac{1}{E_2}; \sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{E_2}{E_1^2}; \sum_{n=0}^{\infty} \overline{po}(n)q^n = \frac{E_2^3}{E_1^2 E_4};$$
$$\sum_{n=0}^{\infty} \overline{pe}(n)q^n = \frac{E_4}{E_2^2}; \sum_{n=0}^{\infty} pod(n)q^n = \frac{E_2}{E_1 E_4}; \sum_{n=0}^{\infty} ped(n)q^n = \frac{E_4}{E_1}.$$

And let $Q_{(p_1,p_2)}(n)$ denote the number of partitions of n into two colors (say, red and blue), where the parts colored red satisfy restrictions of partitions counted by $p_1(n)$, while the parts colored blue satisfy restrictions of partitions counted by $p_2(n)$.

If we denote

$$C_{1^3 4^1 2^{-2}} = \{(\lambda_1, 2\lambda_2, \lambda_3, \lambda_4) | \lambda_1 \in \mathcal{D}, \lambda_2 \in \mathcal{DO}, \lambda_3, \lambda_4 \in \mathcal{P}\},$$

then we can say them as partitions into 4-colors. For $\lambda \in C_{1^3 4^1 2^{-2}}$, we define the sum of parts s, a weight $w_{1^3 4^1 2^{-2}}$ and a multirank $R_{1^3 4^1 2^{-2}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + 2\sigma(\lambda_2) + \sigma(\lambda_3) + \sigma(\lambda_4) \\ w_{1^3 4^1 2^{-2}}(\lambda) &= (-1)^{\#(\lambda_2)} \\ R_{1^3 4^1 2^{-2}}(\lambda) &= \#(\lambda_3) - \#(\lambda_4). \end{aligned}$$

Let $N_{C_{1^3 4^1 2^{-2}}}(m, n)$ denote the number of 4-colored partitions of n if $s(\lambda) = n$ (counted according to the weight $w_{1^3 4^1 2^{-2}}$) with multirank m, and $N_{C_{1^3 4^1 2^{-2}}}(m, t, n)$ denote the number of 4-colored partitions of n with multirank congruent to $m \pmod t$, so that

$$N_{C_{1^3 4^1 2^{-2}}}(m, n) = \sum_{\substack{\lambda \in C_{1^3 4^1 2^{-2}}, s(\lambda)=n, \\ R_{1^3 4^1 2^{-2}}(\lambda)=m}} w_{1^3 4^1 2^{-2}}(\lambda).$$

Since

$$R_{1^3 4^1 2^{-2}}(\lambda_1, 2\lambda_2, \lambda_3, \lambda_4) = -R_{1^3 4^1 2^{-2}}(\lambda_1, 2\lambda_2, \lambda_4, \lambda_3),$$

hence

$$N_{C_{1^3 4^1 2^{-2}}}(m, n) = N_{C_{1^3 4^1 2^{-2}}}(-m, n) \quad \text{and} \quad N_{C_{1^3 4^1 2^{-2}}}(m, t, n) = N_{C_{1^3 4^1 2^{-2}}}(t-m, t, n).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^3 4^1 2^{-2}}}(m,n)z^m q^n = \frac{(-q;q)_\infty (q^2;q^4)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{6}$$

By putting $z = 1$ in the identity (6) we find

$$\sum_{m=-\infty}^{\infty} N_{C_{1^3 4^1 2^{-2}}}(m,n) = Q_{(\overline{p},pod)}(n).$$

Suppose ω is primitive $3th$ root of unity. By letting $z = \omega$ in (6) and using Jacobi triple product identity (1), we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^3 4^1 2^{-2}}}(m,n)\omega^m q^n &= \frac{(-q;q)_\infty (q^2;q^4)_\infty}{(\omega q;q)_\infty (\omega^{-1}q;q)_\infty} \\ &= \frac{(q^2;q^2)_\infty (q^2;q^4)_\infty}{(q^3;q^3)_\infty} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}}{(q^3;q^3)_\infty}.\end{aligned}$$

Since the coefficient of q^n on the right side of the last identity is zero when $n \equiv 1 \pmod 3$, and $1 + \omega + \omega^2$ is a minimal polynomial in $\mathbf{Z}[\omega]$, we must have the result following as

Theorem 3. *For $n \geqslant 0$,*

$$N_{C_{1^3 4^1 2^{-2}}}(0,3,3n+1) = N_{C_{1^3 4^1 2^{-2}}}(1,3,3n+1) = N_{C_{1^3 4^1 2^{-2}}}(2,3,3n+1) = \frac{Q_{(\overline{p},pod)}(3n+1)}{3}.$$

It can also prove the identity in Toh [28]: $Q_{(\overline{p},pod)}(3n+1) \equiv 0 \pmod 3$.

Next we define

$$C_{1^3 2^{-3}} = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}\}.$$

We can say them as partitions into 3-colors. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, we still define the sum of parts s and a multirank $R_{1^3 2^{-3}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3) \\ R_{1^3 2^{-3}}(\lambda) &= \#(\lambda_2) - \#(\lambda_3). \end{aligned}$$

Let $N_{C_{1^3 2^{-3}}}(m,n)$ denote the number of 3-colored partitions of n if $s(\lambda) = n$ with multirank m, and $N_{C_{1^3 2^{-3}}}(m,t,n)$ denote the number of 3-colored partitions of n with multirank congruent to $m \pmod t$, so that

$$N_{C_{1^3 2^{-3}}}(m,n) = \sum_{\substack{\lambda\in C_{1^3 2^{-3}}, s(\lambda)=n,\\ R_{1^3 2^{-3}}(\lambda)=m}} 1.$$

Obviously

$$N_{C_{1^3 2^{-3}}}(m,n) = N_{C_{1^3 2^{-3}}}(-m,n) \quad \text{and} \quad N_{C_{1^3 2^{-3}}}(m,t,n) = N_{C_{1^3 2^{-3}}}(t-m,t,n)$$

due to

$$R_{1^32^{-3}}(\lambda_1,\lambda_2,\lambda_3) = -R_{1^32^{-3}}(\lambda_1,\lambda_3,\lambda_2).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^32^{-3}}}(m,n)z^m q^n = \frac{1}{(q;q^2)_\infty (zq;q^2)_\infty (z^{-1}q;q^2)_\infty}. \tag{7}$$

By putting $z=1$ in the identity (7) we obtain

$$\sum_{m=-\infty}^{\infty} N_{C_{1^32^{-3}}}(m,n) = Q_{(\overline{po},ped)}(n).$$

By setting $z=\omega$ in (7), we get

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^32^{-3}}}(m,n)\omega^m q^n = \frac{1}{(q^3;q^6)_\infty}.$$

Since the coefficient of q^n in the q-expansion of $\frac{1}{(q^3;q^6)_\infty}$ is zero when $n\equiv 1 \pmod 3$, we have the following Theorem.

Theorem 4. *For $n\geqslant 0$,*

$$N_{C_{1^32^{-3}}}(0,3,3n+1) = N_{C_{1^32^{-3}}}(1,3,3n+1) = N_{C_{1^32^{-3}}}(2,3,3n+1) = \frac{Q_{(\overline{po},ped)}(3n+1)}{3}.$$

We define

$$C_{1^24^22^{-2}} = \{(\lambda_1,\lambda_2,2\lambda_3,2\lambda_4) | \lambda_1,\lambda_2\in\mathcal{DO}, \lambda_3,\lambda_4\in\mathcal{P}\}.$$

For $\lambda=(\lambda_1,\lambda_2,2\lambda_3,2\lambda_4)$, we denote the sum of parts s, and a multirank $R_{1^24^22^{-2}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+2\sigma(\lambda_3)+2\sigma(\lambda_4)\\ R_{1^24^22^{-2}}(\lambda) &= \#(\lambda_3)-\#(\lambda_4). \end{aligned}$$

Let $N_{C_{1^24^22^{-2}}}(m,n)$ denote the number of 4-colored partitions of n if $s(\lambda)=n$ with multirank m, and $N_{C_{1^24^22^{-2}}}(m,t,n)$ denote the number of 4-colored partitions of n with multirank congruent to $m \pmod t$, so that

$$N_{C_{1^24^22^{-2}}}(m,n) = \sum_{\substack{\lambda\in C_{1^24^22^{-2}}, s(\lambda)=n,\\ R_{1^24^22^{-2}}(\lambda)=m}} 1.$$

Similarly

$$N_{C_{1^24^22^{-2}}}(m,n) = N_{C_{1^24^22^{-2}}}(-m,n) \quad\text{and}\quad N_{C_{1^24^22^{-2}}}(m,t,n) = N_{C_{1^24^22^{-2}}}(t-m,t,n).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^2 4^2 2_2^{-2}}}(m,n) z^m q^n = \frac{(-q;q^2)_\infty^2}{(zq^2;q^2)_\infty (z^{-1}q^2;q^2)_\infty}. \tag{8}$$

By replacing z by 1 in the identity (8) we discover

$$\sum_{m=-\infty}^{\infty} N_{C_{1^2 4^2 2_2^{-2}}}(m,n) = Q_{(pod,pod)}(n).$$

Theorem 5. *For $n \geqslant 0$,*

$$N_{C_{1^2 4^2 2_2^{-2}}}(0,3,3n+2) = N_{C_{1^2 4^2 2_2^{-2}}}(1,3,3n+2) = N_{C_{1^2 4^2 2_2^{-2}}}(2,3,3n+2) = \frac{Q_{(pod,pod)}(3n+2)}{3}.$$

Chen and Lin [10] has proved $Q_{(pod,pod)}(3n+2) \equiv 0 \pmod 3$.

Proof. Putting $z = \omega$ in (8) and utilizing Jacobi triple product identity (1), gives

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^2 4^2 2_2^{-2}}}(m,n) \omega^m q^n = \frac{(-q;q^2)_\infty^2}{(\omega q^2;q^2)_\infty (\omega^{-1}q^2;q^2)_\infty} = \frac{\sum_{n=-\infty}^{\infty} q^{n^2}}{(q^6;q^6)_\infty}.$$

Since the coefficient of q^n on the right side of the last identity is zero when $n \equiv 2 \pmod 3$, we complete the theorem. □

4 Multiranks and multipartitions congruences modulo 5

In this section, we define statistics that divide the relevant partitions into equinumerous classes and provide the combinatorial interpretation for multipartitions congruences modulo 5 given in Toh [28].

First we denote

$$C_{1^2 4^{-1}} = \{(4\lambda_1, \lambda_2, \lambda_3) | \lambda_1 \in \mathcal{D}, \lambda_2, \lambda_3 \in \mathcal{P}\}.$$

It can be said as partitions into 3-colors. For $\lambda = (4\lambda_1, \lambda_2, \lambda_3) \in C_{1^2 4^{-1}}$, we define the sum of parts s, a weight $w_{1^2 4^{-1}}$ and a multirank $R_{1^2 4^{-1}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= 4\sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3) \\ w_{1^2 4^{-1}}(\lambda) &= (-1)^{\#(\lambda_1)} \\ R_{1^2 4^{-1}}(\lambda) &= \#(\lambda_2) - \#(\lambda_3). \end{aligned}$$

Let $N_{C_{1^24^{-1}}}(m,n)$ denote the number of 3-colored partitions of n if $s(\lambda)=n$ (counted according to the weight $w_{1^24^{-1}}$) with multirank m, and $N_{C_{1^24^{-1}}}(m,t,n)$ denote the number of 3-colored partitions of n with multirank congruent to $m \pmod t$, hence

$$N_{C_{1^24^{-1}}}(m,n)=\sum_{\substack{\lambda\in C_{1^24^{-1}},s(\lambda)=n,\\ R_{1^24^{-1}}(\lambda)=m}} w_{1^24^{-1}}(\lambda).$$

By considering the transformation that interchanges λ_2 and λ_3 we get

$$N_{C_{1^24^{-1}}}(m,n)=N_{C_{1^24^{-1}}}(-m,n),\quad N_{C_{1^24^{-1}}}(m,t,n)=N_{C_{1^24^{-1}}}(t-m,t,n).$$

Then we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^24^{-1}}}(m,n)z^mq^n=\frac{(q^4;q^4)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{9}$$

By putting $z=1$ in the identity (9) we check

$$\sum_{m=-\infty}^{\infty} N_{C_{1^24^{-1}}}(m,n)=Q_{(p,ped)}(n).$$

Suppose ζ is primitive $5th$ root of unity. Substituting $z=\zeta$ into (9), we have

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^24^{-1}}}(m,n)\zeta^mq^n=\frac{(q^4;q^4)_\infty}{[\zeta q,q/\zeta;q]_\infty}=\frac{(q^4;q^4)_\infty\,[q,\zeta^2q,q/\zeta^2;q]_\infty}{(q^5;q^5)_\infty}$$
$$=\frac{\sum_{m=-\infty}^{\infty}(-1)^mq^{12\binom{m}{2}+4m}\sum_{n\geqslant 0}(-1)^n(\zeta^{2n}+\zeta^{2n-2}+\zeta^{-2n})q^{\binom{n+1}{2}}}{(q^5;q^5)_\infty}.$$

The last line depends only on classical identities of Jacobi (1) and (2).

If and only if $n\equiv_5 2$, we have $\zeta^{2n}+\zeta^{2n-2}+\cdots+\zeta^{-2n}=0$. Obviously

$$12\binom{m}{2}+4m\equiv_5\begin{cases}0, & m\equiv_5 0,2;\\ 4, & m\equiv_5 1;\\ 3, & m\equiv_5 3,4;\end{cases}\quad\text{and}\quad \binom{n+1}{2}\equiv_5\begin{cases}0, & n\equiv_5 0,4;\\ 1, & n\equiv_5 1,3;\\ 3, & n\equiv_5 2.\end{cases} \tag{10}$$

The power of q is congruent to 2 modulo 5 only when $12\binom{m}{2}+4m\equiv_5 4$ and $\binom{n+1}{2}\equiv_5 3$ in which case $m\equiv_5 1$ and $n\equiv_5 2$ and the coefficient of q^{5n+2} in the last identity is zero. Since $1+\zeta+\zeta^2+\zeta^3+\zeta^4$ is a minimal polynomial in $\mathbf{Z}[\zeta]$, our main result is as follows.

Theorem 6. *For $n\geqslant 0$,*

$$N_{C_{1^24^{-1}}}(0,5,5n+2)=N_{C_{1^24^{-1}}}(1,5,5n+2)=N_{C_{1^24^{-1}}}(2,5,5n+2)=\frac{Q_{(p,ped)}(5n+2)}{5}.$$

Let

$$C_{1^34^12^{-3}} = \{(\lambda_1, \lambda_2, \lambda_3, 4\lambda_4, 4\lambda_5, 4\lambda_6) | \lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}, \lambda_4 \in \mathcal{D}, \lambda_5, \lambda_6 \in \mathcal{P}\}.$$

We call the elements of $C_{1^34^12^{-3}}$ 6-colored partitions. For $\lambda \in C_{1^34^12^{-3}}$, we define the sum of parts s, a weight $w_{1^34^12^{-3}}$ and a multirank $R_{1^34^12^{-3}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3) + 4\sigma(\lambda_4) + 4\sigma(\lambda_5) + 4\sigma(\lambda_6) \\ w_{1^34^12^{-3}}(\lambda) &= (-1)^{\#(\lambda_4)} \\ R_{1^34^12^{-3}}(\lambda) &= \#(\lambda_2) - \#(\lambda_3) + 2\#(\lambda_5) - 2\#(\lambda_6). \end{aligned}$$

Let $N_{C_{1^34^12^{-3}}}(m, n)$ denote the number of 6-colored partitions of n if $s(\lambda) = n$ (counted according to the weight $w_{1^34^12^{-3}}$) with multirank m, and $N_{C_{1^34^12^{-3}}}(m, t, n)$ denote the number of 6-colored partitions of n with multirank $\equiv t \pmod m$, so that

$$N_{C_{1^34^12^{-3}}}(m, n) = \sum_{\substack{\lambda \in C_{1^34^12^{-3}}, s(\lambda)=n, \\ R_{1^34^12^{-3}}(\lambda)=m}} w_{1^34^12^{-3}}(\lambda).$$

By considering the transformation that interchanges λ_2 and λ_3, λ_5 and λ_6 we obtain

$$N_{C_{1^34^12^{-3}}}(m, n) = N_{C_{1^34^12^{-3}}}(-m, n); \quad N_{C_{1^34^12^{-3}}}(m, t, n) = N_{C_{1^34^12^{-3}}}(t - m, t, n).$$

Then the generating function is

$$\sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} N_{C_{1^34^12^{-3}}}(m, n) z^m q^n = \frac{(q^4; q^4)_\infty}{[q, zq, z^{-1}q; q^2]_\infty \, [z^2 q^4, q^4/z^2; q^4]_\infty}. \tag{11}$$

By putting $z = 1$ in the identity (11) we find

$$\sum_{m=-\infty}^{\infty} N_{C_{1^34^12^{-3}}}(m, n) = Q_{(p,\overline{po})}(n).$$

Theorem 7. *For $n \geqslant 0$,*

$$N_{C_{1^34^12^{-3}}}(0, 5, 5n+4) = N_{C_{1^34^12^{-3}}}(1, 5, 5n+4) = N_{C_{1^34^12^{-3}}}(2, 5, 5n+4) = \frac{Q_{(p,\overline{po})}(5n+4)}{5}.$$

Proof. By replacing z by ζ in (11), we write

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} N_{C_{1^34^12^{-3}}}(m, n) \zeta^m q^n &= \frac{(q^4; q^4)_\infty}{[q, \zeta q, \zeta^{-1} q; q^2]_\infty \, [\zeta^2 q^4, \zeta^{-2} q^4; q^4]_\infty} \\ = \frac{1}{(q^5; q^{10})_\infty (q^{20}; q^{20})_\infty} &\times \left[q^2, \zeta^2 q, \zeta^{-2} q; q^2\right]_\infty \left[\zeta^4 q^4, \zeta^{-4} q^4; q^4\right]_\infty \times \frac{(q^4; q^4)_\infty^2}{(q^2; q^2)_\infty}. \end{aligned}$$

Using modified quintuple product identity (4b) and Jacobi (1), the last two infinite products have the following series representation

$$\sum_{n=-\infty}^{\infty} q^{6\binom{n}{2}+5n}\{\zeta^{6n}+\zeta^{6n-4}+\zeta^{-6n}\}\sum_{m=-\infty}^{\infty} q^{8\binom{m}{2}+2m}.$$

If and only if $n\equiv_5 3$, we have $\zeta^{6n}+\zeta^{6n-4}+\cdots+\zeta^{-6n}=0$. Obviously

$$6\binom{n}{2}+5n\equiv_5\begin{cases}0, & n\equiv_5 0,1;\\ 1, & n\equiv_5 2,4;\\ 3, & n\equiv_5 3;\end{cases}\quad\text{and}\quad 8\binom{m}{2}+2m\equiv_5\begin{cases}0, & m\equiv_5 0,3;\\ 2, & m\equiv_5 1,2;\\ 1, & m\equiv_5 4.\end{cases}\tag{12}$$

We see that in the q-expansion on the right side of the last equation the coefficient of q^n is zero when $n\equiv 4 \pmod 5$. The proof of Theorem 7 has been finished. □

We define

$$C_{1^34^22_{-4}}=\{(\lambda_1,\lambda_2,\lambda_3,2\lambda_4,4\lambda_5,4\lambda_6)|\lambda_1,\lambda_2,\lambda_3\in\mathcal{O},\lambda_4\in\mathcal{D},\lambda_5,\lambda_6\in\mathcal{P}\}.$$

We can say them as partitions into 6-colors. For $\lambda\in C_{1^34^22_{-4}}$, we denote the sum of parts s, a weight $w_{1^34^22_{-4}}$ and a multirank $R_{1^34^22_{-4}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+2\sigma(\lambda_4)+4\sigma(\lambda_5)+4\sigma(\lambda_6)\\ w_{1^34^22_{-4}}(\lambda) &= (-1)^{\#(\lambda_4)}\\ R_{1^34^22_{-4}}(\lambda) &= \#(\lambda_2)-\#(\lambda_3)+\#(\lambda_5)-\#(\lambda_6).\end{aligned}$$

The number of 6-colored partitions of n if $s(\lambda)=n$ (counted according to the weight $w_{1^34^22_{-4}}$) with multirank m is denoted by $N_{C_{1^34^22_{-4}}}(m,n)$, so that

$$N_{C_{1^34^22_{-4}}}(m,n)=\sum_{\substack{\lambda\in C_{1^34^22_{-4}},s(\lambda)=n,\\ R_{1^34^22_{-4}}(\lambda)=m}} w_{1^34^22_{-4}}(\lambda).$$

The number of 6-colored partitions of n with multirank congruent to $m \pmod t$ is denoted by $N_{C_{1^34^22_{-4}}}(m,t,n)$. We have

$$N_{C_{1^34^22_{-4}}}(m,n)=N_{C_{1^34^22_{-4}}}(-m,n),\quad N_{C_{1^34^22_{-4}}}(m,t,n)=N_{C_{1^34^22_{-4}}}(t-m,t,n).$$

The following generating function for $N_{C_{1^34^22_{-4}}}(m,n)$:

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^34^22_{-4}}}(m,n)z^mq^n=\frac{(q^2;q^2)_\infty}{[q,zq,z^{-1}q;q^2]_\infty\,[zq^4,q^4/z;q^4]_\infty}.\tag{13}$$

By setting $z=1$ in the identity (13) we find

$$\sum_{m=-\infty}^{\infty} N_{C_{1^34^22_{-4}}}(m,n)=Q_{(\overline{po},pod)}(n).$$

Theorem 8. *For $n \geqslant 0$,*

$$N_{C_{1^3 4^2 2^{-4}}}(0,5,5n+2)=N_{C_{1^3 4^2 2^{-4}}}(1,5,5n+2)=N_{C_{1^3 4^2 2^{-4}}}(2,5,5n+2)=\frac{Q_{(\overline{po},pod)}(5n+2)}{5}.$$

Proof. The proof of Theorem 8 is similar to Theorem 7. By letting $z=\zeta$ in (13), we get

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^3 4^2 2^{-4}}}(m,n)\zeta^m q^n = \frac{(q^2;q^2)_\infty}{[q,\zeta q,\zeta^{-1}q;q^2]_\infty\,[\zeta q^4,\zeta^{-1}q^4;q^4]_\infty}$$
$$= \frac{[q^2,\zeta^2 q,\zeta^{-2}q;q^2]_\infty\,[q^4,\zeta^2 q^4,\zeta^{-2}q^4;q^4]_\infty}{(q^5;q^{10})_\infty (q^{20};q^{20})_\infty}.$$

Using Jacobi triple product identity (1) and (2), the numerator infinite products have the following series expression

$$\sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}\zeta^{2m}\sum_{n\geqslant 0}(-1)^n q^{4\binom{n+1}{2}}\{\zeta^{2n}+\zeta^{2n-2}+\cdots+\zeta^{-2n}\}.$$

Since $m^2\equiv 0,1,4 \pmod 5$, and $4\binom{n+1}{2}\equiv 0,2,4 \pmod 5$, the power of q is congruent to 2 modulo 5 only when $m\equiv_5 0$ and $n\equiv_5 2$, and if and only if $n\equiv_5 2$, we have $\zeta^{2n}+\zeta^{2n-2}+\cdots+\zeta^{-2n}=0$. Therefore no term involving q^{5n+2} appears on the right-hand side of the last equation, we finish the proof of Theorem 8. $\square$

We consider

$$C_{1^1 2^2 4^{-2}} = \{(\lambda_1,\lambda_2,\lambda_3,4\lambda_4,4\lambda_5,2\lambda_6,2\lambda_7)|\lambda_1,\lambda_4,\lambda_5\in\mathcal{D},\lambda_2,\lambda_3,\lambda_6,\lambda_7\in\mathcal{P},\}.$$

We call them as partitions into 7-colors. For $\lambda\in C_{1^1 2^2 4^{-2}}$, we define the sum of parts s, a weight $w_{1^1 2^2 4^{-2}}$ and a multirank $R_{1^1 2^2 4^{-2}}(\lambda)$, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+4\sigma(\lambda_4)+4\sigma(\lambda_5)+2\sigma(\lambda_6)+2\sigma(\lambda_7)\\ w_{1^1 2^2 4^{-2}}(\lambda) &= (-1)^{\#(\lambda_1)+\#(\lambda_4)+\#(\lambda_5)}\\ R_{1^1 2^2 4^{-2}}(\lambda) &= \#(\lambda_2)-\#(\lambda_3)+2\#(\lambda_6)-2\#(\lambda_7). \end{aligned}$$

Let $N_{C_{1^1 2^2 4^{-2}}}(m,n)$ denote the number of 7-colored partitions of n if $s(\lambda)=n$ (counted according to the weight $w_{1^1 2^2 4^{-2}}$) with multirank m, so that

$$N_{C_{1^1 2^2 4^{-2}}}(m,n) = \sum_{\substack{\lambda\in C_{1^1 2^2 4^{-2}},s(\lambda)=n,\\ R_{1^1 2^2 4^{-2}}(\lambda)=m}} w_{1^1 2^2 4^{-2}}(\lambda).$$

The number of 7-colored partitions of n with multirank $\equiv m \pmod t$ is denoted by $N_{C_{1^1 2^2 4^{-2}}}(m,t,n)$. By interchanging λ_2 and λ_3, λ_6 and λ_7, we also have

$$N_{C_{1^1 2^2 4^{-2}}}(m,n)=N_{C_{1^1 2^2 4^{-2}}}(-m,n);\quad N_{C_{1^1 2^2 4^{-2}}}(m,t,n)=N_{C_{1^1 2^2 4^{-2}}}(t-m,t,n).$$

Then the two variable generating function for $N_{C_{1^1 2^2 4^{-2}}}(m,n)$ is

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^1 2^2 4^{-2}}}(m,n) z^m q^n = \frac{(q;q)_\infty (q^4;q^4)_\infty^2}{[zq, z^{-1}q;q]_\infty [z^2q^2, q^2/z^2;q^2]_\infty}. \tag{14}$$

If we simply put $z=1$ in the identity (14), and read off the coefficients of like powers of q, we find

$$\sum_{m=-\infty}^{\infty} N_{C_{1^1 2^2 4^{-2}}}(m,n) = Q_{(\overline{pe},ped)}(n).$$

Putting $z=\zeta$ in (14) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_{C_{1^1 2^2 4^{-2}}}(m,n)\zeta^m q^n = \frac{(q;q)_\infty (q^4;q^4)_\infty^2}{[\zeta q, \zeta^{-1}q;q]_\infty [\zeta^2q^2, \zeta^{-2}q^2;q^2]_\infty}$$
$$= \frac{[q^2, \zeta^2 q, \zeta^{-2}q;q^2]_\infty \times [q^2,q,q;q^2]_\infty [q^4,q^4;q^4]_\infty}{(q^5;q^5)_\infty}.$$

Using Jacobi triple product identity (1) and Entry 8(x) of [5] P.114, the numerator infinite products have the following series expression

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} \zeta^{2m} \sum_{n=-\infty}^{\infty} (1+3n) q^{6\binom{n}{2}+5n}.$$

If and only if $n \equiv_5 3$, we have $1+3n \equiv_5 0$. Since the coefficient of q^n on the right side of the last identity is $5N$ when $n \equiv_5 3$, and $1+\zeta+\zeta^2+\zeta^3+\zeta^4$ is a minimal polynomial in $\mathbf{Z}[\zeta]$, our main result follows:

Theorem 9. *For $n \geqslant 0$,*

$$N_{C_{1^1 2^2 4^{-2}}}(0,5,5n+3) \equiv_5 N_{C_{1^1 2^2 4^{-2}}}(1,5,5n+3) \equiv_5 \cdots \equiv_5 N_{C_{1^1 2^2 4^{-2}}}(4,5,5n+3).$$

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