On short zero-sum subsequences of zero-sum sequences

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Abstract

Let G be a finite abelian group of exponent $\exp(G)$. By $D(G)$ we denote the smallest integer $d \in \mathbb{N}$ such that every sequence over G of length at least d contains a nonempty zero-sum subsequence. By $\eta(G)$ we denote the smallest integer $d \in \mathbb{N}$ such that every sequence over G of length at least d contains a zero-sum subsequence T with length $|T| \in [1, \exp(G)]$, such a sequence T will be called a short zero-sum sequence. Let $C_0(G)$ denote the set consists of all integer $t \in [D(G)+1, \eta(G)-1]$ such that every zero-sum sequence of length exactly t contains a short zero-sum subsequence. In this paper, we investigate the question whether $C_0(G) \neq \emptyset$ for all non-cyclic finite abelian groups G. Previous results showed that $C_0(G) \neq \emptyset$ for the groups C_n^2 ($n \geqslant 3$) and C_3^3. We show that more groups including the groups $C_m \oplus C_n$ with $3 \leqslant m \mid n$, $C^3_{3^a5^b}$, $C^3_{3\times 2^a}$, $C^4_{3^a}$ and $C^r_{2^b}$ ($b \geqslant 2$) have this property. We

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also determine $C_0(G)$ completely for some groups including the groups of rank two, and some special groups with large exponent.

Keywords: Zero-sum sequence; short zero-sum sequence; short free sequence; zero-sum short free sequence; Davenport constant

1 Introduction

Let G be an additive finite abelian group of exponent $\exp(G)$. We call a zero-sum sequence S over G a *short zero-sum sequence* if $1 \leqslant |S| \leqslant \exp(G)$. Let $\eta(G)$ be the smallest integer d such that every sequence S over G of length $|S| \geqslant d$ contains a short zero-sum subsequence. Let $D(G)$ be the Davenport constant of G, i.e., the smallest integer d such that every sequence over G of length at least d contains a nonempty zero-sum subsequence. Both $D(G)$ and $\eta(G)$ are classical invariants in combinatorial number theory. For detail on terminology and notation we refer to Section 2.

By the definition of $\eta(G)$ we know that for every integer $t \in [1, \eta(G) - 1]$, there is a sequence S over G of length exactly t such that S contains no short zero-sum subsequence. In this paper, we consider the following problem related to $D(G)$ and $\eta(G)$, which was first investigated by Emde Boas in the late sixties. Given a finite abelian group, what are integers $\exp(G) + 1 \leqslant t \leqslant \eta(G) - 1$, if any, such that every zero-sum sequence S over G of length $|S| = t$ contains a short zero-sum subsequence. Denote by $C_0(G)$ the set of all those integers t. It will be readily seen that $C_0(G) \subset [D(G) + 1, \eta(G) - 1]$.

In 1969, Emde Boas and D. Kruyswijk [7] proved that $14 \in C_0(C_3^3)$. In 1997, the second author of this paper showed that $[2q, 3q - 3] \subset C_0(C_q^2)$, where q is a prime power.

Let us first make some easy observations on $C_0(G)$. Note that for every $t \in [1, D(G)]$ there exists a minimal zero-sum sequence over G of length t. So, $C_0(G) \subset [D(G) + 1, \eta(G) - 1]$ follows from the easy fact that $D(G) \geqslant \exp(G)$.

If $G = C_2 \oplus C_{2m}$ then $D(G) + 1 = 2m + 2$ and $\eta(G) - 1 = 2m + 1$. Therefore, by the definition we have $C_0(C_2 \oplus C_{2m}) = \emptyset$. We suggest the following

Conjecture 1. *Let G be a non-cyclic finite abelian group. If $G \neq C_2 \oplus C_{2m}$ then $C_0(G) \neq \emptyset$.*

In this paper we shall determine $C_0(G)$ completely for some groups.

Theorem 2. *Let G be a non-cyclic finite abelian group, and let $r(G)$ be the rank of G. Then,*

1. $C_0(G) = [D(G) + 1, \eta(G) - 1]$ *if* $r(G) = 2$.
2. $C_0(G) = [D(G) + 1, \eta(G) - 1]$ *if* $G = C_{mp^n} \oplus H$ *with p a prime, H a p-group and* $p^n \geqslant D(H)$.
3. $C_0(C_3^4) = \{\eta(C_3^4) - 2, \eta(C_3^4) - 1\} = \{37, 38\}$.

4. $C_0(C_2^r) = \{\eta(C_2^r) - 3, \eta(C_2^r) - 2\}$*, where* $r \geqslant 3$.

We also confirm Conjecture 1 for more groups other than those listed in Theorem 2.

Theorem 3. *If G is one of the following groups then $C_0(G) \neq \emptyset$.*

1. $G = C^3_{3^a5^b}$ *where* $a \geqslant 1$ *or* $b \geqslant 2$.
2. $G = C^3_{3\times 2^a}$ *where* $a \geqslant 4$.
3. $G = C^4_{3^a}$ *where* $a \geqslant 1$.
4. $G = C^r_{2^a}$ *where* $3 \leqslant r \leqslant a$*, or* $a = 1$ *and* $r \geqslant 3$.
5. $G = C^3_k$ *where* $k = 3^{n_1}5^{n_2}7^{n_3}11^{n_4}13^{n_5}$*,* $n_1 \geqslant 1$*,* $n_3 + n_4 + n_5 \geqslant 3$*, and* $n_1 + n_2 \geqslant 11 + 34(n_3 + n_4 + n_5)$.

The rest of this paper is organized as follows: In Section 2 we introduce some notations and some preliminary results; In Section 3 we prove three lemmas connecting $C_0(G)$ with property C; In Section 4 we shall derive some lower bounds on $\min\{C_0(G)\}$; In Section 5 we study $C_0(G)$ with focus on the groups C_3^r; In Section 6 and 7 we shall prove Theorem 2 and Theorem 3, respectively; and in the final Section 8 we give some concluding remarks and some open problems.

2 Notations and some preliminary results

Our notations and terminologies are consistent with [10] and [13]. We briefly gather some key notions and fix the notations concerning sequences over finite abelian groups. Let $\mathbb{Z}$ denote the set of integers. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\}$. Throughout this paper, all abelian groups will be written additively, and for $n, r \in \mathbb{N}$, we denote by C_n a cyclic group with n elements, and denote by C_n^r the direct sum of r copies of C_n.

Let G be a finite abelian group and $\exp(G)$ its exponent. By $r(G)$ we denote the rank of G. A sequence S over G will be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g\in G} g^{\mathsf{v}_g(S)}, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G,$$

and we call

$$|S| = \ell \in \mathbb{N}_0 \quad \text{the } \textit{length} \text{ and} \quad \sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G \quad \text{the } \textit{sum} \text{ of } S.$$

Let $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$. We call S a *square free sequence* if $\mathsf{v}_g(S) \leqslant 1$ for every $g \in G$. So, a square free sequence over G is actually a subset of G. A sequence T

over G is called a *subsequence* of S if $v_g(T) \leqslant v_g(S)$ for every $g \in G$, and denote by $T|S$. For every $r \in [1, \ell]$, define

$$\sum_{\leqslant r}(S) = \{\sigma(T) :\ T \mid S,\ 1 \leqslant |T| \leqslant r\}$$

and define

$$\sum(S) = \{\sigma(T) :\ T \mid S,\ |T| \geqslant 1\}.$$

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$.
- a *short zero-sum sequence* over G if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.
- a *short free* sequence over G if S contains no short zero-sum subsequence.

So, a zero-sum sequence over G which contains no short zero-sum subsequence will be called a zero-sum short free sequence over G.

For every element $g \in G$, we set $g + S = (g + g_1) \cdot \ldots \cdot (g + g_\ell)$. Every map of abelian groups $\varphi : G \to H$ extents to a map from the sequences over G to the sequences over H by $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_\ell)$. If φ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

In the rest of this section we gather some known results which will be used in the sequel.

We shall study $C_0(G)$ by using the following property which was first introduced and investigated by Emde Boas and Kruyswijk [7] in 1969 for the groups C_p^2 with p a prime, and was investigated in 2007 for the groups C_n^r by the second author, Geroldinger and Schmid [12].

Property C: *We say the group C_n^r has property C if $\eta(C_n^r) = c(n-1) + 1$ for some positive integer c, and every short free sequence S over C_n^r of length $|S| = c(n-1)$ has the form $S = \prod_{i=1}^{c} g_i^{n-1}$ where $g_1, \ldots, g_c$ are pairwise distinct elements of C_n^r.*

It is conjectured that every group of the form C_n^r has Property C(see [10], Section 7). We need the following result which states that Property C is multiple.

Lemma 4. *([12], Theorem 3.2) Let $G = C_{mn}^r$ with $m, n, r \in \mathbb{N}$. If both C_m^r and C_n^r have Property C and*

$$\frac{\eta(C_m^r) - 1}{m - 1} = \frac{\eta(C_n^r) - 1}{n - 1} = \frac{\eta(C_{mn}^r) - 1}{mn - 1} = c$$

for some $c \in \mathbb{N}$ then G has Property C.

We also need the following old easy result.

Lemma 5. *([20])* $D(C_n^3) \geqslant 3n - 2$.

Definition 6. Let G be a finite abelian group. Define $g(G)$ to be the smallest integer t such that every square free sequence over G of length t contains a zero-sum subsequence of length $\exp(G)$. Let $f(G)$ be the smallest integer t such that every square free sequence over G of length t contains a short zero-sum subsequence.

We now gather some known results on Property C, $\eta(G), g(G)$ and $f(G)$ which will be used in the sequel.

Lemma 7. *Let $r, t \in \mathbb{N}$, and let $n \geqslant 3$ be an odd integer. Then,*

1. $\eta(C_n^3) \geqslant 8n - 7$. *([6], or [5], Theorem 1.2)*
2. $\eta(C_n^4) \geqslant 19n - 18$. *([5], Theorem 1.3)*
3. $\eta(C_3^3) = 17$. *([19], or [6], page 3)*
4. $\eta(C_3^4) = 39$ *and* $g(C_3^4) = 21$. *([19], or [6], page 3)*
5. $\eta(C_5^3) = 33 = 8 \times 5 - 7$. *([11], Theorem 1.7)*
6. $\eta(C_{2^t}^r) = (2^r - 1)(2^t - 1) + 1$. *([18])*
7. $\eta(C_{3\times 2^\alpha}^3) = 7(3 \times 2^\alpha - 1) + 1$ *where* $\alpha \geqslant 1$. *([11], Theorem 1.8)*
8. C_5^3 *has Property C. ([11], Theorem 1.9)*
9. $\eta(C_3^r) = 2f(C_3^r) - 1$. *([18])*
10. C_3^r *has Property C. ([18])*

Lemma 8. *([5], Lemma 5.4) Let $r \in [3, 5]$, and let S and S' be two square free sequences over C_3^r of length $|S| = |S'| = g(C_3^r) - 1$. Suppose that both S and S' contain no zero-sum subsequence of length 3. Then $S' = \varphi(S) + a$, where φ is an automorphism of C_3^r and $a \in C_3^r$.*

Lemma 9. *([1], Lemma 1) Let T be a square free sequence over C_3^3 of length 8. If T contains no short zero-sum subsequence then there exists an automorphism φ of C_3^3 such that*

$$\varphi(T) = \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\2\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\1\\2\end{pmatrix} \begin{pmatrix}2\\0\\1\end{pmatrix}.$$

Lemma 10. *([3]; [5], page 182) The following square free sequence over C_3^4 of length 20 contains no zero-sum subsequence of length 3.*

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\2\\0\\0\end{pmatrix}\begin{pmatrix}2\\2\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\2\\0\end{pmatrix}\begin{pmatrix}0\\1\\2\\0\end{pmatrix}\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\begin{pmatrix}2\\1\\2\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\2\\2\end{pmatrix}\begin{pmatrix}2\\0\\2\\2\end{pmatrix}\begin{pmatrix}0\\2\\2\\2\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix}\begin{pmatrix}1\\0\\0\\2\end{pmatrix}\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\begin{pmatrix}1\\2\\0\\2\end{pmatrix}\begin{pmatrix}2\\1\\0\\2\end{pmatrix}\begin{pmatrix}1\\1\\1\\2\end{pmatrix}\begin{pmatrix}1\\1\\2\\1\end{pmatrix}.$$

Lemma 11. *([15], Theorem 5.2) Every sequence S over C_n^2 of length $|S| = 3n-2$ contains a zero-sum subsequence of length n or $2n$.*

Lemma 12. *([12], Lemma 4.5) Let G be a finite abelian group, and let H be a proper subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Then $\eta(G) \leqslant (\eta(H)-1)\exp(G/H) + \eta(G/H)$.*

Lemma 13. *Let p be a prime and let H be a finite abelian p-group such that $p^n \geqslant D(H)$. Let $n_1, n_2, m, n \in \mathbb{N}$ with $n_1 \mid n_2$. Then,*

1. *$D(C_{n_1} \oplus C_{n_2}) = n_1 + n_2 - 1$. ([20])*
2. *$D(C_{mp^n} \oplus H) = mp^n + D(H) - 1$. ([7])*
3. *Let $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_r}}$ with $e_i \in \mathbb{N}$. Then, $D(G) = 1 + \sum_{i=1}^r (p^{e_i} - 1)$. ([20])*
4. *$\eta(C_{n_1} \oplus C_{n_2}) = 2n_1 + n_2 - 2$. ([14])*
5. *Let $G = H \oplus C_n$ with $\exp(H) \mid n \geqslant 2$. Then, $\eta(G) \geqslant 2(D(H)-1) + n$. ([5])*

We also need the following easy lemma

Lemma 14. *([16] Lemma 4.2.2) Let G be a finite abelian group. Then, $\mathsf{s}(G) \geqslant \eta(G) + \exp(G) - 1$.*

We shall show that the following property can also be used to study $C_0(G)$.

Property D_0**:** *Let $c, n \in \mathbb{N}$. We say that C_n^r has property D_0 with respect to c if every sequence of the form $g\prod_{i=1}^c g_i^{n-1}$ contains a zero-sum subsequence of length exactly n, where $g, g_i \in C_n^r$ for all $i \in [1, c]$.*

Lemma 15. *([8], page 8) Let $m = 3^a5^b$ with a, b nonnegative integers. Let $n \geqslant 65$ be an odd positive integer such that C_p^3 has Property D_0 with respect to 9 for all prime divisors p of n. If*

$$m \geqslant \frac{2 \times 5^7 n^{17}}{(n^2-7)n - 64}$$

then $\mathsf{s}(C_{mn}^3) = 9mn - 8$.

3 Three lemmas connecting $C_0(G)$ with Property C

Lemma 16. *Let $G = C_n^r$ with $\eta(G) = c(n-1)+1$ for some $c \in \mathbb{N}$. If $c \leqslant n$ and if G has Property C then $\eta(G) - 1 \in C_0(G)$.*

Proof. Let S be a zero-sum sequence over G of length $|S| = \eta(G) - 1 = c(n-1)$. We need to show that S contains a short zero-sum subsequence. If $S = \prod_{i=1}^{c} g_i^{n-1}$ for some $g_i \in G$, then $(n-1)(g_1 + g_2 + \cdots + g_c) = \sigma(S) = 0 = n(g_1 + g_2 + \cdots + g_c)$. It follows that $g_1 + g_2 + \cdots + g_c = 0$. Therefore, $g_1 g_2 \cdot \ldots \cdot g_c$ is a zero-sum subsequence of S of length $c \leqslant n$ and we are done. Otherwise, $S \neq \prod_{i=1}^{c} g_i^{n-1}$ for any $g_i \in G$. It follows from G having Property C that S contains a short zero-sum subsequence. □

Lemma 17. *Let G be a finite abelian group, and let H be a proper subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Suppose that the following conditions hold.*

(i) $\eta(G) = (\eta(H) - 1)\exp(G/H) + \eta(G/H)$;

(ii) $G/H \cong C_n^r$ *has Property C;*

(iii) *There exist $t_1 \in [1, \exp(G/H) - 1]$ and $t_2 \in \{1, 2\}$ such that $t_2 \leqslant t_1$ and such that $[\eta(G/H) - t_1, \eta(G/H) - t_2] \subset C_0(G/H)$.*

Then,

$$[\eta(G) - t_1, \eta(G) - t_2] \subset C_0(G).$$

Proof. To prove this lemma, we assume to the contrary that there is a zero-sum short free sequence S over G of length $\eta(G) - t$ for some $t \in [t_2, t_1]$. Let φ be the natural homomorphism from G onto G/H.

Note that

$$|S| = \eta(G) - t = (\eta(H) - 1)\exp(G/H) + (\eta(G/H) - t). \tag{3.1}$$

This allows us to take an arbitrary decomposition of S

$$S = \left(\prod_{i=1}^{\eta(H)-1} S_i \right) \cdot S' \tag{3.2}$$

with

$$|S_i| \in [1, \exp(G/H)] \tag{3.3}$$

and

$$\sigma(S_i) \in \ker(\varphi) = H \tag{3.4}$$

for every $i \in [1, \eta(H) - 1]$.

Combining (3.1), (3.2), (3.3) and (3.4) we infer that

$$|S'| \geqslant \eta(G/H) - t \geqslant \eta(G/H) - t_1 \tag{3.5}$$

and

$$\sigma\big(\varphi(S')\big) = 0. \tag{3.6}$$

Claim. $\varphi(S')$ contains no zero-sum subsequence of length in $[1, \exp(G/H)]$.
Proof of the claim. Assume to the contrary that, there exists a subsequence $S_{\eta(H)}$ (say) of S' of length $|S_{\eta(H)}| \in [1, \exp(G/H)]$ such that $\sigma(S_{\eta(H)}) \in \ker(\varphi) = H$. It follows that the sequence $U = \prod\limits_{i=1}^{\eta(H)} \sigma(S_i)$ contains a zero-sum subsequence $W = \prod_{i\in I} \sigma(S_i)$ over H with $I \subset [1, \eta(H)]$ and $|W| = |I| \in [1, \exp(H)]$. Therefore, the sequence $\prod_{i\in I} S_i$ is a zero-sum subsequence of S over G with $1 \leqslant |\prod_{i\in I} S_i| \leqslant |I| \exp(G/H) \leqslant \exp(H) \exp(G/H) = \exp(G)$, a contradiction. This proves the claim.

By (3.5), (3.6), the above claim and Condition (iii), we conclude that

$$t_2 = 2$$

and

$$|S'| = \eta(G/H) - 1. \tag{3.7}$$

This together with Condition (ii) implies that

$$\varphi(S') = x_1^{n-1} \cdot \ldots \cdot x_c^{n-1} \tag{3.8}$$

where $c = \frac{\eta(G/H)-1}{n-1}$ and $x_1, \ldots, x_c$ are pairwise distinct elements of the quotient group G/H. So, we just proved that every decomposition of S satisfying conditions (3.3) and (3.4) has the properties (3.5)-(3.8).

Since $t \leqslant t_1 \leqslant \exp(G/H) - 1$, it follows from (3.1), (3.3) and (3.7) that $|S_i| \in [2, \exp(G/H)]$ for all $i \in [1, \eta(H) - 1]$. Moreover, since $t \geqslant t_2 = 2$, it follows that there exists $j \in [1, \eta(H) - 1]$ such that $|S_j| \leqslant \exp(G/H) - 1$. Without loss of generality we assume that

$$|S_1| \in [2, \exp(G/H) - 1].$$

Suppose that there exists $h \in \mathrm{supp}(\varphi(S_1)) \cap \mathrm{supp}(\varphi(S'))$. By (3.8), we have that the sequence $S_1 \cdot S'$ contains a subsequence S_1' with $\varphi(S_1') = h^n$. Let $S'' = S_1 \cdot S' \cdot S_1'^{-1}$. We get a decomposition $S = S_1' \cdot \left(\prod\limits_{i=2}^{\eta(H)-1} S_i\right) \cdot S''$ satisfying (3.3) and (3.4). But $|S''| = |S_1| + |S'| - |S_1'| \leqslant (n-1) + (\eta(G/H) - 1) - n = \eta(G/H) - 2$, a contradiction on (3.7). Therefore,

$$\mathrm{supp}(\varphi(S_1)) \cap \mathrm{supp}(\varphi(S')) = \emptyset.$$

Take a term $g \mid S_1$. Since $\varphi(g) \notin \mathrm{supp}(\varphi(S'))$ and $|S' \cdot g| = \eta(G/H)$, it follows from the above claim that $S' \cdot g$ contains a subsequence S_1' with

$$g \mid S_1' \tag{3.9}$$

and

$$|S_1'| \leqslant \exp(G/H) \tag{3.10}$$

and

$$\sigma(S_1') \in \ker(\varphi). \tag{3.11}$$

Let $S'' = S_1 \cdot S' \cdot S_1'^{-1}$. By (3.8), (3.9), (3.10) and (3.11), we conclude that $|\mathrm{supp}(\varphi(S''))| \geqslant c+1$, a contradiction with (3.8). This proves the lemma. $\square$

From Lemma 17, we immediately obtain the following

Lemma 18. *Let $r \in \mathbb{N}$, and let $G_1 = C_{n_1}^r$, $G_2 = C_{n_2}^r$ and $G = C_{n_1 n_2}^r$. Suppose that the following conditions hold.*

(i) $\frac{\eta(G_1)-1}{n_1-1} = \frac{\eta(G_2)-1}{n_2-1} = \frac{\eta(G)-1}{n_1 n_2 - 1} = c$ *for some* $c \in \mathbb{N}$*;*

(ii) G_2 *has Property C;*

(iii) *There exist $t_1 \in [1, n_2 - 1]$, $t_2 \in \{1, 2\}$ such that $t_2 \leqslant t_1$ and such that $[\eta(G_2) - t_1, \eta(G_2) - t_2] \subset C_0(G_2)$.*

Then,

$$[\eta(G) - t_1, \eta(G) - t_2] \subset C_0(G).$$

4 Some lower bounds on $\min\{C_0(G)\}$

In this section we shall prove the following

Proposition 19. *Let $G = C_n^r$ with $n \geqslant 3, r \geqslant 3$, and let $\alpha_r \equiv -2^{r-1} \pmod n$ with $\alpha_r \in [0, n-1]$. Then,*

1. $C_0(G) \subset [(2^r - 1)(n-1) - \alpha_r + 1, \eta(G) - 1]$ *if* $\alpha_r \neq 0$.
2. $C_0(G) \subset \{(2^r - 1)(n-1) - n, (2^r - 1)(n-1) - n + 1\}$ *if* $\alpha_r = 0$.

Note that $\alpha_r \neq 0$ if and only if $n \neq 2^k$, or $n = 2^k$ and $r - 1 < k$; and $\alpha_r = 0$ if and only if $n = 2^k$ and $k \leqslant r - 1$.

For every $r \in \mathbb{N}$, let

$$G = C_n^r = < e_1 > \oplus \cdots \oplus < e_r >$$

with $< e_i > = C_n$ for every $i \in [1, r]$, and let

$$S_r = \prod_{\emptyset \neq I \subset [1,r]} \left(\sum_{i \in I} e_i \right)^{n-1}.$$

We can regard C_n^r as a subgroup of C_n^{r+1} and therefore S_{r+1} has the following decomposition

$$S_{r+1} = S_r (S_r + e_{r+1}) e_{r+1}^{n-1}.$$

Since the proof of Proposition 19 is somewhat long, we split the proof into lemmas begin with the following easy one

Lemma 20. *S_r is a short free sequence over C_n^r of length $|S_r| = (2^r - 1)(n-1)$ and of sum $\sigma(S_r) = -2^{r-1}(e_1 + \cdots + e_r) = \alpha_r (e_1 + \cdots + e_r)$.*

Proof. Obviously. $\square$

Lemma 21. *Let $G = C_n^r$ with $r \geqslant 2$. Then for every $m \in [1, n-1]$ and every $i \in [1, r]$, the sequence $S_r(e_i^m)^{-1}(me_i)$ contains no short zero-sum subsequence.*

Proof. Without loss of generality, we assume that $i = r$.

Assume to the contrary that $S_r(e_r^m)^{-1}(me_r)$ contains a short zero-sum subsequence U. Since S_r contains no short zero-sum subsequence we infer that $me_r \mid U$. Therefore, $U = (me_r)U_0(U_1 + e_r)e_r^k$ with $U_0 \mid S_{r-1}$ and $U_1 \mid S_{r-1}$ and $k \in [0, n-1-m]$. It follows that U_0U_1 is zero-sum and $1 \leqslant |U_0U_1| \leqslant n-1$. Since every element in $\mathrm{supp}(S_{r-1})$ occurs $n-1$ times in S_{r-1}, it follows from $|U_0U_1| \leqslant n-1$ that $U_0U_1 \mid S_{r-1}$. Therefore, U_0U_1 is a short zero-sum subsequence of S_{r-1}, a contradiction with Lemma 20. $\square$

Let A be a set of zero-sum sequences over G. Define

$$\mathcal{L}(A) = \{|T| : T \in A\}.$$

In this section below we shall frequently use the following easy observation.

Lemma 22. *Let G be a finite abelian group, and let $a, b \in \mathbb{N}$ with $a \leqslant b$. If there exists a set A of zero-sum short free sequences over G such that $[a, b] \subset \mathcal{L}(A)$, then $C_0(G) \cap [a, b] = \emptyset$.*

Proof. It immediately follows from the definition of $C_0(G)$. $\square$

Lemma 23. *Let $G = C_n^r$ with $n, r \geqslant 3$. Then,*

1. $C_0(G) \cap [|S_r| - (3n-3) - \alpha_r, |S_r| - \alpha_r] = \emptyset$ *if* $\alpha_r \neq 0$.
2. $C_0(G) \cap [|S_r| - (3n-3), |S_r| - (n+1)] = \emptyset$ *if* $\alpha_r = 0$.

Proof. Recall that $|S_r| = (2^r - 1)(n-1)$. We split the proof into three steps.

Step 1. In this step we shall prove that

$$C_0(G) \cap [|S_r| - (3n-3) - \alpha_r, |S_r| - (n+1) - \alpha_r] = \emptyset$$

no matter $\alpha_r = 0$ or not.

Let

$$A = \big\{S_r\big((e_1 + \cdots + e_r)^{\alpha_r} W e_3^m\big)^{-1}(me_3) : W \mid S_2, \sigma(W) = 0, m \in [1, n-1]\big\}.$$

It follows from Lemma 21 that every sequence in A is zero-sum short free.

Since $\mathcal{L}(\{W : W \mid S_2, \sigma(W) = 0\}) = [n+1, 2n-1]$, we conclude easily that

$$\mathcal{L}(A) = [|S_r| - (3n-3) - \alpha_r, |S_r| - (n+1) - \alpha_r].$$

Now the result follows from Lemma 22 and Conclusion 2 follows.

Step 2. We show that $C_0(G) \cap [|S_r| - (n + \alpha_r), |S_r| - (r-1)\alpha_r] = \emptyset$ for $\alpha_r \neq 0$.

Let

$$A_1 = \big\{ S_r\big((e_1+e_2)^{\alpha_r} e_3^{\alpha_r} \cdot \ldots \cdot e_r^{\alpha_r} e_1^m\big)^{-1}(me_1) : m \in [1, n-1] \big\}$$

and

$$A_2 = \big\{ S_r\big((e_1+e_2)^{\alpha_r}(e_1+e_3) e_3^{\alpha_r -1} e_4^{\alpha_r} \cdot \ldots \cdot e_r^{\alpha_r} e_1^{n-1}\big)^{-1} \big\}.$$

It is easy to see that every sequence in $A_1 \cup A_2$ is zero-sum short free by Lemma 21 and Lemma 20. Note that

$$\begin{aligned}\mathcal{L}(A_1) \cup \mathcal{L}(A_2) &= [|S_r| - (r-1)\alpha_r - n + 2, |S_r| - (r-1)\alpha_r] \cup \{|S_r| - (r-1)\alpha_r - n + 1\} \\ &= [|S_r| - (r-1)\alpha_r - n + 1, |S_r| - (r-1)\alpha_r].\end{aligned}$$

Since $r \geqslant 3$, we have $|S_r| - (r-1)\alpha_r - n + 1 \leqslant |S_r| - (n + \alpha_r)$. Therefore, $\mathcal{L}(A_1 \cup A_2) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2) \supset [|S_r| - (n+\alpha_r), |S_r| - (r-1)\alpha_r]$. Again the result follows from Lemma 22.

Step 3. We prove $C_0(G) \cap [|S_r| - (r-1)\alpha_r, |S_r| - \alpha_r] = \emptyset$ for $\alpha_r \neq 0$.
Let

$$\begin{aligned}A = \big\{ S_r\big((e_1 + \cdots + e_r)^{k_1}(e_1 \cdot \ldots \cdot e_r)^{k_2}(e_1 + \cdots + e_{k_3}) e_{k_3+1} \cdot \ldots \cdot e_r\big)^{-1} : \\ k_1 \in [0, \alpha_r - 1], k_2 \in [0, \alpha_r - 1], k_1 + k_2 = \alpha_r - 1, k_3 \in [1, r] \big\}.\end{aligned}$$

Then every sequence in A is zero-sum short free by Lemma 21 and by Lemma 20, and

$$\begin{aligned}&\mathcal{L}(A) \\ &\quad = \{|S_r| - k_1 - rk_2 - 1 - (r - k_3) : k_1 + k_2 = \alpha_r - 1, k_2 \in [0, \alpha_r - 1], k_3 \in [1, r]\} \\ &\quad = \{|S_r| - \alpha_r - ((r-1)k_2 + (r - k_3)) : k_2 \in [0, \alpha_r - 1], k_3 \in [1, r]\} \\ &\quad = [|S_r| - r\alpha_r, |S_r| - \alpha_r].\end{aligned}$$

Now the result follows from Lemma 22 and the proof is completed. □

Lemma 24. *Let $n, r \in \mathbb{N}$ with $n \geqslant 3$ and $r \geqslant 3$, and let $G = C_n^r$. If $\alpha_r \neq 0$ then $C_0(G) \subset [(2^r - 1)(n-1) - \alpha_r + 1, \eta(G) - 1]$.*

Proof. It suffices to show that $C_0(G) \cap [n+1, |S_r| - \alpha_r] = \emptyset$.
We proceed by induction on r. Suppose first that $r = 3$.
By Lemma 23 and the definition of $C_0(C_n^3)$, we only need to prove

$$C_0(G) \cap [D(C_n^3) + 1, |S_3| - (3n-3) - \alpha_3 - 1] = \emptyset.$$

By Lemma 5 we have $D(C_n^3) + 1 \geqslant 3n - 1$. So, it suffices to prove that

$$C_0(G) \cap [3n-1, |S_3| - (3n-3) - \alpha_3 - 1] = C_0(G) \cap [3n-1, 4n-4-\alpha_3-1] = \emptyset.$$

If $n = 3$, then $[3n-1, 4n-4-\alpha_3-1] = \emptyset$ and the result follows.

Now assume $n \geqslant 4$. It follows from $\alpha_3 \neq 0$ that $n \geqslant 5$. Thus, $\alpha_3 = n-4$ and $[3n-1, 4n-4-\alpha_3-1] = \{3n-1\}$.

Let $T = (e_1+e_2)^2(e_1+e_3)^{n-1}e_1^{n-1}e_2^{n-2}e_3$. Then T is zero-sum short free over C_n^3 of length $|T| = 3n-1$. Now the result follows from Lemma 22. This completes the proof for $r = 3$.

Now assume that $r \geqslant 4$. By the induction hypothesis there exists a set A_{r-1} of zero-sum short free sequences over C_n^{r-1} such that

$$\mathcal{L}(A_{r-1}) = [n+1, |S_{r-1}| - \alpha_{r-1}].$$

Recall that $C_n^{r-1} \subset C_n^r = C_n^{r-1} \oplus \langle e_r \rangle$. Let

$$A_r = \big\{W_2(W_1+e_r)e_r^{\ell} : W_1 \in A_{r-1}, W_2 \in A_{r-1}, \ell \in [0, n-1], |W_1| + \ell \equiv 0 (\mathrm{mod}\ n)\big\}.$$

Then, every sequence in A_r is zero-sum short free over C_n^r and

$$\begin{aligned}\mathcal{L}(A_r) &= \{|W_2| + |W_1| + \ell : W_1 \in A_{r-1}, W_2 \in A_{r-1}, \ell \in [0, n-1], |W_1| + \ell \equiv 0 (\mathrm{mod}\ n)\}\\ &= \{|W_2| + kn : W_2 \in A_{r-1}, k \in [2, \lceil \frac{|S_{r-1}| - \alpha_{r-1}}{n} \rceil]\\ &\supset [3n+1, 2|S_{r-1}| - 2\alpha_{r-1}].\end{aligned}$$

It follows that

$$\mathcal{L}(A_{r-1}) \cup \mathcal{L}(A_r) \supset [n+1, 2|S_{r-1}| - 2\alpha_{r-1}].$$

Note that

$$\begin{aligned}2|S_{r-1}| - 2\alpha_{r-1} &= |S_r| - (n-1) - 2\alpha_{r-1}\\ &\geqslant |S_r| - 3(n-1).\end{aligned}$$

Therefore,

$$\mathcal{L}(A_{r-1}) \cup \mathcal{L}(A_r) \supset [n+1, |S_r| - 3(n-1)].$$

Now the result follows from Lemma 23. □

Lemma 25. *Let $n, r, k \in \mathbb{N}$ with $k \geqslant 2, r \geqslant k+1$ and $n = 2^k$, and let $G = C_n^r$. Then, $C_0(G) \subset \{(2^r-1)(n-1)-n, (2^r-1)(n-1)-n+1\}$.*

Proof. Since $r \geqslant k+1$ we have that $\alpha_r = 0$.

By Lemma 7 we have

$$|S_r| = (2^r-1)(n-1) = \eta(G) - 1.$$

So, it suffices to show that $C_0(G) \cap ([n+1, \eta(G)-(n+2)] \cup [\eta(G)-n+1, \eta(G)-1]) = \emptyset$.

Since $r \geqslant k+1$ we have

$$\sigma(S_r) = 0.$$

Step 1. We show $C_0(G) \cap [n+1, |S_r| - (n+1)] = \emptyset$.

We proceed by induction on r. Suppose first that $r = k+1$.

If $r = k+1 = 3$, we only need to prove $C_0(G) \cap [3n-1, 4n-5] = \emptyset$ by Lemma 23 and Lemma 5. Let

$$\begin{aligned} A = &\{(e_1+e_2+e_3)(e_1+e_2)^{n-1}(e_1+e_3)^{n-m}(e_2+e_3)e_1^m e_2^{n-1} e_3^{m-2} : m \in [2, n-1]\} \cup \\ &\{(e_1+e_2)^2(e_1+e_3)^{n-1} e_1^{n-1} e_2^{n-2} e_3\}. \end{aligned}$$

Then every sequence in A is zero-sum short free and $\mathcal{L}(A) = [3n-1,\ 4n-3]$ and we are done.

If $r = k+1 > 3$, we have $\alpha_{r-1} \neq 0$ and $r-1 \geqslant 3$, then by Lemma 24 there exists a set A of zero-sum short free sequences over C_n^{r-1} such that $\mathcal{L}(A) \supset [n+1, |S_{r-1}| - \alpha_{r-1}]$.

Let

$$B = A \cup \left\{W_2(W_1+e_r)e_r^\ell : W_1 \in A, W_2 \in A, \ell \in [0, n-1], |W_1| + \ell \equiv 0 (\mathrm{mod}\ n)\right\}.$$

Since

$$|S_{r-1}| - \alpha_{r-1} + |S_{r-1}| - \alpha_{r-1} + \alpha_{r-1} - 1 = |S_r| - 3n/2,$$

we have $\mathcal{L}(B) \supset [n+1, |S_r| - 3n/2]$. It follows from Lemma 23 that $C_0(C_n^r) \cap [n+1, |S_r| - (n+1)] = \emptyset$.

Now assume that $r > k+1$. By the induction hypothesis, we conclude that there exists a set A of zero-sum short free sequences over C_n^{r-1} such that $\mathcal{L}(A) \supset [n+1, |S_{r-1}| - (n+1)]$.

Define a set B of zero-sum short free sequences over C_n^r as follows

$$B = \left\{W_2(W_1+e_r)e_r^\ell : W_1 \in A, W_2 \in A, \ell \in [0, n-1], |W_1| + \ell \equiv 0 (\mathrm{mod}\ n)\right\}.$$

It is easy to see that

$$\mathcal{L}(B) \supset [|S_{r-1}| - n, 2|S_{r-1}| - 2(n+1)] = [|S_{r-1}| - n, |S_r| - (3n+1)].$$

Let

$$\begin{aligned} &C_1 = \{T : T \mid S_2, \sigma(T) = 0\}; \\ &C_2 = \{(e_1+e_3)^{n-m} e_1^{m-1} e_2^{n-1} (e_1+e_2) e_3^m : m \in [1, n-1]\}; \\ &C_3 = \{(e_1+e_2)^2 (e_1+e_3)^{n-1} e_1^{n-1} e_2^{n-2} e_3\}; \\ &C_4 = \{(e_1+e_2+e_3)(e_1+e_2)^{n-1}(e_1+e_3)^{n-m}(e_2+e_3) e_1^m e_2^{n-1} e_3^{m-2} : m \in [2, n-1]\}. \end{aligned}$$

Then every sequence in $\cup_{i=1}^4 C_i$ is zero-sum short free. Clearly,

$$\begin{aligned} \mathcal{L}(C_1) &= [n+1, 2n-1]; \\ \mathcal{L}(C_2) &= [2n, 3n-2]; \\ \mathcal{L}(C_3) &= \{3n-1\}; \\ \mathcal{L}(C_4) &= [3n, 4n-3]. \end{aligned}$$

Let

$$C = \cup_{i=1}^{4} C_i.$$

Then,

$$\mathcal{L}(C) \supset [n+1, 4n-3].$$

Let

$$D = \{S_r T'^{-1} : T' \in C\}.$$

Then every sequence in D is zero-sum short free, and

$$\begin{aligned}\mathcal{L}(D) &\supset [|S_r| - (4n-3), |S_r| - (n+1)] \\ &\supset [|S_r| - 3n, |S_r| - (n+1)].\end{aligned}$$

This completes the proof of Step 1.

Step 2. We prove $C_0(G) \cap [\eta(G) - n + 1, \eta(G) - 1] = \emptyset$.
Let

$$A = \{S_r(e_r^m)^{-1}(me_r) : m \in [1, n-1]\}.$$

Then every sequence in A is zero-sum short free by Lemma 21, and

$$\mathcal{L}(A) = [|S_r| - n + 2, |S_r|] = [\eta(G) - n + 1, \eta(G) - 1].$$

This completes the proof. □

Proof of Proposition 19. 1. It is just Lemma 24.

2. Since $\alpha_r = 0$, we have $n = 2^k$ for some $k \in [2, r-1]$, now the result follows from Lemma 25. □

5 On the groups C_3^r

In this section we shall study $C_0(G)$ with focus on $G = C_3^r$.

Proposition 26. *Let $r, t \in \mathbb{N}$. Then,*

1. $C_0(C_3^3) \subset [\eta(C_3^3) - 4, \eta(C_3^3) - 1]$.
2. $C_0(C_5^3) \subset [\eta(C_5^3) - 5, \eta(C_5^3) - 1]$.
3. $C_0(C_{2^t}^r) \subset \begin{cases} [\eta(C_{2^t}^r) - (2^t - 2^{r-1}), \eta(C_{2^t}^r) - 1], & \text{if} \quad r \leqslant t, \\ [\eta(C_{2^t}^r) - (2^t + 1), \eta(C_{2^t}^r) - 2^t], & \text{if} \quad r > t. \end{cases}$
4. $C_0(C_6^3) \subset \{\eta(C_6^3) - 2, \eta(C_6^3) - 1\}$.

Proof. Conclusions 1, 2 and 4 follow from Lemma 7 and Proposition 19. So, it remains to prove Conclusion 3. If $r \leqslant t$ then applying Proposition 19 with $\alpha_r = 2^t - 2^{r-1}$, it follows from Conclusion 6 of Lemma 7 that $C_0(C_{2^t}^r) \subset [(2^r-1)(2^t-1)-(2^t-2^{r-1})+1, \eta(C_{2^t}^r)-1] = [\eta(C_{2^t}^r) - (2^t - 2^{r-1}), \eta(C_{2^t}^r) - 1]$. If $r > t$ then applying Proposition 19 with $\alpha_r = 0$ we get, $C_0(C_{2^t}^r) \subset [\eta(C_{2^t}^r) - (2^t + 1), \eta(C_{2^t}^r) - 2^t]$. □

Lemma 27. *Let $G = C_3^r$ with $r \geqslant 3$, and let S be a sequence over G. Then,*

1. *If S is a short free sequence over G of length $|S| = \eta(G) - 1$, then $\sum_{\leqslant 2}(S) = C_3^r \setminus \{0\}$.*

2. *Let T be a square free and short free sequence over G, and let $S = T^2$. Then, for every $g \in \mathrm{supp}(S)$ we have, $\sum_{\leqslant 2}(S \cdot g^{-1}) = \sum_{\leqslant 2}(S) \setminus \{2g\}$.*

3. *If every short free sequence of length $\eta(G) - 1$ has sum zero, then $\eta(G) - 2 \in C_0(G)$.*

Proof. Conclusions 1 and 2 are obvious.

To prove Conclusion 3, we assume to the contrary that $\eta(G) - 2 \notin C_0(G)$, i.e., there exists a zero-sum short free sequence S over G of length $|S| = \eta(G) - 2$. By Lemma 7, we have $\eta(G) - 2 = 2(f(G) - 2) + 1$. This forces that $S = g_1^2 \cdot \ldots \cdot g_{f(G)-2}^2 \cdot g_{f(G)-1}$ for some distinct elements $g_1, \ldots, g_{f(G)-1}$ with $g_1 \cdot \ldots \cdot g_{f(G)-1}$ contains no short zero-sum subsequence. Put $T = S \cdot g_{f(G)-1}$. Then $|T| = \eta(G) - 1$. But T contains no short zero-sum subsequence and $\sigma(T) = g_{f(G)-1} \neq 0$, a contradiction. $\square$

Lemma 28. *Every short free sequence over C_3^3 of length 16 has sum zero.*

Proof. Let S be an arbitrary short free sequence over C_3^3 of length $|S| = 16$. From Lemma 7 we obtain that $S = T^2$, where T is a square free and short free sequence over C_3^3 of length 8. It follows from Lemma 9 that $\sigma(T) = 0$. Therefore, $\sigma(S) = 2\sigma(T) = 0$. $\square$

Lemma 29. *The following two conclusions hold.*

1. $\{14, 15\} = \{\eta(C_3^3) - 3, \eta(C_3^3) - 2\} \subset C_0(C_3^3)$.

2. $\{37, 38\} = \{\eta(C_3^4) - 2, \eta(C_3^4) - 1\} \subset C_0(C_3^4)$.

Proof. 1. The conclusion $14 \in C_0(C_3^3)$ is due to Emde Boas and D. Kruyswijk [7]. Now $15 \in C_0(C_3^3)$ follows from Conclusion 3 of Lemma 7, Lemma 27 and Lemma 28.

2. Denote by U the square free sequence over C_3^4 given in Lemma 10. It follows from Conclusion 4 of Lemma 7 that U is a square free sequence of maximum length which contains no zero-sum subsequence of length 3.

Choose an arbitrary square free sequence T over C_3^4 of length $f(C_3^4) - 1$ such that T contains no short zero-sum subsequence. By Lemma 7, we have $|T| = 19$.

Claim. $\sigma(T) \notin -\mathrm{supp}(T) \cup \{0\}$.

Proof of the claim. Put $S = T \cdot 0$. It follows from Conclusion 4 of Lemma 7 that S is a square free sequence over C_3^4 of maximum length which contains no zero-sum subsequence of length 3. By Lemma 8, there exists an automorphism φ of C_3^4 and some $g \in C_3^4$ such that $S = \varphi(U - g)$. Since $0 \mid S$, it follows that $g \mid U$. Thus, $\sigma(T) = \sigma(S) = \sigma(\varphi(U - g)) = \varphi(\sigma(U - g)) = \varphi(\sigma(U) - 20g) = \varphi(\sigma(U) + g)$. It is easy to check that $\sigma(U) = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$. Since $-\sigma(U) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \notin \mathrm{supp}(U)$, it follows that $-\sigma(T) = -\varphi(\sigma(U) + g) = \varphi(-\sigma(U) - g) \notin \varphi(\mathrm{supp}(U) - g) = \mathrm{supp}(S) = \mathrm{supp}(T) \cup \{0\}$. This proves the claim.

From Conclusions 4, 9, 10 of Lemma 7 and the above claim, we derive that every short free sequence over C_3^4 of length $\eta(C_3^4)-1=38$ has a nonzero sum. This is equivalent to that every zero-sum sequence over C_3^4 of length $\eta(C_3^4)-1$ contains a short zero-sum subsequence. Hence, $38=\eta(C_3^4)-1\in C_0(C_3^4)$.

Suppose that $37=\eta(C_3^4)-2\notin C_0(G)$, that is, there exists a zero-sum short free sequence V over C_3^4 of length $|V|=\eta(C_3^4)-2=37$. Since $\mathsf{v}_g(V)\leqslant 2$ for every $g\in\operatorname{supp}(V)$, we have $|\operatorname{supp}(V)|\geqslant 19$. On the other hand, by Conclusion 4 and 9 of Lemma 7, we can derive that $|\operatorname{supp}(V)|\leqslant f(C_3^4)-1=\frac{\eta(C_3^4)-1}{2}=19$. Thus, $V=W^2h^{-1}$, where $h\mid W$ and W is a square free and short free sequence over G of length $f(C_3^4)-1=19$. It follows from $\sigma(V)=0$ that $\sigma(W)=-h\in-\operatorname{supp}(W)$, a contradiction with the claim above. $\square$

Proposition 30. *Let $G=C_3^r$ with $r\geqslant 3$. If there is a short free sequence S over G of length $|S|=\eta(G)-1$ such that $\sigma(S)\neq 0$, then*

1. $|\{\eta(G)-2,\eta(G)-3\}\cap C_0(G)|\leqslant 1.$
2. $|\{\eta(G)-3,\eta(G)-4\}\cap C_0(G)|\leqslant 1.$

Proof. 1. Since $\sigma(S)\neq 0$, it follows from Lemma 27 that there exists a subsequence W of S of length $|W|\in\{1,2\}$ such that $\sigma(S)=\sigma(W)$. Therefore, $\sigma(S\cdot W^{-1})=0$, $|S\cdot W^{-1}|\in\{\eta(G)-3,\eta(G)-2\}$ and $S\cdot W^{-1}$ contains no short zero-sum subsequence. Hence, $\eta(G)-2\notin C_0(G)$ or $\eta(G)-3\notin C_0(G)$.

2. By Conclusion 10 of Lemma 7, we have that $S=T^2$, where T is a square free sequence over G. Choose $g\in\operatorname{supp}(S)$ such that $\sigma(S\cdot g^{-1})\neq 0$. Since $\sigma(S\cdot g^{-1})=\sigma(S)-g\neq 2g$, it follows from Conclusion 2 of Lemma 27 that $\sigma(S\cdot g^{-1})\in\sum_{\leqslant 2}(S\cdot g^{-1})=C_3^r\setminus\{0,2g\}$. Similarly to Conclusion 1, we infer that $\eta(G)-3\notin C_0(G)$ or $\eta(G)-4\notin C_0(G)$. $\square$

Proposition 31. $C_0(C_3^4)=\{37,38\}$.

Proof. By Proposition 19, we have

$$C_0(C_3^4)\subset[30,\eta(C_3^4)-1]=[30,38]. \tag{5.1}$$

We show next that

$$[30,36]\cap C_0(C_3^4)=\emptyset. \tag{5.2}$$

Put

$$T_2=\begin{pmatrix}2\\2\\2\\2\end{pmatrix}^2;$$

$$T_3=\begin{pmatrix}2\\2\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix};$$

$$T_4 = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\2\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix};$$

$$T_5 = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}^2\begin{pmatrix}0\\2\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\end{pmatrix}\begin{pmatrix}0\\2\\2\\2\end{pmatrix};$$

$$T_6 = \begin{pmatrix}0\\2\\0\\0\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix}^2\begin{pmatrix}0\\1\\0\\2\end{pmatrix}\begin{pmatrix}1\\2\\0\\2\end{pmatrix}\begin{pmatrix}2\\1\\0\\2\end{pmatrix};$$

$$T_7 = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\2\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\2\\0\end{pmatrix}\begin{pmatrix}0\\1\\2\\0\end{pmatrix}\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\end{pmatrix}\begin{pmatrix}0\\2\\2\\2\end{pmatrix};$$

$$T_8 = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\2\\0\\0\end{pmatrix}^2\begin{pmatrix}1\\0\\2\\0\end{pmatrix}\begin{pmatrix}0\\1\\2\\0\end{pmatrix}\begin{pmatrix}1\\2\\2\\0\end{pmatrix}\begin{pmatrix}0\\0\\2\\2\end{pmatrix}^2.$$

Let U be the square free sequence given in Lemma 10. Then $\sigma(U) = \begin{pmatrix}2\\2\\2\\2\end{pmatrix}$. Let $S = U^2 \cdot 0^{-2}$. We see that S is a short free sequence of length $38 = \eta(C_3^4) - 1$. By removing T_i from S, we obtain that the resulting sequence S_i is a zero-sum short free sequence of length $\eta(G) - i - 1 = 38 - i$. This proves (5.2). Combining (5.1), (5.2) and Lemma 29, we conclude that $C_0(C_3^4) = \{\eta(G) - 2, \eta(G) - 1\} = \{37, 38\}$. $\square$

6 Proof of Theorem 2

In this section we shall prove Theorem 2 and we need the following lemma.

Lemma 32. *Let p be a prime and let H be a finite abelian p-group such that $p^n \geqslant D(H)$. Then,*

1. *Every sequence S over $C_{p^n} \oplus H$ of length $|S| = 2p^n + D(H) - 2$ contains a zero-sum subsequence T of length $|T| \in \{p^n, 2p^n\}$.*

2. *$\eta(C_{mp^n} \oplus H) \leqslant mp^n + p^n + D(H) - 2$.*

Proof. 1. Let $S = g_1 \cdot \ldots \cdot g_\ell$ be a sequence over $G = C_{p^n} \oplus H$ of length $\ell = |S| = 2p^n + D(H) - 2$. Let $\alpha_i = \begin{pmatrix} 1 \\ g_i \end{pmatrix} \in C_{p^n} \oplus C_{p^n} \oplus H$ with $1 \in C_{p^n}$. By Conclusion 10 of Lemma 13, $\alpha_1 \cdot \ldots \cdot \alpha_\ell$ is a sequence over $C_{p^n} \oplus G$ of length $\ell = p^n + p^n + D(H) - 2 = D(C_{p^n} \oplus G)$. Therefore, $\alpha_1 \cdot \ldots \cdot \alpha_\ell$ contains a nonempty zero-sum subsequence W(say). By the making of α_i we infer that $|W| = p^n$ or $|W| = 2p^n$. Let T be the subsequence of S which corresponds to W. Then T is a zero-sum subsequence of S of length $|T| \in \{p^n, 2p^n\}$.

2. We first consider the case that $m = 1$. Let $G = C_{p^n} \oplus H$. We want to prove that $\eta(G) \leqslant 2p^n + D(H) - 2$.

Let $S = g_1 \cdot \ldots \cdot g_\ell$ be a sequence over $G = C_{p^n} \oplus H$ of length $\ell = |S| = 2p^n + D(H) - 2$. We need to show that S contains a short zero-sum subsequence. It follows from Conclusion 1 that S contains a zero-sum subsequence T of length $|T| \in \{p^n, 2p^n\}$. If $|T| = p^n$ then T itself is a short zero-sum sequence over G and we are done. Otherwise, since $p^n \geqslant D(H)$, it follows from Conclusion 3 of Lemma 13 that $|T| = 2p^n > p^n + D(H) - 1 = D(G)$. Therefore, T contains a nonempty proper zero-sum subsequence T'. Now either T' or TT'^{-1} is a short zero-sum subsequence of S. This proves that $\eta(C_{p^n} \oplus H) \leqslant 2p^n + D(H) - 2$. By Lemma 12, we have

$$\begin{aligned}
\eta(C_{mp^n} \oplus H) &\leqslant (\eta(C_m) - 1)\exp(C_{p^n} \oplus H) + \eta(C_{p^n} \oplus H) \\
&\leqslant (m-1)p^n + 2p^n + D(H) - 2 \\
&= mp^n + p^n + D(H) - 2.
\end{aligned}$$

□

Lemma 33. *Let G be a finite abelian group. Then $[D(G) + 1, \min\{2\exp(G) + 1, \eta(G) - 1\}] \subset C_0(G)$.*

Proof. If $[D(G)+1, \min\{2\exp(G)+1, \eta(G)-1\}] = \emptyset$ then the conclusion of this lemma hold true trivially. Now assume that $[D(G)+1, \min\{2\exp(G)+1, \eta(G)-1\}] \neq \emptyset$. Let S be an arbitrary zero-sum sequence over G of length $|S| \in [D(G)+1, \min\{2\exp(G)+1, \eta(G)-1\}]$. It suffices to show that S contains a short zero-sum subsequence. Since $|S| \geqslant D(G) + 1$, it follows that S contains a zero-sum subsequence T of length $|T| \in [1, |S| - 1]$. Then $\sigma(ST^{-1}) = 0$. Since $|S| \leqslant 2\exp(G) + 1$, we infer that $|T| \in [1, \exp(G)]$ or $|ST^{-1}| \in [1, \exp(G)]$. This proves the lemma. □

Proof of Theorem 2, 1. By the definition of $C_0(G)$ we have, $C_0(G) \subset [D(G)+1, \eta(G)-1]$. So, we need to show

$$[D(G) + 1, \eta(G) - 1] \subset C_0(G).$$

Suppose first that

$$G = C_n \oplus C_n.$$

By Conclusions 1 and 4 of Lemma 13, we have $D(G) = 2n - 1$ and $\eta(G) = 3n - 2$. Let S be a zero-sum sequence over G of length $|S| \in [2n, 3n - 3]$. We need to show S contains a short zero-sum subsequence. We may assume that

$$\mathsf{v}_0(S) = 0.$$

Let $T = S \cdot 0^{3n-2-|S|}$. Then $|T| = 3n-2$ and T contains a zero-sum subsequence T' of length $|T'| \in \{n, 2n\}$ by Lemma 11. If $|T'| = n$ then $T'0^{-\mathsf{v}_0(T')}$ is a short zero-sum subsequence of S and we are done. So, we may assume that $|T'| = 2n$. Let $T'' = TT'^{-1}$. Now T'' is a zero-sum subsequence of T of length $|T''| = n-2$. If T'' contains at least one nonzero element then $T''0^{-\mathsf{v}_0(T'')}$ is a short zero-sum subsequence of S and we are done. So, we may assume that $T'' = 0^{n-2}$. This forces that $T' = S$. It follows from $D(G) = 2n-1$ that S contains a zero-sum subsequence S_0 of length $|S_0| \in [1, 2n-1]$. Therefore, either S_0 or SS_0^{-1} is a short zero-sum subsequence of S.

Now suppose that

$$G = C_n \oplus C_m$$

with $n \mid m$ and

$$n < m.$$

By Conclusions 1 and 4 of Lemma 13, we have that $D(G) = n+m-1 < 2m$ and $2m+1 > 2n+m-2 = \eta(G)$. It follows from Lemma 33 that $[D(G)+1, \eta(G)-1] \subset C_0(G)$.

2. By Conclusion 2 of Lemma 13 and Conclusion 2 of Lemma 32, we have that $D(C_{mp^n} \oplus H) = mp^n + D(H) - 1$ and $\eta(C_{mp^n} \oplus H) \leqslant mp^n + p^n + D(H) - 2$.

Suppose $m \geqslant 2$. Then $\eta(C_{mp^n} \oplus H) \leqslant 2mp^n$. Similarly to the proof of Conclusion 1, we can prove that $[D(C_{mp^n} \oplus H) + 1, \eta(C_{mp^n} \oplus H) - 1] \subset C_0(G)$, and we are done. So, we may assume

$$m = 1.$$

Then $\eta(C_{p^n} \oplus H) \leqslant 2p^n + D(H) - 2$ and the proof is similar to that of 1 by using Conclusion 1 of Lemma 32.

3. It is just Proposition 31.

4. Observe that $\sum\limits_{g \in C_2^r \setminus \{0\}} g = 0$. Then, any square free sequence S over C_2^r with $\mathsf{v}_0(S) = 0$ and $|S| \in \{2^r-3, 2^r-2\}$ must have a nonzero sum. It follows from Conclusion 6 of Lemma 7 that $\{\eta(C_2^r)-3, \eta(C_2^r)-2\} = \{2^r-3, 2^r-2\} \subset C_0(C_2^r)$. So, $C_0(C_2^r) = \{2^r-3, 2^r-2\} = \{\eta(C_2^r)-3, \eta(C_2^r)-2\}$ follows from Proposition 19. □

7 Proof of Theorem 3

Lemma 34. *If* $\frac{\eta(C_m^r)-1}{m-1} = \frac{\eta(C_n^r)-1}{n-1} = c$ *for some* $c \in \mathbb{N}$ *and if* $\eta(C_{mn}^r) \geqslant c(mn-1)+1$ *then* $\eta(C_{mn}^r) = c(mn-1)+1$.

Proof. The lemma follows from Lemma 12. □

Lemma 35. $C_{2^t}^r$ *has Property C.*

Proof. It follows from Lemma 4 and Conclusion 6 of Lemma 7 by induction on t. □

Proposition 36. *Let* $n = 3m$*, where* m *is an odd positive integer. Then,*

1. *If* $\eta(C_m^3) = 8m-7$ *then* $\eta(C_n^3) - 2 \in C_0(C_n^3)$.

2. *If* $\eta(C_m^4) = 19m - 18$ *then* $\{\eta(C_n^4) - 2, \eta(C_n^4) - 1\} \subset C_0(C_n^4)$.

Proof. 1. By Conclusion 3 of Lemma 7 and Lemma 12, we have

$$\begin{aligned}\eta(C_n^3) &\leqslant (\eta(C_3^3) - 1) \cdot m + \eta(C_m^3) \\ &= 16m + 8m - 7 \\ &= 8n - 7.\end{aligned}$$

Combined with Conclusion 1 of Lemma 7, we have

$$\frac{\eta(C_n^3) - 1}{n - 1} = \frac{\eta(C_m^3) - 1}{m - 1} = \frac{\eta(C_3^3) - 1}{3 - 1} = 8. \tag{7.1}$$

Now we show $\eta(C_n^3) - 2 \in C_0(C_n^3)$ by applying Lemma 18 with $G_2 = C_3^3$ and $t_1 = t_2 = 2$. Conditions (i)-(iii) of Lemma 18 are verified by (7.1), Conclusion 10 of Lemma 7, and Conclusion 1 of Lemma 29 respectively. We are done.

2. The proof is similar to that of Conclusion 1. $\square$

Proposition 37. *Let* $\alpha, \beta \in \mathbb{N}_0$ *with* $\alpha \geqslant 1$. *Then,*

1. *If* $\alpha + \beta \geqslant 2$ *then* $\{\eta(C_{3^\alpha 5^\beta}^3) - 2, \eta(C_{3^\alpha 5^\beta}^3) - 1\} \subset C_0(C_{3^\alpha 5^\beta}^3)$.
2. $\{\eta(C_{3^\alpha}^4) - 2, \eta(C_{3^\alpha}^4) - 1\} \subset C_0(C_{3^\alpha}^4)$.

Proof. 1. By Conclusions 1, 3 and 5 of Lemma 7 and Lemma 34, we conclude that

$$\frac{\eta(C_{3^s 5^t}^3) - 1}{3^s 5^t - 1} = 8 \tag{7.2}$$

for every $s, t \in \mathbb{N}_0$ with $s + t \geqslant 1$. Combined with Proposition 36, we have $\eta(C_{3^\alpha 5^\beta}^3) - 2 \in C_0(C_{3^\alpha 5^\beta}^3)$.

By Lemma 4, Conclusions 8, 10 of Lemma 7 and (7.2), we have $C_{3^\alpha 5^\beta}^3$ has Property C. Since $\alpha + \beta \geqslant 2$, we have $8 < 3^\alpha 5^\beta$. Therefore, it follows from (7.2) and Lemma 16 that $\eta(C_{3^\alpha 5^\beta}^3) - 1 \in C_0(C_{3^\alpha 5^\beta}^3)$. We are done.

2. By Conclusion 2 of Lemma 29, we need only to consider the case that $\alpha > 1$. By Conclusions 2 and 4 of Lemma 7 and Lemma 34, we can derive

$$\frac{\eta(C_{3^{\alpha-1}}^4) - 1}{3^{\alpha-1} - 1} = 19.$$

Combined with Conclusion 2 of Proposition 36, we have $\{\eta(C_{3^\alpha}^4) - 2, \eta(C_{3^\alpha}^4) - 1\} \subset C_0(C_{3^\alpha}^4)$, done. $\square$

Proposition 38. *Let* $m = 3^\alpha 5^\beta$ *with* $\alpha \in \mathbb{N}$ *and* $\beta \in \mathbb{N}_0$. *Let* $n \geqslant 65$ *be an odd positive integer such that* C_p^3 *has Property* D_0 *with respect to 9 for all prime divisors* p *of* n. *If*

$$m \geqslant \frac{6 \times 5^7 n^{17}}{(n^2 - 7)n - 64} + 3$$

then $\eta(C_{mn}^3) - 2 \in C_0(C_{mn}^3)$.

Proof. Let $m' = \frac{m}{3}$. Then $m' = 3^{\alpha-1}5^{\beta} \geqslant \frac{2\times 5^7 n^{17}}{(n^2-7)n-64}$ and $\alpha - 1 \geqslant 0$.

By Lemma 15 and Lemma 14 we have $s(C^3_{m'n}) = 9m'n - 8$ and $\eta(C^3_{m'n}) \leqslant 8m'n - 7$. It follows from Lemma 7 that $\eta(C^3_{m'n}) = 8m'n - 7$. Since $\eta(C^3_3) = 8 \times 3 - 7$ and $\eta(C^3_{m'n}) = 8m'n - 7$, it follows from Lemma 34 that $\eta(C^3_{mn}) = 8mn - 7$. What's more, C^3_3 has Property C and $\eta(C^3_3) - 2 \in C_0(C^3_3)$ by Lemma 29. Therefore, $\eta(C^3_{mn}) - 2 \in C_0(C^3_{mn})$ by Lemma 18. $\square$

Proof of Theorem 3.

1. If $a \geqslant 1$ then it follows from Proposition 37 and Lemma 29. Now assume $b \geqslant 2$. Since $\eta(C^3_{3^a5^b}) = 8(3^a5^b - 1) + 1$, it follows from Lemma 16 that $\eta(C^3_{3^a5^b}) - 1 \in C_0(C^3_{3^a5^b})$.

2. Let $G_1 = C^3_{3\times 2^{a-3}}$ and $G_2 = C^3_8$. By Lemma 35, Conclusions 6, 7 and 8 of Lemma 7, we have that $\eta(G_1) = 7(3 \times 2^{a-3} - 1) + 1$, $\eta(G_2) = 7 \times (8 - 1) + 1$ and G_2 has Property C. Therefore, $\eta(C^3_8) - 1 \in C_0(C^3_8)$ by Lemma 16. So, $\eta(C^3_{3\times 2^a}) - 1 \in C_0(C^3_{3\times 2^a})$ by Lemma 18.

3. The result follows from Proposition 37.

4. Let $G = C^r_{2^a}$ with $3 \leqslant r \leqslant a$. By Lemma 35 and Conclusions 6 of Lemma 7, we have $\eta(C^r_{2^a}) = (2^r - 1)(2^a - 1) + 1$ and $C^r_{2^a}$ has Property C. Since $2^r - 1 < 2^a$, it follows from Lemma 16 that $\eta(C^r_{2^a}) - 1 \in C_0(C^r_{2^a})$.

   If $G = C^r_2$ and $r \geqslant 3$, then it follows from Conclusion 4 of Theorem 2.

5. Let $m = 3^{n_1}5^{n_2}$ and $n = 7^{n_3}11^{n_4}13^{n_5}$. It follows from $n_3 + n_4 + n_5 \geqslant 3$ that $n > 65$. By the hypothesis of $n_1 + n_2 \geqslant 11 + 34(n_3 + n_4 + n_5)$ we infer that, $m = 3^{n_1}5^{n_2} \geqslant 3^{n_1+n_2} \geqslant 3^{11}3^{34(n_3+n_4+n_5)} > 4 \times 5^8 \times 13^{14(n_3+n_4+n_5)} \geqslant 4 \times 5^8 n^{14} > \frac{6\times 5^7 n^{17}}{(n^2-7)n-64} + 3$. Since it has been proved that every prime $p \in \{3, 5, 7, 11, 13\}$ has Property D_0 with respect to 9 in [8], it follows from Proposition 38 that $\eta(C^3_k) - 2 \in C_0(C^3_k)$. $\square$

8 Concluding Remarks and Open Problems

Proposition 39. *Let G be a non-cyclic finite abelian group with* $\exp(G) = n$. *Then $C_0(G) \cup \{\eta(G)\}$ doesn't contain $n + 1$ consecutive integers.*

Proof. Assume to contrary that $[t, t + n] \subset C_0(G) \cup \{\eta(G)\}$ for some $t \in \mathbb{N}$. By the definition of $C_0(G)$ we have that $t + n - 1 < \eta(G)$. So, we can choose a short free sequence T over G of length $|T| = t + n - 1$. It follows from $t + n - 1 \in C_0(G) \cup \{\eta(G)\}$ that $\sigma(T) \neq 0$. Let $g = \sigma(T)$ and let $S = T \cdot (-g)$. Since $|S| = t + n \in C_0(G) \cup \{\eta(G)\}$, S contains a short zero-sum subsequence U with $(-g) \mid U$. Note that $t \leqslant |S \cdot U^{-1}| \leqslant t + n - 2$ and $\sigma(S \cdot U^{-1}) = 0$. It follows from $[t, t + n] \subset C_0(G) \cup \{\eta(G)\}$ that $S \cdot U^{-1}$ contains a short zero-sum subsequence, which is a contradiction with $S \cdot U^{-1} \mid T$. $\square$

Proposition 39 just asserts that $C_0(G)$ can't contain any interval of length more than $\exp(G)$. Proposition 19 shows that $C_0(C^r_n)$ could not contain integers much smaller than $\eta(C^r_n) - 1$. So, it seems plausible to suggest

Conjecture 40. *Let $G \neq C_2 \oplus C_{2m}, m \in \mathbb{N}$ be a non-cyclic finite abelian group. Then* $C_0(G) \subset [\eta(G) - (\exp(G) + 1), \eta(G) - 1]$.

Conjecture 40 and Conjecture 1 suggest the following

Conjecture 41. *Let $G \neq C_2 \oplus C_{2m}, m \in \mathbb{N}$ be a non-cyclic finite abelian group. Then* $1 \leqslant |C_0(G)| \leqslant \exp(G)$.

Conjecture 42. $C_0(G) = [\min\{C_0(G)\}, \max\{C_0(G)\}]$.

The following notation concerning the inverse problem on $\mathsf{s}(G)$ was introduced in [10].

Property D: *We say the group C_n^r has property D if $\mathsf{s}(C_n^r) = c(n-1)+1$ for some positive integer c, and every sequence S over C_n^r of length $|S| = c(n-1)$ which contains no zero-sum subsequence of length n has the form $S = \prod_{i=1}^{c} g_i^{n-1}$ where $g_1, \ldots, g_c$ are pairwise distinct elements of C_n^r.*

Conjecture 43. *([10], Conjecture 7.2) Every group C_n^r has Property D.*

It has been proved in ([10], Section 7) that Conjecture 43, if true would imply

Conjecture 44. *Every group C_n^r has Property C.*

Suppose that Conjecture 44 holds true for all groups of the form C_n^r. For fixed $n, r \in \mathbb{N}$ and any $a \in \mathbb{N}$ we have that $\eta(C_{n^a}^r) = c(n^a, r)(n-1)+1$, where $c(n^a, r) \in \mathbb{N}$ depends on n^a and r. By Lemma 12 we obtain that the sequence $\{c(n^a, r)\}_{a=1}^{\infty}$ is decreasing. Therefore, $c(n^a, r) \leqslant n^a$ for all sufficiently large a. Hence, by Lemma 16 we infer that $\eta(C_{n^a}^r) - 1 \in C_0(C_{n^a}^r)$ for all sufficiently large $a \in \mathbb{N}$.

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References

[1] G. Bhowmik and J.C. Schlage-Puchta, Davenport' s constant for groups of the form $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, *Additive Combinatorics* **43** (2007), 307–326.

[2] R. Chi, S.Y. Ding, W.D. Gao, A. Geroldinger and W.A. Schmid, On zero-sum subsequences of restricted size IV, *Acta Math. Hungar.* **107** (2005), 337–344.

[3] B.L. Davis, D. Maclagan and R. Vakil, The card game set, *Math. Intelligencer* **25** (2003), 33–40.

[4] Y. Edel, Sequences in abelian groups G of odd order without zero-sum subsequences of length exp(G), *Des. Codes Cryptogr.* **47** (2008), 125–134.

[5] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin and L. Rackham, Zero-sum problems in finite abelian groups and affine caps, *Q. J. Math.* **58** (2007), 159–186.

[6] C. Elsholtz, Lower bounds for multidimentional zero sums, *Combinatorica* **24** (2004), 351–358.

[7] P. van Emde Boas and D. Kruyswijk, A combinatorial problem on finite abelian groups, II. *Math. Centre Report* ZW-1969-008.

[8] Y.S. Fan, W.D. Gao and Q.H. Zhong, On the Erdos-Ginzburg-Ziv constant of finite abelian groups of high rank, *J. Number Theory* **131** (2011), 1864–1874.

[9] W.D. Gao, On a combinatorial problem connected with factorizations, *Colloq. Math.* **72** (1997), 251–268.

[10] W.D. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups : a survey, *Expo. Math.* **24** (2006), 337–369.

[11] W.D. Gao, Q.H. Hou, W.A. Schmid and R. Thangadurai, On short zero-sum subsequences II, *Integers* **7** (2007), A21.

[12] W.D. Gao, A. Geroldinger and W.A. Schmid, Inverse zero-sum problems, *Acta Arith.* **128** (2007), 245–279.

[13] A. Geroldinger and F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, *Pure and Applied Mathematics*, vol. 278, Chapman & Hall/CRC, (2006).

[14] A. Geroldinger, Additive group theory and non-unique factorizations, *Combinatorial Number Theory and Additive Group Theory* (2009), 1–86.

[15] A. Geroldinger, D.J. Grynkiewicz and W.A. Schmid, Zero-sum problems with congruence conditions, *Acta Math. Hungar.* **131** (2011), 323–345.

[16] A. Geroldinger and I.Z. Ruzsa, Combinatorial number theory and additive group theory, Birkhauser (2009).

[17] H. Harborth, Ein Extremalproblem für Gitterpunkte, *J. Reine Angew. Math.* **262** (1973), 356–360.

[18] H. Harborth, Ein Extremalproblem für Gitterpunkte, Ph.D. Thesis, Technische Univesität Braunschweig (1982).

[19] A. Kemnitz, On a lattice point problem, *Ars Combin.* 16b (1983), 151–160.

[20] J.E. Olson, A combinatorial problem on finite abelian groups, I; II, *J. Number Theory* **1** (1969), 8–10; 195–199.