Planar graphs of maximum degree six without 7-cycles are class one

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Abstract

It is conjectured by Vizing (1965) that every planar graphs graph G with maximum degree $6 \leqslant \Delta \leqslant 7$ is class one. The case $\Delta = 7$ was confirmed independently by Sanders and Zhao (2001), and by Zhang (2000). In this paper, we prove that every planar graph G with $\Delta = 6$ and without 7-cycles is class one.

Keywords: Planar graph; edge coloring; class one; cycle

1 Introduction

In this paper, all graphs under consideration are simple and finite. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. Let $V(G), E(G), F(G)$ and $\Delta(G)$ (or Δ for short) be the vertex set, edge set, face set, and maximum degree of a given plane graph G, respectively. Let C_n denote a cycle of length n. We say that G is C_n*-free* if G contains no C_n as a subgraph.

An *edge k-coloring* of a graph G is a function $\phi : E(G) \mapsto \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. The *edge chromatic number*, denoted $\chi'(G)$, of a graph G is the smallest integer k such that G has an edge k-coloring. The celebrated Vizing's Theorem says that the edge chromatic number of a simple graph G is equal to Δ or $\Delta + 1$. G is *class one* if $\chi'(G) = \Delta$ and *class two* if $\chi'(G) = \Delta + 1$. A class two graph is *critical* if $\chi'(G - e) < \chi'(G)$ for any edge e of G. A critical graph G is Δ-*critical* if it has maximum degree Δ.

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In 1965, Vizing[6] proposed the following well-known Planar Graph Coloring conjecture:

Conjecture 1 *Every planar graph G with $\Delta = 6, 7$ is class one.*

The case $\Delta = 7$ was confirmed independently by Sanders and Zhao [4], and by Zhang [9]. This result was further extended by Sanders and Zhao [5] to a graph with $\Delta = 7$ which can be embedded in a surface of characteristic zero. The case $\Delta = 6$ remains open.

In[10], Zhou proved that every planar graph G with $\Delta = 6$ is class one if it is C_3-free, or C_4-free, or C_5-free. Li, Luo and Niu[2] generalized Zhou's results to the surface of Euler characteristic at least -3 or -1. Bu and Wang[1] proved that planar graphs G with $\Delta = 6$ and without 6-cycles, or without two adjacent 3-cycles are class one. Wang and Chen [7] proved that planar graphs with $\Delta = 6$ and without a 5-cycle with a chord is class one. More recently, Wang, Chen and Wang [8] further proved that planar graphs with $\Delta = 6$ and without a 6-cycle with a chord is class one.

In this paper, we prove the following result, which extends a result in [1] and [10]:

Theorem 1. *If G is a planar graph with $\Delta = 6$ and without 7-cycles, then $\chi'(G) = \Delta$.*

To show Theorem 1, we need to introduce some notation. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_k]$ if $u_1, u_2, \ldots, u_k$ are the vertices of $b(f)$ in a cyclic order. For $x \in V(G) \cup F(G)$, let $d(x)$ denote the degree of x in G. A vertex of degree k (at least k, at most k, respectively) is called a *k-vertex* (or *k^+-vertex, k^--vertex*, respectively). Similarly, we can define a k-face, k^+-face and k^--face. Let $v \in V(G)$. If a k-vertex u is adjacent to v, then u is called a *k-neighbor* of v, and we use $d_k(v)$ to denote the number of k-neighbors of v. Similarly, we can define $d_{k^+}(v)$ and $d_{k^-}(v)$. For $i \geqslant 3$, let $m_i(v)$ denote the number of i-faces incident to v. Moreover, $m_{i^+}(v)$ and $m_{i^-}(v)$ can be defined analogously. Let $N(v)$ denote the set of neighbors of v, and let $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$, let $N(S) = \cup_{u \in S} N(u)$.

The following is the outstanding Vizing's Adjacent Lemma (we denote it by **VAL** for short).

Lemma 2. ([6]) *If G is a Δ-critical graph and xy is an edge of G, then $d(x)+d(y) \geqslant \Delta+2$ and x is adjacent to at least $(\Delta - d(y) + 1)$ Δ-vertices. Furthermore, every vertex is adjacent to at least two Δ-vertices.*

Let G be a 6-critical graph and $v \in V(G)$. Then the assertions (P1) to (P5) below follow automatically from Lemma 2.

(P1) If $d(v) = 2$, then $d_6(v) = 2$.

(P2) If $d(v) = 3$, then $d_{4^-}(v) = 0$ and $d_5(v) \leqslant 1$.

(P3) If $d(v) = 4$, then $d_{3^-}(v) = 0$ and $d_4(v) \leqslant 1$; and if $d_4(v) = 1$, then $d_6(v) = 3$.

(P4) If $d(v) = 5$, then $d_2(v) = 0$ and $d_3(v) \leqslant 1$; moreover, $d_6(v) = 4$ if $d_3(v) = 1$, and $d_6(v) \geqslant 3$ if $d_4(v) \geqslant 1$.

(P5) If $d(v) = 6$, then $d_2(v) \leqslant 1$; moreover, $d_6(v) = 5$ if $d_2(v) = 1$, $d_6(v) \geqslant 4$ if $d_3(v) \geqslant 1$, and $d_6(v) \geqslant 3$ if $d_4(v) \geqslant 1$.

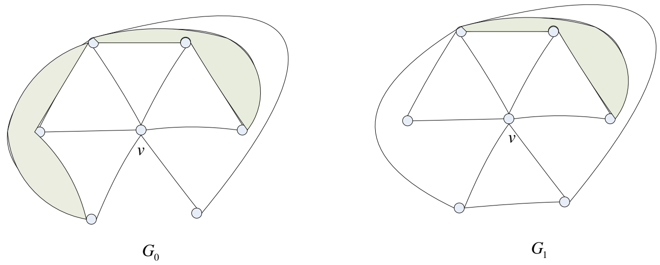

Figure 1: The configurations in Claim 5, where v is a bad 6-vertex.

The vertex v is called *bad* if either $d(v) = 6$, $m_3(v) = 4$ and $m_4(v) = 2$, or $d(v) = m_3(v) = 5$ and $d_6(v) = 2$, or $d(v) = m_3(v) = 4$. It is easy to see that a bad 5-vertex v satisfies $d_5(v) = 3$ by (P4).

Lemma 3. ([9]) *Let G be a Δ-critical graph. If $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$, then the following hold:*
(1) *Every vertex in $N(x, y)\backslash\{x, y\}$ is a Δ-vertex.*
(2) *Every vertex in $N(N(x, y))\backslash\{x, y\}$ is of degree at least $\Delta - 1$.*
(3) *If $d(x), d(y) < \Delta$, then every vertex in $N(N(x, y))\backslash\{x, y\}$ is a Δ-vertex.*

Lemma 4. ([3]) *Let G be a critical graph and x be a 3-vertex in G. If x is adjacent to three Δ-vertices, then at least one Δ-vertex in $N(x)$ is adjacent to only one $(\Delta - 1)^-$-vertex which is x.*

2 Proof of Theorem 1.1

Let G be a planar graph with $\Delta = 6$ and without 7-cycles that is embedded in the plane. Assume to the contrary that G is class two. Without loss of generality, we may assume that G is 6-critical. Then G is 2-connected, implying that the boundary of each face forms a cycle and every edge lies on the boundaries of two faces.

We first investigate structural properties of G and then use Euler's formula and the discharging technique to derive a contradiction. Given a k-vertex $v \in V(G)$, let $v_0, v_1, \cdots, v_{k-1}$ be the neighbors of v in clockwise order. For $0 \leqslant i \leqslant k - 1$, let f_i be the incident face of v with vv_i and vv_{i+1} as boundary edges, where indices are taken modulo k.

Claim 5. *Let v be a 6-vertex with $m_3(v) = 4$. Then one of the following cases holds:*
(1) *v is bad such that $G(N[v]) \cong G_0$ or G_1, as shown in Figure 1;*
(2) *$m_4(v) = 1$ and $m_{8^+}(v) = 1$;*
(3) *$m_5(v) = 1$ and $m_{6^+}(v) = 1$;*
(4) *$m_{6^+}(v) = 2$.*

Proof. Since $m_3(v) = 4$, the proof is split into the following three cases by symmetry.

Case 1 $d(f_i) = 3$ for $i = 1, 2, 3, 4$.

Since G contains no 7-cycles, it is easy to see that $v_0v_1, v_0v_5 \notin E(G)$. Let $f_5 = [vv_5y_1 \cdots y_pv_0]$ and $f_0 = [vv_0z_1 \cdots z_qv_1]$. Then $p, q \geqslant 1$ and $p, q \neq 4$. By symmetry, we may assume that $q \geqslant p$. If $p \geqslant 3$, then (4) holds. If $p = 2$ and $q \geqslant 3$, then (3) holds. If $p = 1$ and $q \geqslant 5$, then (2) holds. Otherwise, it suffices to consider the following two subcases to derive (1):

Subcase 1.1 $p = 1$ and $1 \leqslant q \leqslant 3$.

Since G is C_7-free, it follows that $y_1 \in \{v_2, v_3, v_4\}$.

(1.1a) $y_1 = v_4$, implying that $d(v_5) = 2$.

• Assume that $q = 1$. Then $z_1 \in \{v_2, v_3, v_4\}$ as before. If $z_1 = v_2$, then $d(v_1) = 2$. If $z_1 = v_4$, then $d(v_0) = 2$. Thus, $N(v)$ has at least two 2-vertices in both cases, contradicting (P5). If $z_1 = v_3$, we get a 7-cycle $C_7 = vv_5v_4v_0v_3v_2v_1v$, a contradiction.

• Assume that $q = 2$. Then at least one of z_1, z_2 belongs to $\{v_2, v_3\}$ since otherwise we get a 7-cycle $C_7 = vv_0z_1z_2v_1v_2v_3v$. If $z_2 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_2 = v_3$, then $z_1 \neq v_4$, and we get $C_7 = v_0z_1v_3v_2v_1vv_4v_0$. If $z_1 = v_2$, we get $C_7 = vv_1z_2v_2v_3v_4v_5v$. If $z_1 = v_3$, we have two possibilities. When $z_2 = v_2$, we have $d(v_1) = 2$, contradicting (P5). When $z_2 \neq v_2$, we get $C_7 = v_1z_2v_3v_4v_5vv_2v_1$.

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$, for otherwise we take $C_7 = vv_0z_1z_2z_3v_1v_2v$. If $v_2 = z_3$, then $d(v_1) = 2$, contradicting (P5). If $v_2 = z_2$, we take $C_7 = vv_1z_3v_2v_3v_4v_5v$. If $v_2 = z_1$, then $z_2, z_3 \notin N(v)$ by the planarity of G, hence we have $C_7 = vv_1z_3z_2v_2v_3v_4v$.

(1.1b) $y_1 = v_3$.

• Assume that $q = 1$. Then $z_1 \in \{v_2, v_3\}$ by the planarity of G and the hypothesis that G is C_7-free. If $z_1 = v_2$, we get $C_7 = vv_1v_2v_0v_3v_4v_5v$. If $z_1 = v_3$, we have $G(N[v]) \cong G_0$, hence the conclusion (1) holds.

• Assume that $q = 2$. We note that at least one of z_1, z_2 belongs to $\{v_2, v_3\}$ by the planarity of G and the fact that G is C_7-free. If $z_1 = v_2$, we get $C_7 = vv_1z_2v_2v_3v_4v_5v$. If $z_1 = v_3$, we assert that $z_2 \neq v_2$, otherwise it follows that $d(v_1) = 2$ and $d(v_3) = 3$, contradicting (P5). Thus, we get $C_7 = vv_2v_1z_2v_3v_4v_5v$. If $z_2 = v_2$, we derive by (P5) that $z_1 \neq v_3$, hence $C_7 = vv_0z_1v_2v_3v_4v_5v$. If $z_2 = v_3$, it is easy to deduce that $d(v_3) \geqslant 7$, contradicting the assumption that $\Delta = 6$.

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$ since G is C_7-free. If $z_3 = v_2$, then $z_2 \neq v_3$ by (P5). When $z_1 \neq v_3$, we get $C_7 = vv_0z_1z_2v_2v_3v_4v$. When $z_1 = v_3$, we get $C_7 = vv_1v_2z_2v_3v_4v_5v$. If $z_2 = v_2$, we get $C_7 = vv_1z_3v_2v_3v_4v_5v$. If $z_1 = v_2$, we get $C_7 = vv_1z_3z_2v_2v_3v_4v$.

(1.1c) $y_1 = v_2$.

• If $q = 1$, then $z_1 = v_2$ by the planarity of G and the hypothesis that G is C_7-free. Hence $d(v_1) = d(v_0) = 2$, contradicting (P5).

• Assume that $q = 2$. By the planarity of G, $v_3 \notin \{z_1, z_2\}$. If $v_2 \notin \{z_1, z_2\}$, G contains a 7-cycle $C_7 = vv_0z_1z_2v_1v_2v_3v$. If $z_1 = v_2$, we get $C_7 = vv_1z_2v_2v_3v_4v_5v$. If $z_2 = v_2$, we get $C_7 = vv_0z_1v_2v_3v_4v_5v$.

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$ since G is C_7-free. If $z_3 = v_2$, we get $C_7 = vv_0z_1z_2v_2v_3v_4v$. If $z_2 = v_2$, we get $C_7 = vv_1z_3v_2v_3v_4v_5v$. If $z_1 = v_2$, we get $C_7 = vv_1z_3z_2v_2v_3v_4v$.

Subcase 1.2 $p = q = 2$.

Since G is C_7-free, we see that at least one of y_1, y_2 belongs to $\{v_3, v_4\}$, and at least one of z_1, z_2 belongs to $\{v_2, v_3\}$. Furthermore, $y_2 \neq v_1$ and $z_1 \neq v_5$.

• If $y_2 = v_4$, then we get $C_7 = vv_5y_1v_4v_3v_2v_1v$. If $z_1 = v_2$, we have a similar proof. So assume that $z_1 \neq v_2$.

• Assume that $y_1 = v_4$. Then $d(v_5) = 2$. If $y_2 = v_3$, then $d(v_4) = 3$, contradicting (P5). If $y_2 = v_2$, then $z_2 = v_2$ and $d(v_1) = 2$, contradicting (P5). If $y_2 \notin \{v_2,\ v_3\}$, we get $C_7 = vv_0y_2v_4v_3v_2v_1v$.

If $z_2 = v_2$, we have a similar proof. Hence assume that $z_2 \neq v_2$.

• Assume that $y_1 = v_3$. If $z_2 = v_3$, then $d(v_3) \geqslant 7$, a contradiction. If $z_1 = v_3$, then we get $C_7 = vv_2v_1z_2v_3v_4v_5v$.

• Assume that $y_2 = v_3$. Then we get $C_7 = vv_1v_2v_3y_1v_5v_4v$.

Case 2 $d(f_i) = 3$ for $i = 1, 2, 3, 5$.

Since G is C_7-free, we see $v_0v_1, v_0v_4, v_1v_5, v_4v_5 \notin E(G)$. Let $f_4 = [vv_4y_1 \cdots y_pv_5]$ and $f_0 = [vv_0z_1 \cdots z_qv_1]$ such that $q \geqslant p \geqslant 1$ and $p, q \neq 4$. If one of (2)-(4) holds, we are done. Otherwise, similar to Case 1, it suffices to consider the following two subcases:

Subcase 2.1 $p = 1$ and $1 \leqslant q \leqslant 3$.

Since G is C_7-free, it follows that $y_1 \in \{v_2, v_3\}$.

(2.1a) $y_1 = v_3$, implying that $d(v_4) = 2$. Then $z_1 \neq v_5$, for otherwise $d(v_0) = 2$, contradicting (P5).

• Assume that $q = 1$. Then $z_1 \in \{v_2, v_3\}$ by the same reason. If $z_1 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_1 = v_3$, then $G(N[v]) \cong G_1$, hence (1) holds.

• Assume that $q = 2$. Since G is C_7-free, at least one of z_1, z_2 coincides with v_2 or v_3. If $z_2 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_1 = v_2$, we get $C_7 = vv_0v_5v_3v_2z_2v_1v$. If $z_2 = v_3$, we get $C_7 = vv_5v_0z_1v_3v_2v_1v$. If $z_1 = v_3$, we get $C_7 = vv_0v_5v_3z_2v_1v_2v$.

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$ by the previous analysis. If $z_3 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_2 = v_2$, we get $C_7 = vv_0v_5v_3v_2z_3v_1v$. If $z_1 = v_2$, we get $C_7 = vv_5v_3v_2z_2z_3v_1v$.

(2.1b) $y_1 = v_2$.

• Assume that $q = 1$. By the planarity of G and the foregoing argument, $z_1 \notin N(v)\backslash\{v_2\}$. If $z_1 = v_2$, we have $G(N[v]) \cong G_1$ and therefore (1) holds.

• Assume that $q = 2$. It follows that $v_2 \in \{z_1, z_2\}$ since G is C_7-free. If $z_1 = v_2$, then $d(v_2) \geqslant 7$, a contradiction. If $z_2 = v_2$, then $z_1 \neq v_5$, and we get $C_7 = vv_5v_0z_1v_2v_3v_4v$.

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$. If $z_1 = v_2$, we get $C_7 = vv_1z_3z_2v_2v_3v_4v$. If $z_3 = v_2$, we get $C_7 = vv_0z_1z_2v_2v_3v_4v$. If $z_2 = v_2$, we have two possibilities. When $z_1 \neq v_5$, we get $C_7 = vv_5v_0z_1v_2v_3v_4v$. When $z_1 = v_5$, we have $d(v_0) = 2$ and $d(v_5) = 3$, contradicting (P5).

Subcase 2.2 $p = q = 2$.

We see that at least one of y_1, y_2 (z_1, z_2, respectively) coincides with v_2 or v_3.

• Assume that $y_1 = v_3$. Then $d(v_4) = 2$. By the previous discussion, $y_2 \neq v_1$. If $y_2 = v_0$, then $d(v_5) = 2$, contradicting (P5). If $y_2 = v_2$, then $d(v_3) = 3$, contradicting (P5). This shows that $y_2 \notin \{v_0, v_1, v_2\}$, hence we get $C_7 = vv_0v_5y_2v_3v_2v_1v$.

If $z_2 = v_2$, we have a similar proof.

• Assume that $y_2 = v_3$. If $z_1 = v_2$, we get $C_7 = vv_1z_2v_2v_3y_1v_4v$. If $z_1 = v_3$, we get $C_7 = vv_4y_1v_3z_2v_1v_2v$. If $z_2 = v_3$, we claim that $z_1 \neq v_5$, for otherwise $d(v_0) = 2$ and $d(v_5) = 3$, contradicting (P5). Thus, we get $C_7 = vv_5v_0z_1v_3y_1v_4v$.

If $z_1 = v_2$, we have a similar proof.

• Assume that one of y_1, y_2 coincides with v_2. By the planarity of G, none of z_1, z_2 coincides with v_3. Thus, we get $C_7 = vv_0z_1z_2v_1v_2v_3v$.

Case 3 $d(f_i) = 3$ for $i = 1, 2, 4, 5$.

Since G is C_7-free, we see that $v_0v_1, v_0v_3, v_1v_4, v_3v_4 \notin E(G)$. Let $f_3 = [vv_3y_1 \cdots y_pv_4]$ and $f_0 = [vv_0z_1 \cdots z_qv_1]$ such that $q \geqslant p \geqslant 1$ and $p, q \neq 4$. If one of (2)-(4) holds, we are done. Otherwise, it suffices to consider the following two subcases:

Subcase 3.1 $p = 1$ and $1 \leqslant q \leqslant 3$.

Since G is C_7-free, it follows that $y_1 \in \{v_2, v_5\}$. Without loss of generality, we assume that $y_1 = v_2$. Then $d(v_3) = 2$.

• If $q = 1$, then $z_1 \in \{v_2, v_5\}$ similarly, so $d(v_1) = 2$ or $d(v_0) = 2$, contradicting (P5).

• Assume that $q = 2$. We note that $v_3 \notin \{z_1, z_2\}$ by the plane embedding of G. Thus, $v_2 \in \{z_1, z_2\}$. If $z_1 = v_2$, we get $C_7 = vv_0v_5v_4v_2z_2v_1v$. If $z_2 = v_2$, then $d(v_1) = 2$, contradicting (P5).

• Assume that $q = 3$. Then $v_2 \in \{z_1, z_2, z_3\}$. If $z_3 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_1 = v_2$, we get $C_7 = vv_5v_4v_2z_2z_3v_1v$. If $z_2 = v_2$, we get $C_7 = vv_0v_5v_4v_2z_3v_1v$.

Subcase 3.2 $p = q = 2$.

If $y_1, y_2 \notin \{v_2, v_5\}$, we get $C_7 = vv_2v_3y_1y_2v_4v_5v$. If $z_1, z_2 \notin \{v_2, v_5\}$, we get a similar 7-cycle. Otherwise, by symmetry, we assume that the following two possibilities:

• Assume that $y_1 = v_2$. Then $d(v_3) = 2$. Obviously, $y_2 \neq v_1$ since $v_1v_4 \notin E(G)$. If $y_2 = v_5$, then $d(v_4) = 2$, contradicting (P5). If $y_2 \neq v_0$, we get $C_7 = vv_0v_5v_4y_2v_2v_1v$. So suppose that $y_2 = v_0$. In this case, we must have $v_2 \in \{z_1, z_2\}$. If $z_2 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $z_1 = v_2$, we get $C_7 = vv_1z_2v_2v_0v_5v_4v$.

• Assume that $y_2 = v_2$. We get $C_7 = vv_3y_1v_2v_4v_5v_0v$. □

For a bad 6-vertex v, Claim 5 asserts that $G(N[v]) \cong G_0$ or G_1, see Figure 1. We say that a 6-vertex in the induced subgraph $G[N(v)]$ other than v is a *master* of v, and v is the *slave* of its master. Clearly, if u is a master of some 6-vertex, then $m_3(u) = 2$, $m_4(u) = 2$, and u has only one slave.

Claim 6. *Let v be a 6-vertex with $m_3(v) = 3$. If $m_4(v) = 3$, then $d_{3^-}(v) = 0$.*

Proof. Assume to the contrary that $d_{3^-}(v) > 0$. If f_i is a 4-face, we let $f_i = [vv_iy_iv_{i+1}]$, where indices are taken modulo 6.

Case 1 $d(f_i) = 3$ for $i = 1, 3, 5$.

Then $d(f_j) = 4$ for $j = 0, 2, 4$. We note that if $y_j \in \{v_{j-1}, v_{j+2}\}$, then $d(v_j) = 2$ or $d(v_{j+1}) = 2$ for each $j = 0, 2, 4$. By (P5), at least two of y_j's, say y_2 and y_4, do not coincide with v_{j-1} and v_{j+2}. If $y_2 = v_5$, we get $C_7 = vv_1v_2v_5y_4v_4v_3v$. If $y_4 = v_2$, we have a similar proof. If $y_2 \neq y_4$, we get $C_7 = vv_2y_2v_3v_4y_4v_5v$. Assume that $y_2 = y_4$, which implies that $y_2 \notin \{v_0, v_1\}$. If $y_0 \in \{v_2, v_5\}$, say $y_0 = v_2$, then we get $C_7 = vv_5v_0v_2y_2v_3v_4v$. Otherwise, $y_0 \notin \{v_2, v_5\}$, we have two possibilities. If $y_0 \neq y_2$, then we get $C_7 = vv_0y_0v_1v_2y_2v_3v$. If $y_0 = y_2$, i.e., y_0, y_2, y_4 identify to one vertex, then it is easy to see that $d(v_i) \geqslant 3$ for all $i = 0, 1, \cdots, 5$. Since $d_{3^-}(v) > 0$, we may assume, without loss of generality, that $d(v_1) = 3$. By (P2), $d(v_2) \geqslant 5$. However, v_2 is a cut vertex of G, contradicting the fact that G is 2-connected.

Case 2 $d(f_i) = 3$ for $i = 1, 2, 4$.

Then $d(f_j) = 4$ for $j = 0, 3, 5$. If $y_3 \notin N(v)$, we get $C_7 = vv_1v_2v_3y_3v_4v_5v$. So assume that $y_3 \in N(v)$. We have the following subcases:

- If $y_3 = v_0$, we get $C_7 = vv_1v_2v_3v_0v_4v_5v$.
- Assume that $y_3 = v_5$, then $d(v_4) = 2$. If $y_0 \notin N(v)$, we get $C_7 = vv_0y_0v_1v_2v_3v_5v$. Otherwise, $y_0 \in \{v_2, v_3, v_5\}$ by the plane embedding of G. If $y_0 = v_2$, then $d(v_1) = 2$, contradicting (P5). If $y_0 = v_5$, then it follows that $y_5 \notin N(v)$ by the plane embedding of G, hence we construct $C_7 = vv_0y_5v_5v_3v_2v_1v$. So assume that $y_0 = v_3$. If $y_5 \neq v_3$, we get $C_7 = vv_0y_5v_5v_3v_2v_1v$. If $y_5 = v_3$, then $d(v_0) = 2$, contradicting (P5).
- Assume that $y_3 = v_2$. Then $d(v_3) = 2$. If $y_5 \notin N(v)$, we get $C_7 = vv_0y_5v_5v_4v_2v_1v$. Otherwise, we have $y_5 \in \{v_1, v_2, v_4\}$ by the planarity of G. If $y_5 = v_4$, then $d(v_5) = 2$, contradicting (P5). If $y_5 = v_1$, or $y_5 = v_2$, then $y_0 \notin N(v)$ by (P5) and the planarity of G, thus we have $C_7 = vv_0y_0v_1v_2v_4v_5v$.
- Assume that $y_3 = v_1$. If $y_5 \notin N(v)$, we get $C_7 = vv_0y_5v_5v_4v_1v_2v$. Otherwise, $y_5 \in \{v_1, v_4\}$ by the planarity of G. If $y_5 = v_4$, then $d(v_5) = 2$. When $y_0 = v_4$, we have $d(v_0) = 2$, contradicting (P5). When $y_0 \neq v_4$, we get $C_7 = vv_4v_0y_0v_1v_2v_3v$. If $y_5 = v_1$, then it is easy to see that $y_0 \notin N(v)$, and hence $d(v_1) \geqslant 7$, a contradiction.

Case 3 $d(f_i) = 3$ for $i = 1, 2, 3$.

Then $d(f_j) = 4$ for $j = 0, 4, 5$. If $y_4 \notin N(v)$, we get $C_7 = vv_5y_4v_4v_3v_2v_1v$. Otherwise, it suffices to handle the case $y_4 \in \{v_0, v_1, v_2, v_3\}$.

- If $y_4 = v_0$, we get $C_7 = vv_1v_2v_3v_4v_0v_5v$. If $y_0 = v_5$, we have a similar proof.
- Assume that $y_4 = v_3$, then $d(v_4) = 2$. If $y_5 \notin N(v)$, we get $C_7 = vv_0y_5v_5v_3v_2v_1v$. Otherwise, suppose that $y_5 \in \{v_1, v_2, v_3\}$ by the plane embedding of G. If $y_5 = v_3$, then $d(v_5) = 2$, contradicting (P5). If $y_5 = v_1$, then $y_0 \notin N(v)$, we get $C_7 = vv_0y_0v_1v_2v_3v_4v$. Finally, suppose that $y_5 = v_2$. If $y_0 = v_2$, then $d(v_0) = 2$, contradicting (P5). Otherwise, we can construct a 7-cycle as above.

If $y_0 = v_2$, we have a similar proof.

• Assume that $y_4 = v_1$. If $y_5 \notin N(v)$, we get $C_7 = vv_0y_5v_5v_1v_2v_3v$. Otherwise, $y_5 = v_1$ by the planarity of G. Noting that $y_0 \notin N(v)$, we can construct a 7-cycle as above.

If $y_0 = v_4$, we have a similar proof.

• If $y_4 = v_2$, then $y_0 = v_3$ by symmetry, which is impossible by the planarity of G. □

Claim 7. *If v is a 6-vertex adjacent to a bad 5-vertex, then $m_{8^+}(v) \geqslant 2$.*

Proof. Assume that $v_0, v_1, \ldots, v_5$ are the neighbors of v in clockwise order, and v_1 is a bad 5-vertex. Then $m_3(v_1) = 5$, $d_6(v_1) = 2$ and $d_5(v_1) = 3$ by definition. This implies that $v_1v_2, v_1v_0 \in E(G)$. Let x and y be the other two neighbors of v_1 such that v, v_0, x, y, v_2 are arranged around v in clockwise order. The proof is split into three cases as follows:

Case 1 $v_3 = x$.

Since G is C_7-free, it is easy to inspect that neither v_4 nor v_5 is adjacent to a vertex in $\{x, v_0\}$. Moreover, since G is simple and G is embedded in the plane, both v_4 and v_5 can not identify to x, hence $v_1v_4, v_1v_5 \notin E(G)$. Our goal is to show that $d(f_i) \geqslant 8$ for $i = 3, 5$.

Let $f_3 = [vxu_1 \cdots u_sv_4]$. Obviously, $d(f_3) = s + 3 \geqslant 4$, and $v_1, v_2, y \notin \{u_1, u_2, \cdots, u_s\}$. If $s = 1$, we get $C_7 = vv_4u_1xv_1yv_2v$. If $s = 2$, we get $C_7 = vv_4u_2u_1xyv_2v$. If $s = 3$, we get $C_7 = vv_4u_3u_2u_1xv_1v$. Since G is C_7-free, $s \neq 4$. Therefore, $s \geqslant 5$, that is $d(f_3) \geqslant 8$.

Let $f_5 = [vv_5z_1 \cdots z_lv_0]$. Then $d(f_5) = l + 3 \geqslant 4$, $l \neq 4$, and $v_1, v_2, y \notin \{z_1, z_2, \cdots, z_l\}$. If $l = 1$, we get $C_7 = vv_5z_1v_0v_1yv_2v$. If $l = 2$, we get $C_7 = vv_5z_1z_2v_0v_1v_2v$. If $l = 3$, we get $C_7 = vv_5z_1z_2z_3v_0v_1v$. Therefore, $s \geqslant 5$, that is $d(f_3) \geqslant 8$.

If $v_5 = y$, we have a similar discussion.

Case 2 $v_3 = y$.

By Case 1, we assume that $v_5 \neq y$. We are going to show that $d(f_i) \geqslant 8$ for $i = 2, 5$.

Firstly, we show that $d(f_2) \geqslant 8$. If $d(f_2) = 3$, then $d(v_2) = 3$, contradicting the definition of a bad 5-vertex. So assume that $f_2 = [vv_2u_1 \cdots u_sy]$, where $s \geqslant 1$ and $s \neq 4$. Note that $v_0, v_1, x \notin \{u_1, u_2, \cdots, u_s\}$. If $s = 1$, we get $C_7 = vv_0xyu_1v_2v_1v$. If $s = 2$, we get $C_7 = vv_1xyu_2u_1v_2v$. If $s = 3$, we get $C_7 = vv_1yu_3u_2u_1v_2v$. This shows that $s \geqslant 5$, namely $d(f_2) \geqslant 8$.

Secondly, we show that $d(f_5) \geqslant 8$. If $v_5 = x$, then we can show that $d(f_5) \geqslant 8$ as above. Otherwise, assume that $v_5 \neq x$. Since $v_0v_5 \notin E(G)$, we see that $d(f_5) \geqslant 4$. Again, let $f_5 = [vv_5z_1 \cdots z_lv_0]$ with $l \geqslant 1$. Then $v_1, v_2 \notin \{z_1, z_2, \cdots, z_l\}$ by the planarity of G. It suffices to inspect that $l \notin \{1, 2, 3\}$. In fact, if $l = 3$, we get $C_7 = vv_5z_1z_2z_3v_0v_1v$. If $l = 2$, we get $C_7 = vv_5z_1z_2v_0v_1v_2v$. So assume that $l = 1$. Since $xv_5 \notin E(G)$, we derive that $z_1 \neq x$. We get $C_7 = vv_5z_1v_0xv_1v_2v$.

If $v_5 = x$, we have a similar proof.

Case 3 $v_3, v_5 \notin \{x, y\}$.

To show that $d(f_i) \geqslant 8$ for $i = 2, 5$, it suffices to inspect $d(f_2)$ by symmetry. If $d(f_2) = 3$, then C_7 exists obviously. So assume that $d(f_2) \geqslant 4$, and let $f_2 = [vv_2u_1 \cdots u_kv_3]$ with $k \geqslant 1$ and $k \neq 4$. We first note that $v_1 \notin \{u_1, u_2, \cdots, u_k\}$. If $k = 3$, we get $C_7 = vv_1v_2u_1u_2u_3v_3v$. If $k = 1$, then u_1 is identical to at most one of x, y, v_0, so C_7 can be always constructed. Assume that $k = 2$. If $u_1 = y$, then $d(v_2) = 3$, contradicting the definition of v_1. Thus, suppose that $u_1 \neq y$. If $u_2 \neq y$, we get $C_7 = vv_1yv_2u_1u_2v_3v$. If $u_2 = y$, then $u_1 \notin N(v_1)$, we get $C_7 = vv_1v_2u_1yxv_0v$. Thus, $k \geqslant 5$, that is, $d(f_2) \geqslant 8$. □

Claim 8. *If v is a 5-vertex adjacent to a bad 4-vertex, then $m_3(v) \leqslant 4$; moreover, if $m_3(v) = 4$, then $m_{8^+}(v) \geqslant 1$.*

Proof. Assume that $v_0, v_1, \ldots, v_4$ are the neighbors of v in clockwise order, and v_1 is a bad 4-vertex. Then $m_3(v_1) = 4$, implying that both f_0 and f_1 are 3-faces, hence $v_1v_2, v_1v_0 \in E(G)$. Let x be the neighbors of v_1 different from v_0, v_1, v_2. By symmetry, the proof can be split into two cases below.

Case 1 $x \notin \{v_3, v_4\}$.

We are going to show that $m_3(v) \leqslant 3$ in this case. Assume to the contrary that $m_3(v) \geqslant 4$. Without loss of generality, we assume that $d(f_2) = 3$ (otherwise, $d(f_4) = 3$.) Then at least one of f_3 and f_4 is a 3-face. If $d(f_3) = 3$, we get $C_7 = vv_0xv_1v_2v_3v_4v$. If $d(f_4) = 3$, we get $C_7 = vv_4v_0xv_1v_2v_3v$. We always obtain a contradiction.

Case 2 $v_3 = x$.

Clearly, $v_4 \neq x$ as G is simple. If $d(f_2) = 3$, then $d(v_2) = 3$, contradicting (P3). Thus, $d(f_2) \geqslant 4$, and therefore it follows that $m_3(v) \leqslant 4$. To complete the proof, assume that $m_3(v) = 4$. This implies that both f_3 and f_4 are 3-faces, hence $v_0v_4, v_4x \in E(G)$. Let $f_2 = [vv_2u_1 \cdots u_kx]$ with $k \geqslant 1$ and $k \neq 4$. It is easy to see that $v_0, v_1, v_4 \notin \{u_1, u_2, \cdots, u_k\}$ by the planarity of G. If $k = 1$, we get $C_7 = vv_4v_0v_1xu_1v_2v$. If $k = 2$, we get $C_7 = vv_0v_1xu_2u_1v_2v$. If $k = 3$, we get $C_7 = vv_1xu_3u_2u_1v_2v$. Thus, $k \geqslant 5$, i.e., $d(f_2) \geqslant 8$. □

Claim 9. *No two bad 5-vertices are adjacent.*

Proof. Assume that v is a bad 5-vertex with neighbors $v_0, v_1, \cdots, v_4$ in clockwise order which is adjacent to a bad 5-vertex, say v_1. Let x and y be the other neighbors of v_1 such that v, v_0, x, y, v_2 are arranged around v_1 in clockwise order. If at least one of x and y does not belong to $N(v)$, then a 7-cycle is easily established. Otherwise, it is easy to derive that $x = v_3$ and $y = v_4$, which is impossible by the planarity of G. □

Using Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (d(f) - 8) = -16. \tag{1}$$

We define an initial weight function w by $w(v) = 3d(v) - 8$ for a vertex $v \in V(G)$, and $w(f) = d(f) - 8$ for a face $f \in F(G)$. It follows from equality (1) that the total sum of weights is -16. Then, we will define appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function w' is produced. However the total sum of weights is kept fixed when the discharging is in process. Nevertheless, we can show that $w'(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction:

$$0 \leqslant \sum_{x \in V(G) \bigcup F(G)} w'(x) = \sum_{x \in V(G) \bigcup F(G)} w(x) = -16,$$

and hence demonstrates that no such counterexample can exist.

Our discharging rules are defined as follows.

(R1) Every vertex v sends $\frac{5}{3}$ to each incident 3-face, 1 to each incident 4-face, $\frac{3}{5}$ to each incident 5-face, and $\frac{1}{3}$ to each incident 6-face.

(R2) Every 6-vertex v sends $\frac{7}{3}$ to each adjacent 2-vertex, $\frac{2}{d_3(v)+d_4(v)+d_5(v)}$ to each adjacent 3-, 4- or 5-vertex, with one exception: if v is adjacent to a 3-vertex x and a 5-vertex y with $xy \in E(G)$, then v sends 2 to x and $\frac{1}{3}$ to y.

(R3) Every master sends 1 to its slave.

Let $\alpha(v)$ denote the resultant weight of a vertex v after (R1)-(R3) are carried out. Then we do the following additional assignments:

(R4) Every 5-vertex v with $\alpha(v) > 0$ sends $\frac{\alpha(v)}{d_3(v)+d_4(v)}$ to each adjacent 3- or 4-vertex.

(R5) Every 6-vertex v with $\alpha(v) > 0$ sends $\frac{\alpha(v)}{d_5(v)}$ to each adjacent bad 5-vertex.

For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y according to our discharging rules. A vertex x is called *small* if $2 \leqslant d(x) \leqslant 5$.

Observation 1 *Under (R2), every 6-vertex sends at most $\frac{7}{3}$ to its adjacent small vertices.*

Proof. Let v be a 6-vertex in G, and let $s(v)$ denote the sum of weights that v has sent to its small adjacent vertices according to (R2). It suffices to inspect that $s(v) \leqslant \frac{7}{3}$.

If v is adjacent to a 2-vertex u, then $d_6(v) = 5$ by (P5), i.e., $d_3(v) + d_4(v) + d_5(v) = 0$. By (R2), v sends $\frac{7}{3}$ to u and nothing to other neighbors. Therefore, $s(v) = \frac{7}{3}$. Otherwise, assume that $d_2(v) = 0$. If v is adjacent to a 3-vertex x and a 5-vertex y with $xy \in E(G)$, then v cannot be adjacent to other small vertices by (P5). That is, $d_3(v) = d_5(v) = 1$ and $d_4(v) = 0$. By (R1), $s(v) = \tau(v \to x) + \tau(v \to y) = 2 + \frac{1}{3} = \frac{7}{3}$. If v is not adjacent to such vertices x and y, then $s(v) \leqslant 2$ by (R2). □

Observation 2 *Let v be a 6-vertex and u a small vertex adjacent to v. Then, after (R2) was carried out, we have the following:*

(1) *If $d(u) = 3$, then $\tau(v \to u) \geqslant 1$.*

(2) *If $d(u) = 4$, then $\tau(v \to u) \geqslant \frac{2}{3}$.*

(3) *If $d(u) = 5$ and v is adjacent to a 3-vertex x such that $ux \in E(G)$, then $\tau(v \to u) = \frac{1}{3}$; otherwise $\tau(v \to u) \geqslant \frac{1}{2}$.*

Proof. Suppose that v is a 6-vertex adjacent to a vertex u with $3 \leqslant d(u) \leqslant 5$.

(1) Assume that $d(u) = 3$. By (P5), we see that $d_2(v) = 0$, $d_6(v) \geqslant 4$, hence $d_3(v) + d_4(v) + d_5(v) \leqslant 2$. If v is adjacent to a 5-vertex y such that $uy \in E(G)$, we have $\tau(v \to u) = 2$ by (R2). Otherwise, $\tau(v \to u) = 2/(d_3(v) + d_4(v) + d_5(v)) \geqslant 1$.

(2) Assume that $d(u) = 4$. It is easy to see that $d_2(v) = 0$ by (P5). If $d_3(v) \geqslant 1$, then it follows from the proof of (1) that $\tau(v \to u) \geqslant 1$. Thus, assume that $d_3(v) = 0$. Since $d_4(v) \geqslant 1$, (P5) asserts that $d_6(v) \geqslant 3$, which implies that $d_4(v) + d_5(v) \leqslant 3$. Consequently, $\tau(v \to u) = 2/(d_4(v) + d_5(v)) \geqslant \frac{2}{3}$.

(3) Assume that $d(u) = 5$. Again, (P5) guarantees that $d_2(v) = 0$. If v is adjacent to a 3-vertex x such that $ux \in E(G)$, then $\tau(v \to u) = \frac{1}{3}$ by (R2). Otherwise, when $d_3(v) + d_4(v) \geqslant 1$, the result follows from the proofs of Cases (1) and (2). So assume that $d_3(v) = d_4(v) = 0$. By **VAL**, $d_6(v) \geqslant 2$, so $d_5(v) \leqslant 4$. This shows $\tau(v \to u) = 2/d_5(v) \geqslant \frac{1}{2}$. □

We carry out (R1)-(R5) in G. Let w' denote the resultant weight function after discharging was finished. It remains to verify that $w'(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Let $f \in F(G)$. Since G is 2-connected, $b(f)$ forms a cycle. This means that f is incident to $d(f)$ distinct vertices. Since G is C_7-free, $d(f) \neq 7$. If $d(f) = 3$, then each of its boundary vertices gives it exactly $\frac{5}{3}$ by (R1). Thus, $w'(f) = 3 - 8 + 3 \times \frac{5}{3} = 0$. If $4 \leqslant d(f) \leqslant 6$, we have a similar examination. If $d(f) \geqslant 8$, then $w'(f) = w(f) = d(f) - 8 \geqslant 0$.

Let $v \in V(G)$. Then $2 \leqslant d(v) \leqslant 6$. Let $v_0, v_1, \cdots, v_{d(v)-1}$ denote the neighbors of v in clockwise order. For $0 \leqslant i \leqslant d(v) - 1$, we use f_i to represent the incident face of v with vv_i and vv_{i+1} as boundary edges, where indices are taken modulo $d(f)$. The proof is divided into the following five cases.

(1) $d(v) = 2$.

It is easy to see that $d_6(v) = 2$ by (P5), $m_3(v) \leqslant 1$ since G is simple. By (R1) and (R2), we get that $w'(v) \geqslant (3 \times 2 - 8) + 2 \times \frac{7}{3} - \frac{5}{3} - 1 = 0$.

(2) $d(v) = 3$.

Then $w(v) = 1$. By (P2), $d_{4^-}(v) = 0$, hence $d_5(v) + d_6(v) = 3$. Without loss of generality, assume that $d(v_1) \leqslant d(v_2) \leqslant d(v_0)$. By **VAL**, $d(v_1) \geqslant 5$ and $d(v_2) = d(v_0) = 6$.

If $d(v_1) = 6$, then each v_i, for $i = 0, 1, 2$, is adjacent to at most two 5^--vertices by **VAL**, and at least one v_i is adjacent to only one 5^--vertex by Lemma 4. Thus, by (R1), (R2) and Observation 2(1), $w'(v) \geqslant 1 + 2 + 2 \times 1 - 3 \times \frac{5}{3} = 0$.

Assume that $d(v_1) = 5$. Then $d_6(v_1) = 4$. If $m_3(v) = 0$, then $w'(v) \geqslant 1 + 2 \times 1 - 3 = 0$ by (R1) and Observation 2(1).

- Assume that $m_3(v) = 1$. If f_1 is a 3-face, then $\tau(v_2 \to v) = 2$ by (R2). If f_0 is a 3-face, then $\tau(v_0 \to v) = 2$ by (R2). It turns out that $w'(v) \geqslant 1 + 2 + 1 - \frac{5}{3} - 2 \times 1 = \frac{1}{3}$ by (R1) and Observation 2(1). If f_2 is a 3-face, then $m_3(v_1) \leqslant 3$. After (R1)-(R3), $\alpha(v_1) \geqslant (3 \times 5 - 8) + 4 \times \frac{1}{2} - 3 \times \frac{5}{3} - 2 \times 1 = 2$ by Observation 2(3). By (R4), $\tau(v_1 \to v) \geqslant 2$. Hence $w'(v) \geqslant 1 + 2 \times 1 + 2 - \frac{5}{3} - 2 \times 1 = \frac{1}{3}$.
- Assume that $m_3(v) \geqslant 2$. If both f_0 and f_1 are 3-faces, then $\tau(v_i \to v) = 2$ for $i = 0, 2$ by (R2), hence $w'(v) \geqslant 1 + 2 \times 2 - 3 \times \frac{5}{3} = 0$. Otherwise, we may suppose that $d(f_1) = d(f_2) = 3$ and $d(f_0) \geqslant 4$ by symmetry. By (R2), $\tau(v_2 \to v) = 2$. Since $m_3(v_1) \leqslant 4$, after (R1)-(R3), we obtain that $\alpha(v_1) \geqslant 7 - 4 \times \frac{5}{3} - 1 + 3 \times \frac{1}{2} = \frac{5}{6}$ by Observation 2(3). By (R4), $\tau(v_1 \to v) \geqslant \frac{5}{6}$. Therefore, $w'(v) \geqslant 1 + 2 + \frac{5}{6} + 1 - 2 \times \frac{5}{3} - 1 = \frac{1}{2}$.

(3) $d(v) = 4$.

Then $w(v) = 4$, and $d_2(v) = d_3(v) = 0$ by (P3). By **VAL**, $d_6(v) \geqslant 2$. If $m_3(v) \leqslant 2$, then $w'(v) \geqslant 4 + 2 \times \frac{2}{3} - 2 \times \frac{5}{3} - 2 \times 1 = 0$ by (R1) and Observation 2(2).

- Assume that $m_3(v) = 3$, say $d(f_i) = 3$ for $i = 1, 2, 3$, and $d(f_0) \geqslant 4$. If $d_6(v) \geqslant 3$, then $w'(v) \geqslant 4 + 3 \times \frac{2}{3} - 3 \times \frac{5}{3} - 1 = 0$. So suppose that $d_6(v) = 2$ by **VAL**, which implies that $d_5(v) = 2$. By symmetry, we need to consider the following four possibilities:

If $d(v_1) = d(v_2) = 5$, then $d_6(v_1) = 3$, $d_5(v_1) = d_4(v_1) = 1$, and $d_3(v_1) = 0$. After

(R1)-(R3), $\alpha(v_1) \geqslant 7 - 4 \times \frac{5}{3} - 1 + 3 \times \frac{1}{2} = \frac{5}{6}$. By (R4), $\tau(v_1 \to v) \geqslant \frac{5}{6}$ and hence $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + \frac{5}{6} - 3 \times \frac{5}{3} - 1 = \frac{1}{6}$ by (R1) and Observation 2(2).

If $d(v_0) = d(v_1) = 5$, then $d_6(v_i) \geqslant 3$, $d_3(v_i) = 0$, and $d_4(v_i) \leqslant 2$ for $i = 0, 1$. After (R1)-(R3), $\alpha(v_i) \geqslant 7 - 4 \times \frac{5}{3} - 1 + 3 \times \frac{1}{2} = \frac{5}{6}$. By (R4), $\tau(v_i \to v) \geqslant \frac{5}{12}$ for $i = 0, 1$. Hence $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + 2 \times \frac{5}{12} - 3 \times \frac{5}{3} - 1 = \frac{1}{6}$.

If $d(v_1) = d(v_3) = 5$, then for $i \in \{1, 3\}$, $d_6(v_i) \geqslant 3$, $d_4(v_i) \leqslant 2$, $d_3(v_i) = 0$. Note that v_1 is adjacent to a 6-vertex which is adjacent to the 4-vertex v, and v_3 is adjacent to two 6-vertices which are adjacent to the 4-vertex v. After (R1)-(R3), $\alpha(v_1) \geqslant 7 - 4 \times \frac{5}{3} - 1 + 2 \times \frac{1}{2} + \frac{2}{3} = 1$, and $\alpha(v_3) \geqslant 7 - 5 \times \frac{5}{3} + \frac{1}{2} + 2 \times \frac{2}{3} = \frac{1}{2}$ by Observation 2 and its proof. By (R4), $\tau(v_1 \to v) \geqslant \frac{1}{2}$, and $\tau(v_3 \to v) \geqslant \frac{1}{4}$, hence $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + \frac{1}{2} + \frac{1}{4} - 3 \times \frac{5}{3} - 1 = \frac{1}{12}$ by (R1) and Observation 2(2).

If $d(v_2) = d(v_3) = 5$, then for $i \in \{2, 3\}$, $d_6(v_i) = 3$, $d_4(v_i) = d_5(v_i) = 1$, $d_3(v_i) = 0$, and v_i is adjacent to a 6-vertex which is adjacent to the 4-vertex v. After (R1)-(R3), $\alpha(v_i) \geqslant 7 - 5 \times \frac{5}{3} + 2 \times \frac{1}{2} + \frac{2}{3} = \frac{1}{3}$ by Observation 2 and its proof. By (R4), $\tau(v_i \to v) \geqslant \frac{1}{3}$ for $i = 2, 3$, hence $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + 2 \times \frac{1}{3} - 3 \times \frac{5}{3} - 1 = 0$ by (R1) and Observation 2(2).

• Assume that $m_3(v) = 4$. Then v is bad. If $d_6(v) = 4$, then $w'(v) \geqslant 4 + 4 \times \frac{2}{3} - 4 \times \frac{5}{3} = 0$ by (R1) and Observation 2(2). If v is adjacent to a 4-vertex, say v_1, then by Lemma 3, every vertex in $N(N(v, v_1)) \setminus \{v, v_1\}$ is of degree 6. It follows that $\tau(v_3 \to v) = 2$ and $\tau(v_i \to v) = 1$ for $i = 0, 2$, hence $w'(v) \geqslant 4 + 2 + 2 \times 1 - 4 \times \frac{5}{3} = \frac{4}{3}$. So assume that $d_4(v) = 0$, $2 \leqslant d_6(v) \leqslant 3$ by **VAL**, hence $1 \leqslant d_5(v) \leqslant 2$. Let $d(v_1) = 5$. Then $d_6(v_1) \geqslant 3$. By Claim 8, y sends at most 7 (=max$\{3 \times \frac{5}{3} + 2, 4 \times \frac{5}{3}\}$) to incident faces. We see that $\alpha(v_1) \geqslant 7 - 7 + 3 \times \frac{1}{2} = \frac{3}{2}$ by Observation 2(3). By (R4), $\tau(v_1 \to v) \geqslant \frac{3}{4}$. If $d_5(v) = 1$, then $d_6(v) = 3$, and therefore $w'(v) \geqslant 4 + 3 \times \frac{2}{3} + \frac{3}{4} - 4 \times \frac{5}{3} = \frac{1}{12}$.

Now, assume that $d_5(v) = 2$. We have two subcases to be handled. If $d(v_2) = 5$, then $d_6(v_i) = 3$, $d_4(v_i) = 1$ and $d_3(v_i) = 0$ for $i = 1, 2$. After (R1)-(R3), $\alpha(v_i) \geqslant 7 - 7 + 3 \times \frac{1}{2} = \frac{3}{2}$. By (R4), $\tau(v_i \to v) \geqslant \frac{3}{2}$ for $i = 1, 2$. Hence $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + 2 \times \frac{3}{2} - 4 \times \frac{5}{3} = \frac{5}{3}$. If $d(v_3) = 5$, then $d_6(v_i) \geqslant 3$, $d_4(v_i) \leqslant 2$, $d_3(v_i) = 0$ for $i = 1, 3$. Thus, $\alpha(v_i) \geqslant 7 - 7 + 3 \times \frac{1}{2} = \frac{3}{2}$, $\tau(v_i \to v) \geqslant \frac{3}{4}$ for $i = 1, 3$, and $w'(v) \geqslant 4 + 2 \times \frac{2}{3} + 2 \times \frac{3}{4} - 4 \times \frac{5}{3} = \frac{1}{6}$ accordingly.

(4) $d(v) = 5$.

Then $w(v) = 7$. We note that $d_6(v) \geqslant 2$, $d_2(v) = 0$, and $d_3(v) \leqslant 1$ by **VAL**. If $m_3(v) \leqslant 4$, then $\alpha(v) \geqslant 7 + 2 \times \frac{1}{3} - 4 \times \frac{5}{3} - 1 = 0$ by (R1) and Observation 2(3). So assume that $m_3(v) = 5$. If $d_3(v) = 1$, then $d_6(v) = 4$ and henceforth $\alpha(v) \geqslant 7 + 4 \times \frac{1}{3} - 5 \times \frac{5}{3} = 0$. Furthermore, assume that $d_3(v) = 0$. If $d_6(v) \geqslant 3$, then $\alpha(v) \geqslant 7 + 3 \times \frac{1}{2} - 5 \times \frac{5}{3} = \frac{1}{6}$ by Observation 2(3). Otherwise, we conclude that $d_6(v) = 2$ and $d_5(v) = 3$. This shows that v is a bad 5-vertex. Let y be an arbitrary 6-vertex adjacent to v. By Claim 7, $m_{8^+}(y) \geqslant 2$. Thus, $\alpha(y) \geqslant (3 \times 6 - 8) - 4 \times \frac{5}{3} - \frac{7}{3} = 1$ by Observation 1. Note that y is adjacent to at most three bad 5-vertices by Claim 9. Thus, $\tau(y \to v) \geqslant \frac{1}{3}$ by (R5), and hence $w'(v) \geqslant 7 + 2 \times \frac{1}{2} + 2 \times \frac{1}{3} - 5 \times \frac{5}{3} = \frac{1}{3}$.

(5) $d(v) = 6$.

We see that $w(v) = 10$, $d_6(v) \geqslant 2$, $d_2(v) \leqslant 1$, and $m_3(v) \leqslant 4$ since G contains no 7-cycles. Instead of showing that $w'(v) \geqslant 0$, we need only to prove that $\alpha(v) \geqslant 0$. If $m_3(v) \leqslant 1$, then $\alpha(v) \geqslant 10 - \frac{7}{3} - 1 \times \frac{5}{3} - 5 \times 1 = 1$ by (R1) and Observation 1. If $m_3(v) = 2$ and v is not the master of any vertex, then $\alpha(v) \geqslant 10 - \frac{7}{3} - 2 \times \frac{5}{3} - 4 \times 1 = \frac{1}{3}$.

If $m_3(v) = 2$, and moreover v is the master of some vertex, then by the analysis following the proof of Claim 5, we know that $m_4(v) = 2$ and v has exactly one slave. Thus, $\alpha(v) \geqslant 10 - \frac{7}{3} - 2 \times \frac{5}{3} - 2 \times 1 - 2 \times \frac{3}{5} - 1 = \frac{2}{15}$ by (R1) and (R3). If $m_3(v) = 3$, then by Claim 6, $m_4(v) \leqslant 2$, or $m_4(v) = 3$ and $d_2(v) = d_3(v) = 0$. For the former, $\alpha(v) \geqslant 10 - \frac{7}{3} - 3 \times \frac{5}{3} - 2 \times 1 - \frac{3}{5} = \frac{1}{15}$. For the latter, v sends at most 2 to adjacent small vertices, thus $\alpha(v) \geqslant 10 - 2 - 3 \times \frac{5}{3} - 3 \times 1 = 0$.

Finally, assume that $m_3(v) = 4$. If $m_4(v) = 2$, then v is a slave of some vertex, hence $\alpha(v) \geqslant 10 + 1 - \frac{7}{3} - 4 \times \frac{5}{3} - 2 \times 1 = 0$ by (R3) and Observation 1. If $m_4(v) = 1$, then by Claim 5(2), $m_{8^+}(v) = 1$ and hence $\alpha(v) \geqslant 10 - \frac{7}{3} - 4 \times \frac{5}{3} - 1 = 0$. If $m_4(v) = 0$, then by Claim 5(3) and (4), $m_5(v) = 1$ and $m_{6^+}(v) = 1$, or $m_{6^+}(v) = 2$. Thus, we always have $\alpha(v) \geqslant 10 - \frac{7}{3} - 4 \times \frac{5}{3} - \frac{3}{5} - \frac{1}{3} = \frac{1}{15}$. □

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