Roman domination number of the Cartesian products of paths and cycles

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Abstract

Roman domination is an historically inspired variety of domination in graphs, in which vertices are assigned a value from the set $\{0, 1, 2\}$ in such a way that every vertex assigned the value 0 is adjacent to a vertex assigned the value 2. The *Roman domination number* is the minimum possible sum of all values in such an assignment. Using an algebraic approach we present an $O(C)$-time algorithm for computing the Roman domination numbers of special classes of graphs called polygraphs, which include rotagraphs and fasciagraphs. Using this algorithm we determine formulas for the Roman domination numbers of the Cartesian products of the form $P_n \square P_k$, $P_n \square C_k$, for $k \leqslant 8$ and $n \in \mathbb{N}$, and $C_n \square P_k$ and $C_n \square C_k$, for $k \leqslant 6$ and $n \in \mathbb{N}$, for paths P_n and cycles C_n. We also find all special graphs called Roman graphs in these families of graphs.

1 Introduction

The domination number of a graph and its many variations have been extensively studied in the literature along with their algorithmic aspects [24, 23]. It is well known that the problem of computing the domination number of an arbitrary graph is NP-complete [24]. It is therefore worthwhile to construct algorithms for computing domination numbers of

graphs in various subclasses of graphs. Among them, grids (i.e. Cartesian products of paths) are widely studied because they represent interconnection models of multiprocessors in VLSI systems. The domination numbers of the Cartesian products, for several fixed values of k, were computed for $P_n \Box P_k$ in [1, 8, 11, 19, 20], for $C_n \Box C_k$ in [14, 35, 52] and for $P_n \Box C_k$ in [42]. A general $O(\log n)$ algorithm based on path algebra in [36], can be used to compute the domination number of $P_n \Box P_k$, for any fixed k. This algorithm can also be used to compute distance based invariants [32] and domination numbers [53] in polygraphs in constant time, that is, the algorithm can find closed formulas for arbitrary values of n. The existence of an algorithm that can provide closed formulas for the domination numbers of all grid graphs ($P_n \Box P_k$) has been observed or claimed in [17, 40].

An interesting variety of domination that is popular because of its historical significance is called *Roman domination* [43, 48]. In the 4th century Emperor Constantine, in order to defend the Roman Empire, decreed that two types of armies should be placed in cities in such a way that the entire Empire could be secured. The first type of army was highly trained and mobile, and could move from city to city to defend against any attack. The second type of army was a local militia that was permanently stationed at a given city. The Emperor decreed that no mobile army could ever leave a city to defend another if in doing so it left the originating city undefended. Thus, two armies were stationed at some cities, only a local militia at others, and other cities had no army.

While the problem is still of interest in military operations research [2], it also has applications in cases where a time-critical service is to be provided with some backup. For example, a fire station should never send all emergency vehicles to answer a call.

In Roman domination, every vertex is assigned a value from the set $\{0, 1, 2\}$ in such a way that every vertex assigned the value 0 is adjacent to at least one vertex assigned the value 2. The *Roman domination number* equals the minimum sum of all values in such an assignment. A formal definition of the Roman domination number of a graph, first given in [9], is presented in Section 2.

Since the problem of computing the Roman domination number of an arbitrary graph is NP-complete [12], it is worthwhile to compute the Roman domination number of various classes of graphs [9, 18, 25, 39, 38, 46, 47]. A Vizing-like conjecture for Roman domination [49], some properties of minimum Roman dominating functions [7, 15, 29, 41, 50] and algorithmic aspects of Roman domination [16] have also been studied. One of the open problems in [9] was that of computing the Roman domination numbers of arbitrary grid graphs. The Roman domination numbers of grid graphs $P_n \Box P_k$ have been determined for $k \leqslant 4$ and $n \in \mathbb{N}$ in [9, 12]. A linear-time algorithm for computing the Roman domination number of $P_n \Box P_k$, for any fixed k, is given in [12].

In this paper we use the path algebra approach [36] to design an $O(\log n)$ algorithm for computing the Roman domination numbers of arbitrary grid graphs, and show how it can be converted to a constant time algorithm that provides closed formulas for these numbers. More precisely, the time complexity of this algorithm is independent of n and has superpolynomial time complexity in terms of k. We use the algorithm to find formulas for Roman domination numbers of $P_n \Box P_k$ and $P_n \Box C_k$ for $k \leqslant 8$ and $n \in \mathbb{N}$, and for $C_n \Box P_k$ and $C_n \Box C_k$ for $k \leqslant 6$ and $n \in \mathbb{N}$.

In Section 2 we present the concept of polygraphs, which has been widely used in chemical graph theory and elsewhere [4, 21, 31, 33]. In Section 3 we summarize a general framework for solving different problems in the classes of polygraphs, called rotagraphs and faciagraphs, that is presented in [36]. Section 4 contains the algorithm for computing the Roman domination numbers of rotagraphs and faciagraphs. Section 5 summarizes the results obtained by this algorithm, and Section 6 presents a listing of all known Roman graphs in the classes of rotagraphs and faciagraphs studied. We conclude with a couple of remarks and open problems on the topic in Section 7.

2 Preliminaries

We consider finite undirected and directed graphs. The term *graph* denotes an undirected graph, while the term *digraph* denotes a directed graph. An edge between two vertices u and v in a graph is denoted uv, while an arc from a vertex u to a vertex v in digraph is denoted (u, v). Let P_n denote a *path* on n vertices and C_n denote a *cycle* on n vertices.

In a graph $G = (V, E)$, a set $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to a vertex in D. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a γ–set.

Roman domination was formally defined in [9] as follows: For a graph $G = (V, E)$, let $f : V \longrightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$. Let $|V_i| = n_i$ for $i = 0, 1, 2$. Note that there exists a 1–1 correspondence between the functions $f : V \longrightarrow \{0, 1, 2\}$ and ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) if every vertex in V_0 is adjacent to a vertex in V_2. The weight of f is defined as:

$$w(f) = \sum_{v \in V} f(v) = n_1 + 2n_2.$$

The *Roman domination number*, $\gamma_R(G)$, equals the minimum weight of an RDF of G. We will also say that a function $f = (V_0, V_1, V_2)$ is a γ_R-*function*, if it is an RDF and $w(f) = \gamma_R(G)$. Obviously, $\gamma(G) \leqslant \gamma_R(G) \leqslant 2\gamma(G)$. The only graphs that satisfy $\gamma_R(G) = \gamma(G)$ are edgeless graphs. A graph is called a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Finding classes of Roman graphs was one of the open problems posed in [9]. For instance, among paths, graphs P_{3k} and P_{3k+2} are Roman graphs. Paths P_{3k+1} are examples of graphs which are not Roman graphs: $\gamma_R(P_{3k+1}) = 2k + 1 < 2\gamma(P_{3k+1}) = 2k + 2$.

The difference between γ_R and 2γ can be arbitrarily large. Consider, for example, the family of *subdivided stars* $S(K_{1,n})$, which are obtained from the family of stars $K_{1,n}$ by subdividing every edge. It can be seen that $\gamma_R(S(K_{1,n})) = 2 + n$, while $2\gamma(S(K_{1,n})) = 2n$. The construction of a γ-set and γ_R–function for $S(K_{1,5})$ is given in Figure 1, where the full circs in the subdivided star on the left are elements of a γ-set, while the full circs in the subdivided star on the right are elements of V_1 and the double circ vertex in the center is the only element of V_2 in the γ_R-function.
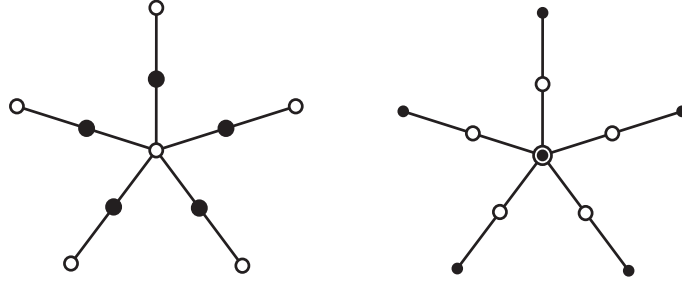
Figure 1: A minimum dominating set and a γ_R–function of $S(K_{1,5})$.

The *Cartesian product* of graphs G and H, denoted $G\Box H$, is a graph with a vertex set $V(G) \times V(H)$ and $(g,h)(g',h') \in E(G\Box H)$ if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Cartesian product graphs include the grid graphs, which are products of paths $P_n\Box P_k$, and tori, which are products of cycles $C_n\Box C_k$.

Let $G_1,\ldots,G_n$ be arbitrary mutually disjoint graphs and $X_1,\ldots,X_n$ a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$ ($X_i \subseteq V(G_i) \times V(G_{i+1})$ for $i = 1,\ldots,n$). For convenience we also set $G_{n+1} = G_1$. A *polygraph* $\Omega_n = \Omega_n(G_1,\ldots G_n; X_1,\ldots X_n)$ over monographs $G_1,\ldots,G_n$ is defined in the following way:

$$V(\Omega_n) = V(G_1) \cup \ldots \cup V(G_n),$$

$$E(\Omega_n) = E(G_1) \cup X_1 \cup \ldots \cup E(G_n) \cup X_n.$$

For a polygraph Ω_n and for $i = 1,\ldots,n$ we also define

$$D_i = \{u \in V(G_i) \mid \exists v \in G_{i+1} : uv \in X_i\},$$

$$R_i = \{u \in V(G_{i+1}) \mid \exists v \in G_i : uv \in X_i\}.$$

In general, $R_i \cap D_{i+1}$ does not have to be empty. If all graphs G_i are isomorphic to a fixed graph G (i.e. there exists an isomorphism $\varphi_i : V(G_i) \longrightarrow V(G)$ for $i = 1,\ldots,n+1$, and $\varphi_{n+1} = \varphi_1$) and all sets X_i are equal to a fixed set $X \subseteq V(G) \times V(G)$ ($(u,v) \in X \iff \left(\varphi_i^{-1}(u), \varphi_{i+1}^{-1}(v)\right) \in X_i$ for all i), we call such a graph *rotagraph*, $\omega_n(G; X)$. A rotagraph without edges between the first and the last copy of G is a *fasciagraph*, $\psi_n(G; X)$. More precisely, in a fasciagraph, $X_n = \emptyset$ and $X_1 = X,\ldots,X_{n-1} = X$. In a rotagraph as well as in a fasciagraph, all sets D_i and R_i are equal to fixed sets D and R, respectively ($D_i = \varphi_i^{-1}(D)$ and $R_i = \varphi_{i+1}^{-1}(R)$). Of course, in the case of fasciagraphs, $D_n = \emptyset$ and $R_n = \emptyset$. Observe that the Cartesian product of paths P_n and P_k, $P_n\Box P_k$, is an example of a fasciagraph and that the Cartesian product of cycles C_n and C_k, $C_n\Box C_k$, is an example of a rotagraph. A graph $P_n\Box C_k$ can be treated either as a fasciagraph or as a rotagraph.

3   Path algebras and the algorithm

Let us now summarize a general framework for solving different problems on the class of fasciagraphs and rotagraphs, which was proposed in [36] and also used in [53]:

A semiring $\mathcal{P} = (P, \oplus, \circ, e^\oplus, e^\circ)$ is a set P on which two binary operations, $\oplus$ and $\circ$ are defined such that:

1. $(P, \oplus)$ is a commutative monoid with $e^\oplus$ as a unit;

2. $(P, \circ)$ is a monoid with e° as a unit;

3. $\circ$ is left- and right-distributive over $\oplus$;

4. $\forall x \in P, x \circ e^\oplus = e^\oplus = e^\oplus \circ x$.

An idempotent semiring is called a *path algebra*. It is easy to see that a semiring is a path algebra if and only if $e^\circ \oplus e^\circ = e^\circ$ holds for e°, the unit of the monoid $(P, \circ)$. An important example of a path algebra for our work is $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$. Here $\mathbb{N}_0$ denotes the set of nonnegative integers and $\mathbb{N}$ the set of positive integers. For more examples of path algebras we refer to [6].

Let $\mathcal{P} = (P, \oplus, \circ, e^\oplus, e^\circ)$ be a path algebra and let $\mathcal{M}_n(\mathcal{P})$ be the set of all $n \times n$ matrices over P. Let $A, B \in \mathcal{M}_n(\mathcal{P})$ and define operations $\oplus$ and $\circ$ in the usual way:

$$
\begin{aligned}
(A \oplus B)_{ij} &= A_{ij} \oplus B_{ij}, \\
(A \circ B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}.
\end{aligned}
$$

$\mathcal{M}_n(\mathcal{P})$ equipped with the above defined operations is a path algebra with the zero and the unit matrix as units of the semiring. In our example $\mathcal{P}_1 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$, all elements of the zero matrix are ∞, the unit of the monoid $(P, \min)$, and the unit matrix is a diagonal matrix with diagonal elements equal to $e^\circ = 0$ and all other elements equal to $e^\oplus = \infty$.

Let $\mathcal{P}$ be a path algebra and let G be a labeled digraph, that is a digraph together with a labeling function ℓ which assigns to every arc of G an element of P. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The labeling ℓ of G can be extended to walks in the following way: For a walk $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \ldots (v_{i_{k-1}}, v_{i_k})$ of G let

$$
\ell(Q) = \ell(v_{i_0}, v_{i_1}) \circ \ell(v_{i_1}, v_{i_2}) \circ \ldots \circ \ell(v_{i_{k-1}}, v_{i_k}).
$$

Let S_{ij}^k be the set of all walks of order k from v_i to v_j in G and let $A(G)$ be the matrix defined by:

$$
A(G)_{ij} = \begin{cases} \ell(v_i, v_j); & \text{if } (v_i, v_j) \text{ is an arc of } G \\ e^\oplus; & \text{otherwise.} \end{cases}
$$

It is well-known (see for example [6]) that

$$
\left(A(G)^k \right)_{ij} = \bigoplus_{Q \in S_{ij}^k} \ell(Q).
$$

Let $\omega_n(G; X)$ be a rotagraph and $\psi_n(G; X)$ a fasciagraph. Set $U = D_i \cup R_i = D \sqcup R$ and let $N = 2^{|U|}$. Define a labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$ as follows: The vertex set

of $\mathcal{G}$ is formed by the subsets of U which will be denoted V_i. An arc joins a subset V_i with a subset V_j if V_i is not in a "conflict" with V_j. Here a conflict of V_i with V_j means that using V_i and V_j as a part of a solution in consecutive copies of G would violate the problem assumption. For instance, if we look for a domination number of a graph, such a conflict would be a nonempty intersection between sets V_i and V_j, or if we look for an independence number of a graph, such a conflict would be an edge between sets V_i and V_j. Let finally $\ell : E(\mathcal{G}) \longrightarrow P$ be a labeling of $\mathcal{G}$ where $\mathcal{P}$ is a path algebra on the set P. The general scheme for the algorithm that solves different problems on polygraphs as proposed in [36] is:

Algorithm 1 [36]

1. Select the appropriate path algebra $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$.

2. Determine an appropriate labeling ℓ of a graph $\mathcal{G}(G; X)$.

3. In $\mathcal{M}(\mathcal{P})$ calculate $A(\mathcal{G})^n$.

4. Among admissible coefficients of $A(\mathcal{G})^n$ select one which optimizes the corresponding goal function.

It is well known that, in general, Step 3 of the algorithm can be implemented to run in $O(\log n)$-time [44]. However, computing the powers of $A(\mathcal{G})^n = A_n$ in $O(C)$-time is possible using special structure of the matrices in some cases, including the distance based invariants [32], the domination numbers [53], and others [31, 33, 54]. Here we prove that $A(\mathcal{G})^n = A_n$ can be computed in $O(C)$-time for the Roman domination number (see Section 4).

4 The Roman domination number of fasciagraphs and rotagraphs

Let $\omega_n(G; X)$ be a rotagraph and $\psi_n(G; X)$ a fasciagraph as defined above. Set $U = D_i \cup R_i = D \sqcup R$. Keep in mind that $D_i \subseteq G_i$ and $R_i \subseteq G_{i+1}$, but since $R_i = R$ and $D_i = D$ for all i, we can write $U = D_i \cup R_i = D \sqcup R$. A labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$ is a graph with a vertex set:

$$V(\mathcal{G}) = \{(V_i, W_i) \mid V_i, W_i \subseteq U, V_i \cap W_i = \emptyset\}$$

For convenience we sometimes refer to a vertex of $\mathcal{G}$ shortly by $v_i = (V_i, W_i)$. In particular, $v_0 = (V_0, W_0)$ stands for $(\emptyset, \emptyset)$.

Let $v_i, v_j \in V(\mathcal{G})$ and consider for a moment $\psi_3(G; X)$. Let $V_i \cup W_i \subseteq D_1 \cup R_1$ and $V_j \cup W_j \subseteq D_2 \cup R_2$. Let $\gamma_{R_{i,j}}(G; X)$ be the weight of a γ_R-function of a graph $G_2 \setminus ((((V_i \cup W_i) \cap R_1) \cup (D_2 \cap (V_j \cup W_j))))$, such that $V_i \cup V_j \subseteq V'$ and $W_i \cup W_j \subseteq W'$,
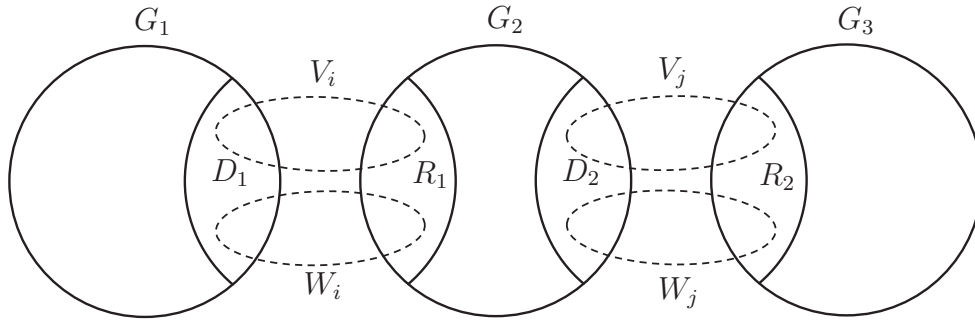
Figure 2: $\psi_3(G; X)$ with the above notation.

where $(V(G_2) \setminus (V' \cup W'), V', W')$ is an RDF of a graph G_2. For consistency, we introduce an arc between vertices v_i and v_j only if $R \cap V_i \cap W_j \cap D = \emptyset$ and $R \cap W_i \cap V_j \cap D = \emptyset$. Set

$$\ell(v_i, v_j) = |R \cap V_i| + 2\,|R \cap W_i| + |V_j \cap D| + 2\,|W_j \cap D| -$$
$$- |R \cap V_i \cap V_j \cap D| - 2\,|R \cap W_i \cap W_j \cap D| + \gamma_{R_{i,j}}(G; X). \tag{1}$$

Then we have an algorithm which computes the Roman domination number of rotagraphs and fasciagraphs:

Algorithm 2

1. For a path algebra select $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

2. Label $\mathcal{G} = \mathcal{G}(G; X)$ as defined in (1).

3. In $\mathcal{M}(\mathcal{P})$ calculate $A\,(\mathcal{G})^n$.

4. Let $\gamma_R\,(\psi_n(G; X)) = (A\,(\mathcal{G})^n)_{00}$ and $\gamma_R\,(\omega_n(G; X)) = \min_i\,(A\,(\mathcal{G})^n)_{ii}$.

Theorem 4.1 *The Algorithm 2 correctly computes the Roman domination number of rotagraphs and fasciagraphs:*

$$\gamma_R\,(\psi_n(G; X)) = (A\,(\mathcal{G})^n)_{00} \tag{2}$$

$$\gamma_R\,(\omega_n(G; X)) = \min_i\,(A\,(\mathcal{G})^n)_{ii} \tag{3}$$

in $O(\log n)$-time.

Proof. Let G_1 and G_2 be arbitrary graphs, X_1 a set of edges between vertices of G_1 and G_2 and let $\Omega_2(G_1, G_2; X_1, \emptyset)$ be a polygraph. Let also $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$

be a path algebra and let $\mathcal{G}'$ be a labeled digraph for Ω_2 defined as above. Then, by the definition of the labeling, we have

$$\gamma_R \left(\Omega_2 \left(G_1, G_2; X_1, \emptyset \right) \right) = \left[A(G_1) + A(G_2) \right]_{00}$$

$$= \left[\min_{v_k \in V(\mathcal{G})} \{ \ell(0, v_k) + \ell(v_k, 0) \} \right]_{00}.$$

Let $G_1 = G$, $X_1 = X$ and $G_2 = \psi_{n-1}(G; X)$. Then (2) follows by induction.

For (3), similarly, consider $\Omega_2(G_1, G_2; X_1, X_2)$ and let $G_1 = G$, $X_1 = X_2 = X$ and $G_2 = \psi_{n-1}(G; X)$.

The time complexity of the algorithm was already discussed for a general case in Section 3. $\qquad\square$

As mentioned before we prove that calculating the powers of matrices $A(\mathcal{G})$, $A(\mathcal{G})^n = A_n$ (and therefore implementing the algorithm) is possible in $O(C)$-time based on the following lemma:

Lemma 4.2 *Let $k = |V(\mathcal{G}(G; X))|$ and $K = |V(G)|$. Then there is an index $q \leqslant (4K + 2)^{k^2}$ such that $A_q = A_p + C$ for some index $p < q$ and some constant matrix C. Let $P = q - p$. Then for every $r \geqslant p$ and every $s \geqslant 0$ we have*

$$A_{r+sP} = A_r + sC.$$

Proof. First observe that for any $l \geqslant 1$, the difference between any pair of entries of A_l, both different from ∞, is bounded by $4K$. Assume $(A_l)_{ij} \neq \infty$. Then

$$(A_l)_{ij} = \gamma_R \left((V(G_1) \setminus (V_i \cup W_i)) \cup V(G_2) \cup \ldots \cup V(G_{l-1}) \cup (V(G_l) \setminus (V_j \cup W_j)) \right)$$
$$\leqslant \gamma_R(\psi_l(G; X)).$$

Since $V_i \cap W_i = \emptyset$ and $V_j \cap W_j = \emptyset$ it follows that

$$|R \cap V_i| + 2\,|R \cap W_i| + |V_j \cap D| + 2\,|W_j \cap D| \leqslant 4|V(G)|.$$

According to (1) we have

$$\ell(v_i, v_j) \leqslant 4|V(G)| + \gamma_{R_{i,j}} = 4|V(G)| + (A_l)_{ij}$$
$$(A_l)_{ij} \geqslant \ell(v_i, v_j) - 4|V(G)| \geqslant \gamma_R \left(\psi_l(G; X) \right) - 4|V(G)|.$$

Therefore
$$\gamma_R \left(\psi_l(G; X) \right) - 4|V(G)| \leqslant (A_l)_{ij} \leqslant \gamma_R(\psi_l(G; X)).$$

For $l \geqslant 1$, let $K_l = \min\{(A_l)_{ij}\}$ and let $A_l' = A_l - (K_l)J$, where J is the matrix with all entries equal to 0 (recall that we are still in the path algebra $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$). Since the difference between any two elements of A_l, different from ∞, cannot be greater than $4K$, the entries of A_l' can only have values $0, 1, \ldots, 4K, \infty$. Hence there are indices $p < q \leqslant (4K + 2)^{k^2}$ such that $A_p' = A_q'$. This proves the first part of the lemma.

The equality $A_{r+sP} = A_r + sC$ follows from the fact that for arbitrary matrices D, E and a constant matrix C we have $(D + C) \circ E = D \circ E + C$, where $+$ is the ordinary matrix addition, i.e. $(A + B)_{ij} = A_{ij} + B_{ij}$ for all i, j:

$$((D + C) \circ E)_{ij} = \min_k\{((D)_{ik} + C) + (E)_{kj}\} = \min_k\{(D)_{ik} + (E)_{kj}\} + C$$
$$(D \circ E + C)_{ij} = \min_k\{(D)_{ik} + (E)_{kj}\} + (C)_{ij} = \min_k\{(D)_{ik} + (E)_{kj}\} + C.$$

Therefore, let $A_q = A_p + C$ for some index $p < q$ and some constant matrix $C = [c]_{ij}$. Then

$$A_{q+1} = (A_p + C) \circ A$$
$$= (A_p \circ A) + C$$
$$= A_{p+1} + C.$$

Let $P = q - p$ and $r \geqslant p$. Then also

$$A_{r+P} = A_{r+q-p} = A_q \circ A_{r-p}$$
$$= (A_p + C) \circ A_{r-p} = A_{p+r-p} + C$$
$$= A_r + C,$$

and by induction on s we have

$$A_{r+sP} = A_{r+P+(s-1)P} = (A_r + C) \circ A_{(s-1)P} = A_{r+(s-1)P} + C$$
$$= A_r + (s-1)C + C = A_r + sC$$

for every $s \geqslant 0$. $\qquad\qquad\square$

Remark 4.3 *Consider the Roman domination number of a fasciagraph $P_n \square P_k$, where k is a fixed number. Then $q < (4k + 2)^{2^{3k}}$. In Table 1 we give the upper bounds for q for the investigated grids. Observe that already in the cases of very small monographs, enormously large q are obtained. That is why the second part of Lemma 4.2 is useful for practical purposes - once a period is detected, it cannot change. As we implemented the improved algorithm, the period was always found much sooner. Exact values are presented in the third column of Table 1.*

Hence, Lemma 4.2 says that if we assume the size of G is a given constant (and n is a variable), then the algorithm will run in constant time. But it is important to emphasize that the algorithm is useful for practical purposes only if the number of vertices of the monograph G is relatively small, since the time complexity is in general exponential in the number of vertices of the monograph G.

Table 1: The upper bound for q and actual values computed by the algorithm.

$P_n \square P_k$	the upper bound for q from Lemma 4.2	value computed by the algorithm
$k = 2$	10^{64}	3
$k = 3$	$14^{512} \approx 6 \cdot 10^{586}$	6
$k = 4$	$18^{4096} \approx 4 \cdot 10^{5141}$	9
$k = 5$	22^{32768}	11
$k = 6$	26^{262144}	9
$k = 7$	$30^{2097152}$	16
$k = 8$	$34^{16777216}$	11

5 Cartesian products of paths and cycles

Our aim is to calculate the Roman domination number for graphs $P_n \square P_k$, $P_n \square C_k$, $C_n \square P_k$ and $C_n \square C_k$ for some fixed values of k. These graphs are isomorphic to special classes of fasciagraphs and rotagraphs (i.e. fasciagraphs and rotagraphs where $G = P_k$ or $G = C_k$ and where X is a matching between two copies of G). Therefore, calculating exact formulas for their Roman domination number is possible because of the Algorithm 2 together with Lemma 4.2. But with increasing k matrices $A(G)^n$ become bigger and bigger. Therefore we had to improve straightforward implementation of the algorithm. Instead of calculating all elements of the powers of matrices $A(G)^n$, we calculated only those rows which are important for the result. Then we checked the difference of the new row against the previously stored rows until a constant difference was detected. This yields a correct result because of the next lemma, adaptation of a lemma from [54].

Lemma 5.1 *Assume that the j-th row of A_{n+P} and A_n differ for a constant, $a_{ji}^{(n+P)} = a_{ji}^{(n)} + C$ for all i. Then $\min_i a_{ji}^{(n+P)} = \min_i a_{ji}^{(n)} + C$.*

Proof. Let $a_{jl}^{(n+P)} = \min_i a_{ji}^{(n+P)}$ and assume that there exists $t \neq l$ such that $a_{jl}^{(n)} > a_{jt}^{(n)}$. It follows that
$$a_{jt}^{(n+P)} = a_{jt}^{(n)} + C < a_{jl}^{(n)} + C = a_{jl}^{(n+P)},$$
which contradicts the minimality of $a_{jl}^{(n+P)}$. $\qquad\square$

Obviously, such an improvement is crucial in the case of fasciagraphs: recall that $\gamma_R (\psi_n(G; X)) = (A(\mathcal{G})^n)_{00}$. For computing 00-th element of A_n we only need the first rows of the matrices A_j for $2 \leqslant j \leqslant n - 1$:

$$A_{n_{0j}} = \min_i \left\{ A_{n-1_{0i}} + A_{ij} \right\}.$$

This is not the case for rotagraphs because $\gamma_R\left(\omega_n(G;X)\right) = \min_i\left(A\left(\mathcal{G}\right)^n\right)_{ii}$.

By implementation of the improved algorithm we got formulas presented in the following subsections. For each case also constructions of γ_R-functions are presented. In every figure that follows we only emphasized vertices of V_1 and V_2 of a γ_R-function of a depicted graph in a way that a single full circ represents a vertex of V_1 and a double circ represents a vertex of V_2.

5.1 $\gamma_R(P_n \square P_k)$ for $k \in \{5, 6, 7, 8\}$

The Roman domination number of grid graphs was studied in [9, 12] and the following results were established:

$$\gamma_R\left(P_n\right) = \gamma_R\left(C_n\right) = \left\lceil \frac{2n}{3} \right\rceil$$

$$\gamma_R\left(P_n \square P_2\right) = n + 1$$

$$\gamma_R\left(P_n \square P_3\right) = \begin{cases} \left\lfloor \frac{3n}{2} \right\rfloor + 2; & \text{if } n \in \{4k + 3 \mid k \in \mathbb{N} \cup \{0\}\} \\ \left\lfloor \frac{3n}{2} \right\rfloor + 1; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(P_n \square P_4\right) = \begin{cases} 2n + 1; & \text{if } n \in \{1, 2, 3, 5, 6\} \\ 2n; & \text{otherwise} \end{cases}$$

No formulas were given for $k > 4$. However, in [12] the author also proposed an algorithm for computing $\gamma_R\left(P_n \square P_k\right)$ for any fixed value of k in $O(n)$-time. By implementing the improved Algorithm 2 as already discussed above, we obtained formulas given below. We also looked for the γ_R-functions of graphs $P_n \square P_k$ for the investigated k and all $n \in \mathbb{N}$. They are depicted in Figures 3 to 6.

$$\gamma_R\left(P_n \square P_5\right) = \begin{cases} 8; & \text{if } n = 3 \\ \left\lfloor \frac{12n}{5} \right\rfloor + 2; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(P_n \square P_6\right) = \begin{cases} \left\lfloor \frac{14n}{5} \right\rfloor + 2; & \text{if } n < 5 \text{ or } n \in \{5k, 5k + 3, 5k + 4 \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{14n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(P_n \square P_7\right) = \begin{cases} \left\lfloor \frac{16n}{5} \right\rfloor + 2; & \text{if } n \in \{1, 2, 4, 7, 5k \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{16n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(P_n \square P_8\right) = \begin{cases} 9; & \text{if } n = 2 \\ 16; & \text{if } n = 4 \\ \left\lfloor \frac{18n}{5} \right\rfloor + 4; & \text{if } n \in \{5k + 3 \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{18n}{5} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

The Roman domination numbers for small grids are presented in Table 2.

Table 2: Roman domination number of some $P_n \square P_k$.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	10
2		3	4	5	6	7	8	9	10	11	12	13	14	15	16
3			6	7	8	10	12	13	14	16	18	19	20	22	24
4				8	11	13	14	16	18	20	22	24	26	28	30
5					14	16	18	21	23	26	28	30	33	35	38
6						19	22	24	27	30	33	36	38	41	44
7							24	28	31	34	38	41	44	47	50
8								32	35	39	42	46	50	53	57

5.2 $\gamma_R(P_n \square C_k)$ for $k \in \{3, 4, 5, 6, 7, 8\}$

In the literature we found no results on the Roman domination numbers of graphs $P_n \square C_k$. Our formulas are given below and constructions for each case are depicted in Figures 7 to 12.

$$\gamma_R (P_n \square C_3) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$$

$$\gamma_R (P_n \square C_4) = \begin{cases} 3; & \text{if } n = 1 \\ 2n; & \text{otherwise} \end{cases}$$

$$\gamma_R (P_n \square C_5) = 2n + 2$$

$$\gamma_R (P_n \square C_6) = \left\lfloor \frac{8n}{3} \right\rfloor + 2$$

$$\gamma_R (P_n \square C_7) = \begin{cases} 3n + 2; & \text{if } n \in \{1, 2, 4\} \\ 3n + 3; & \text{otherwise} \end{cases}$$

$$\gamma_R (P_n \square C_8) = \begin{cases} 8; & \text{if } n = 2 \\ \left\lfloor \frac{7n}{2} \right\rfloor + 2; & \text{if } n \in \{3, 4, 8\} \\ \left\lfloor \frac{7n}{2} \right\rfloor + 3; & \text{otherwise} \end{cases}$$

5.3 $\gamma_R(C_n \square P_k)$ for $k \in \{2, 3, 4, 5, 6\}$ and $n \geqslant 3$

We implemented this case as a rotagraph. From (3) we know that the calculations for rotagraphs take much more time than the calculations for fasciagraphs. Therefore we covered only the cases for $k \in \{2, 3, 4, 5, 6\}$. As in former cases, formulas for the Roman domination number are presented below and constructions can be found in Figures 13 to

18.

$$\gamma_R\left(C_n \square P_2\right) = \begin{cases} n; & \text{if } n \in \{4k \mid k \in \mathbb{N}\} \\ n+1; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \square P_3\right) = \begin{cases} 5; & n = 3 \\ \left\lceil \frac{3n}{2} \right\rceil; & \text{if } n \in \{4k, 4k+1 \mid k \in \mathbb{N}\} \\ \left\lceil \frac{3n}{2} \right\rceil + 1; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \square P_4\right) = \begin{cases} 7; & \text{if } n = 3 \\ 2n; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \square P_5\right) = \begin{cases} \left\lceil \frac{12n}{5} \right\rceil + 1; & \text{if } n \in \{5k+2 \mid k \in \mathbb{N}\} \\ \left\lceil \frac{12n}{5} \right\rceil; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \square P_6\right) = \begin{cases} \left\lfloor \frac{14n}{5} \right\rfloor; & \text{if } n \in \{5k \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{14n}{5} \right\rfloor + 1; & \text{if } n \in \{5k+4 \mid k \in \mathbb{N}_0\} \\ \left\lfloor \frac{14n}{5} \right\rfloor + 2; & \text{otherwise} \end{cases}$$

5.4 $\gamma_R(C_n \square C_k)$ for $k \in \{3, 4, 5, 6\}$ and $n \geqslant 3$

In [18], the authors showed that $\gamma_R\left(C_{5n} \square C_{5m}\right) = 10mn$, which is consistent with our calculations. We found no other formulas for the Roman domination numbers of graphs $C_n \square C_k$. Constructions for each case can be found in Figures 19 to 23.

$$\gamma_R\left(C_n \square C_3\right) = \left\lceil \frac{3n}{2} \right\rceil$$

$$\gamma_R\left(C_n \square C_4\right) = 2n$$

$$\gamma_R\left(C_n \square C_5\right) = \begin{cases} 2n; & \text{if } n \in \{5k \mid k \in \mathbb{N}\} \\ 2n+2; & \text{otherwise} \end{cases}$$

$$\gamma_R\left(C_n \square C_6\right) = \begin{cases} \left\lfloor \frac{8n}{3} \right\rfloor; & \text{if } n \in \{6k \mid k \in \mathbb{N}\} \\ \left\lfloor \frac{8n}{3} \right\rfloor + 1; & \text{if } n \in \{6k+5, 18k+3, 18k+8, 18k+13 \mid k \in \mathbb{N}_0\} \\ \left\lfloor \frac{8n}{3} \right\rfloor + 2; & \text{otherwise} \end{cases}$$

6 Roman graphs

Combining results obtained here and known results on the domination number of graphs [1, 8, 19, 35, 42] we also looked for Roman graphs (graphs, satisfying $\gamma_R(G) = 2\gamma(G)$). In the cases where the domination numbers of graphs have not been calculated yet, we also

refer to a simple observation: a graph G cannot be a Roman graph if $\gamma_R(G)$ is an odd number. It is also known that a graph is a Roman graph if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$ ([9], Proposition 16). Except in cases $P_n \square C_5$, $P_n \square C_8$, $C_n \square P_5$, $C_n \square C_5$ and $C_n \square C_6$, the following is a complete listing of all Roman graphs in the classes of graphs that we have studied.

1. Roman graphs among $P_n \square P_k$:

$k = 1$: $n \in \{3l + 2, 3l + 3 \mid l \in \mathbb{N}_0\}$

$k = 2$: n odd

$k = 3$: $n \in \{4l + 1, 4l + 2, 4l + 3 \mid l \in \mathbb{N}\}$

$k = 4$: $n \in \mathbb{N} \setminus \{1, 2, 3, 5, 6, 9\}$

$k = 5$: $n \in \{1, 2, 3, 7, 5l, 5l + 1 \mid l \in \mathbb{N}\}$

$k = 6$: $n \in \{1, 3, 5, 7, 8, 12, 15, 22\}$

$k = 7$: $n \in \{2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 16\}$

$k = 8$: $n \in \{1, 4, 6, 7, 8\}$

2. Roman graphs among $P_n \square C_k$:

$k = 3$: $n \in \{4l + 1, 4l + 2 \mid l \in \mathbb{N}_0\}$

$k = 4$: $n \geqslant 2$

$k = 5$: $n \in \{1, 2, 3\}$

$k = 6$: $n \in \{1, 3, 4, 6, 6l + 1, 6l + 3, 6l + 4, 6l + 6 \mid l \in \mathbb{N}\}$

$k = 7$: $n \in \{2, 4, 2l + 1 \mid l \in \mathbb{N}_0\}$

$k = 8$: $n \in \{1, 2, 3, 4, 5, 6\}$

3. Roman graphs among $C_n \square P_k$:

$k = 2$: $n \in \{4l, 4l + 1, 4l + 3 \mid l \in \mathbb{N}\}$

$k = 3$: $n \geqslant 4$

$k = 4$: $n \in \mathbb{N} \setminus \{3, 5, 9\}$

$k = 5$: $n \in \{3, 4, 7, 8, 10l, 10l + 4, 10l + 7, 10l + 8 \mid l \in \mathbb{N}\}$

$k = 6$: $n \in \{3, 4, 6, 8, 11, 14, 18, 28\}$

4. Roman graphs among $C_n \square C_k$:

$k = 3$: $n \in \{4l, 4l + 1 \mid l \in \mathbb{N}\}$

$k = 4$: $n \in \mathbb{N}$

$k = 5$: $n \in \{3, 4, 5l, 5l + 1, 5l + 2, 5l + 4 \mid l \in \mathbb{N}\}$

$$k = 6: \quad n \in \{6l, 6l + 4, 18l + 1, 18l + 5, 18l + 7, \mid l \in \mathbb{N}\}$$

Remark 6.1 *It was proven in [42] that* $n + \left\lceil \frac{n}{5} \right\rceil \leqslant \gamma\left(C_n \square P_5\right) \leqslant n + \left\lceil \frac{n}{4} \right\rceil$. *In fact, we show here that since* $C_{10i} \square P_5$, $C_{10i+4} \square P_5$, $C_{10i+7} \square P_5$ *and* $C_{10i+8} \square P_5$ *are Roman graphs (see Figure 17), their domination number equals* $\frac{\gamma_R}{2}$. *More precisely,*

- $\gamma_R(C_{10i} \square P_5) = \left\lceil \frac{12(10i)}{5} \right\rceil = 24i$ *hence* $\gamma(C_{10i} \square P_5) = 12i = 10i + 2i = n + \frac{n}{5}$;

- $\gamma_R(C_{10i+4} \square P_5) = \left\lceil \frac{12(10i+4)}{5} \right\rceil = 24i + 10$ *hence* $\gamma(C_{10i+4} \square P_5) = 12i + 5 = (10i + 4) + (2i + 1) = n + \left\lceil \frac{n}{5} \right\rceil$;

- $\gamma_R(C_{10i+7} \square P_5) = \left\lceil \frac{12(10i+7)}{5} \right\rceil = 24i + 17 + 1 = 24i + 18$ *hence* $\gamma(C_{10i+7} \square P_5) = 12i + 9 = (10i + 7) + (2i + 2) = n + \left\lceil \frac{n}{5} \right\rceil$;

- $\gamma_R(C_{10i+8} \square P_5) = \left\lceil \frac{12(10i+8)}{5} \right\rceil = 24i + 20$ *hence* $\gamma(C_{10i+8} \square P_5) = 12i + 10 = (10i + 8) + (2i + 2) = n + \left\lceil \frac{n}{5} \right\rceil$.

7 Summary and open problems

Let us conclude with some remarks and suggestions for future work on this topic.

1. All the implementations of the algorithm presented here were performed on a personal computer or on a small computer cluster. Because of the time and space complexity limitations we could therefore not calculate the Roman domination numbers of the Cartesian products of paths and cycles for larger monographs although theoretically we can get the exact formula for $\gamma_R(P_n \square P_k)$ for any value of k. It would be interesting to find another way to determine $\gamma_R(P_n \square P_k)$ for any $k, n \in \mathbb{N}$. Recall that Chang's conjecture about the domination number of grid graphs was open for quite a long time before Gonçalves, Pinlou, Rao and Thomassé [19] proved it. Considering the fact that the nice expression $\gamma(P_n \square P_k) = \left\lfloor \frac{(k+2)(n+2)}{5} \right\rfloor - 4$ holds for quite "large" grids, i.e. for $P_n \square P_k$ for any $16 \leqslant k \leqslant n$ we do not think we have enough information to make analogous conjecture for Roman domination on grid graphs. From the results that have been established, it seems that the Roman domination number of $P_n \square P_k$ could be of the form $\left\lfloor \frac{2(k+1)n}{5} \right\rfloor + C(k)$, where $C(k)$ is some constant, dependent of k.

2. As can be seen from γ_R-functions in Figures 3 to 23, our graphs (except $C_n \square C_6$ for some n) have property that $V_1 \cup V_2$ is an independent set. Note that in some cases two vertices of V_2 are adjacent in the figure because we wanted to show that they are Roman graphs. But in each case, one of the two adjacent vertices, which are assigned the value 2, can be replaced with two vertices of V_1 in a way that $V_1 \cup V_2$ is an independent set. Such graphs were called Roman domination perfect graphs

in [30] where it was shown that if a graph does not contain one of the forbidden induced subgraphs from a list, then it is Roman domination perfect graph. Note that all our graphs contain one of the forbidden induced subgraphs of that list, but are still Roman domination perfect graphs.

3. The same approach as used in this paper can provide algorithms for computing the Roman domination numbers of the other known graph products [22] of paths and cycles. Only bounds are known for the Roman domination number of the strong product of arbitrary graphs [51] and also $\gamma_R(P_n \boxtimes P_k)$ is not determined yet. To the best of our knowledge, no one has even tried to work on the Roman domination number of the direct product of graphs. The approach used here would make it possible to determine $\gamma_R(P_n \times P_k)$ for some fixed values of k, and more general results would be welcome. Roman domination number of the fourth well known graph product, the lexicographic product, was established in [37] using a new concept of the so-called dominating couples.

4. Several new graph invariants based on the Roman domination have been studied, for instance weak Roman domination [10, 27, 45], k-Roman domination [26], Roman k-domination [34] and others [3, 5, 13, 28]. We believe that a similar approach could yield constant time algorithms for most or even all of these graph invariants on fasciagraphs and rotagraphs.

Acknowledgments

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$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

$n = 5k + 2$
$\gamma_R = 12k + 6$

$n = 5k + 3$
$\gamma_R = 12k + 9$

$n = 5k + 4$
$\gamma_R = 12k + 11$

$n = 5k$
$\gamma_R = 12k + 2$

$n = 5k + 1$
$\gamma_R = 12k + 4$

Figure 3: $P_n \square P_5$

$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

$n = 5k + 2$
$\gamma_R = 14k + 8$

$n = 5k + 3$
$\gamma_R = 14k + 10$

$n = 5k + 4$
$\gamma_R = 14k + 13$

$n = 5k$
$\gamma_R = 14k + 2$

$n = 5k + 1$
$\gamma_R = 14k + 5$

Figure 4: $P_n \square P_6$

$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$ $n = 7$

$n = 5k + 3$
$\gamma_R = 16k + 12$

$n = 5k + 4$
$\gamma_R = 16k + 15$

$n = 5k$
$\gamma_R = 16k + 2$

$n = 5k + 1$
$\gamma_R = 16k + 6$

$n = 5k + 2$
$\gamma_R = 16k + 9$
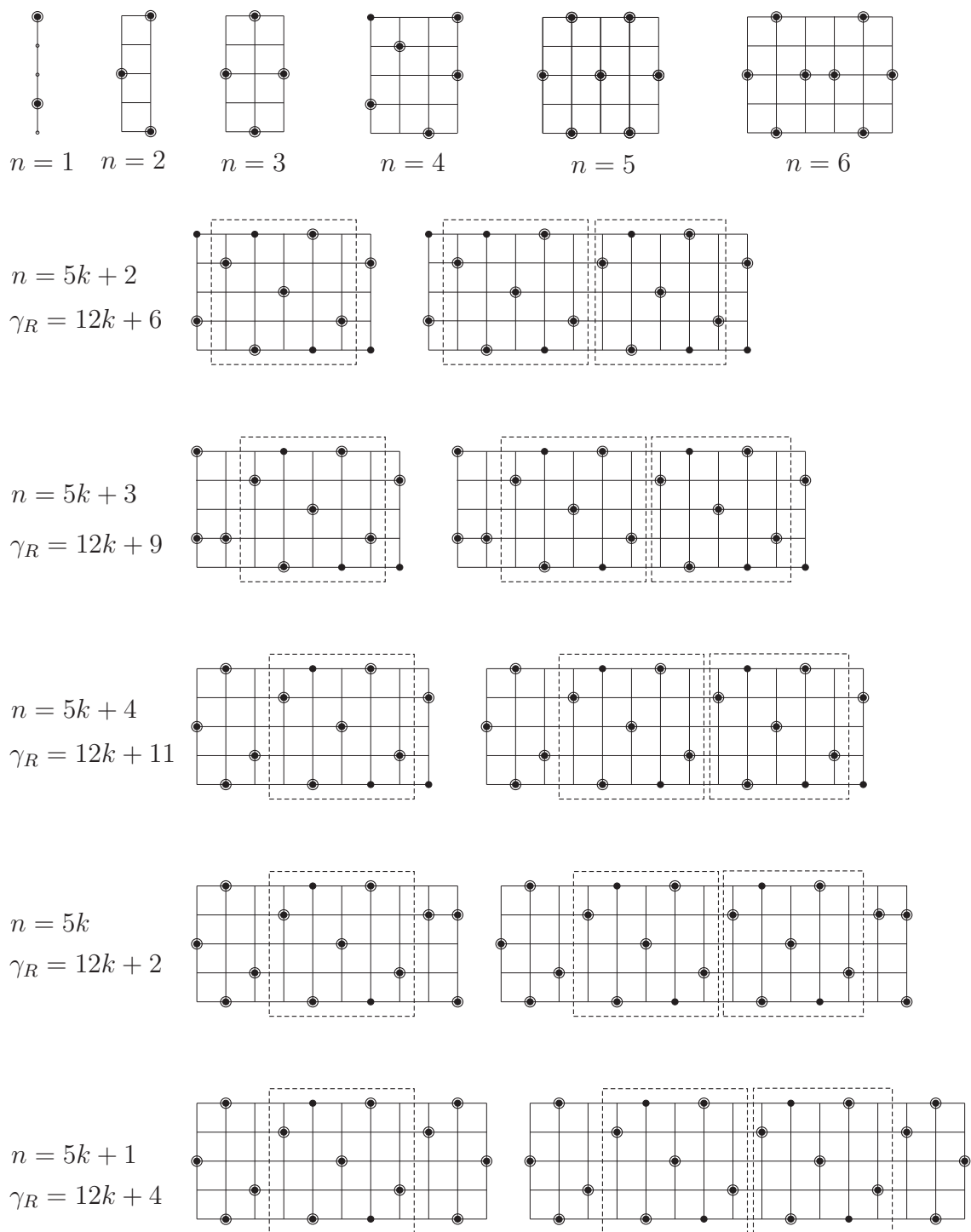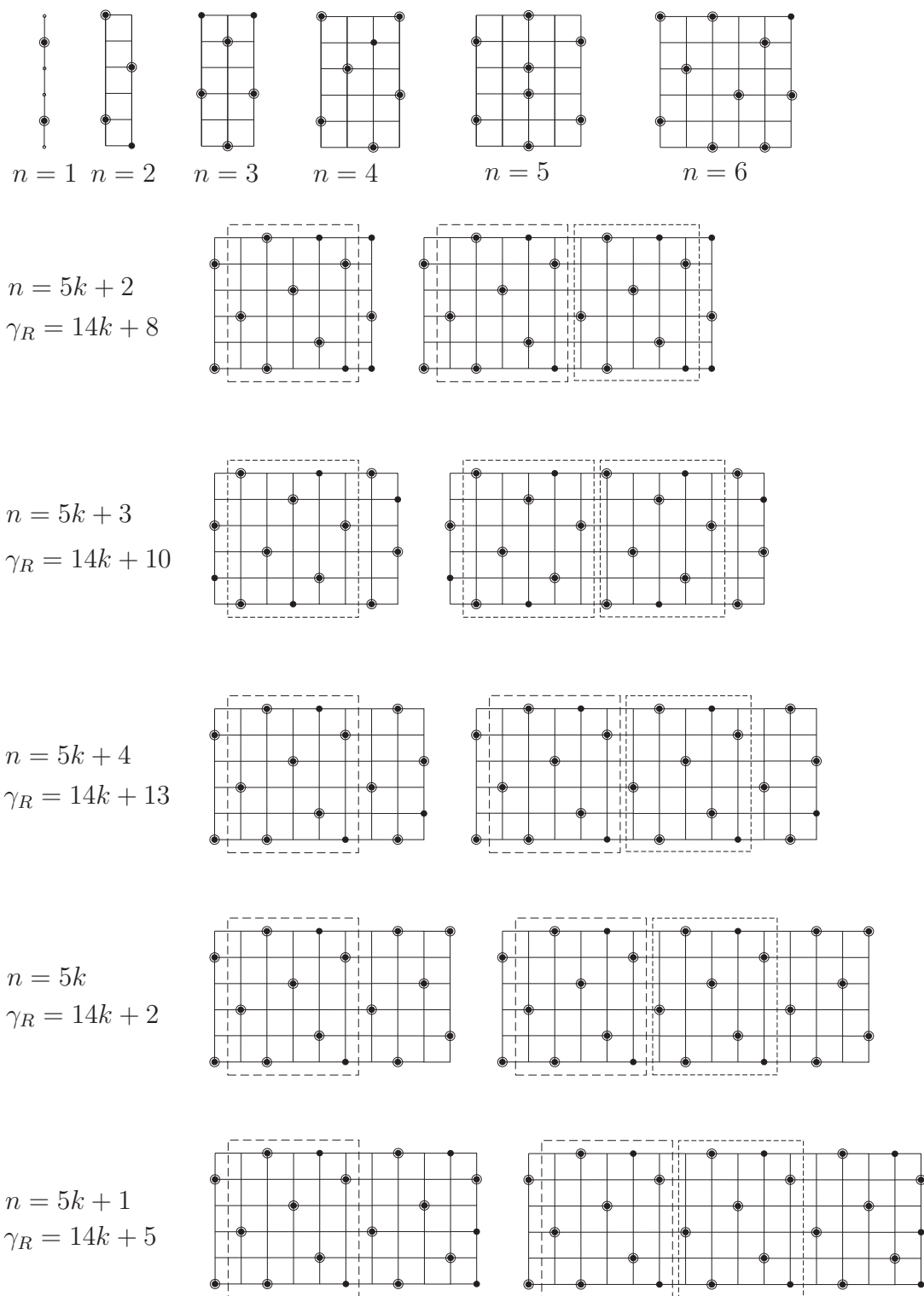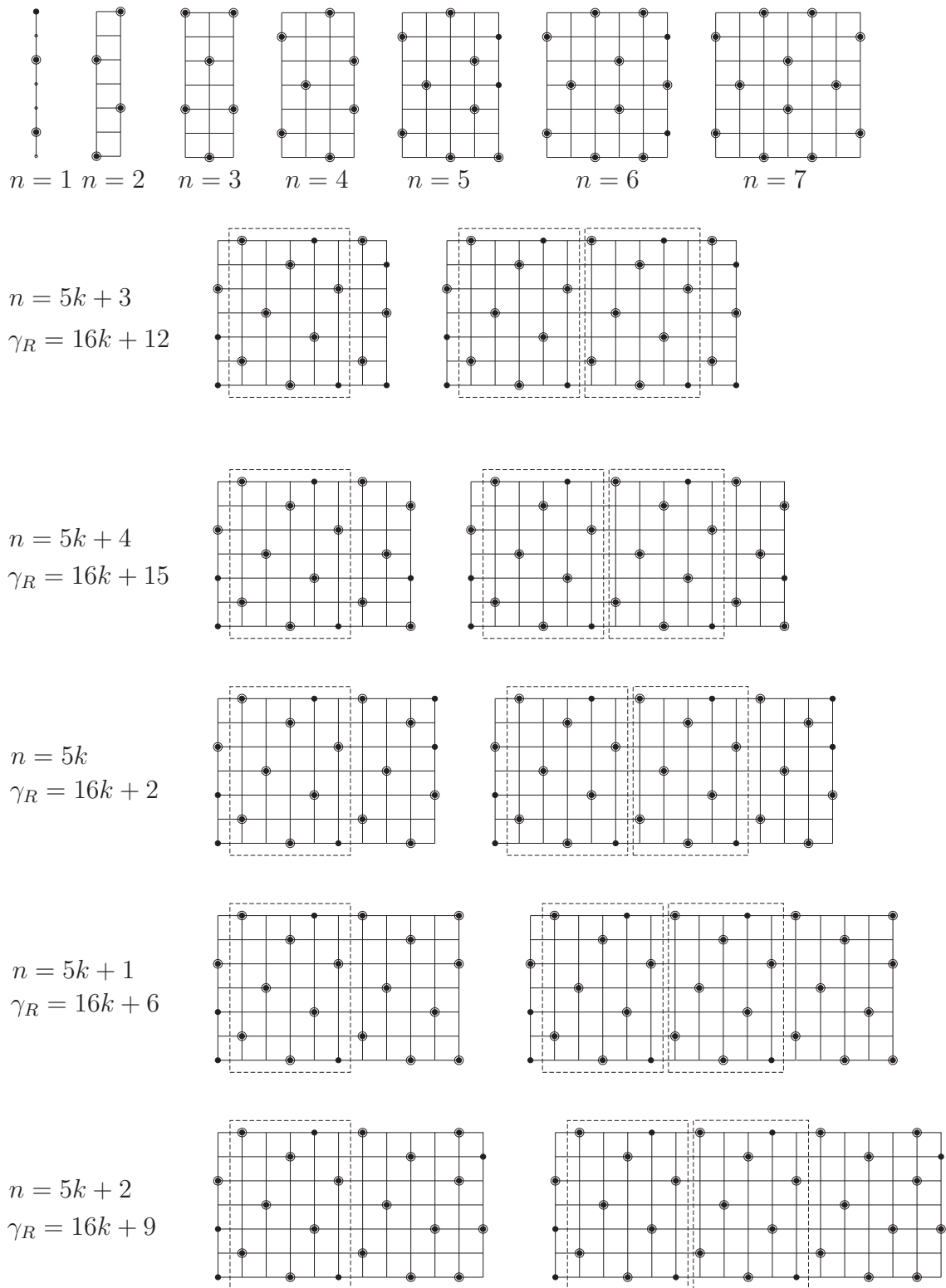
Figure 5: $P_n \square P_7$

Figure 6: $P_n \square P_8$

$n = 1 \qquad n = 2 \qquad n = 3 \qquad n = 4$

$n = 4k + 1$
$\gamma_R = 6k + 2$

$n = 4k + 2$
$\gamma_R = 6k + 4$

$n = 4k + 3$
$\gamma_R = 6k + 5$

$n = 4k$
$\gamma_R = 6k + 1$

Figure 7: $P_n \square C_3$

$n = 1 \qquad n = 2 \qquad n = 3 \qquad n = 4 \qquad n = 5$
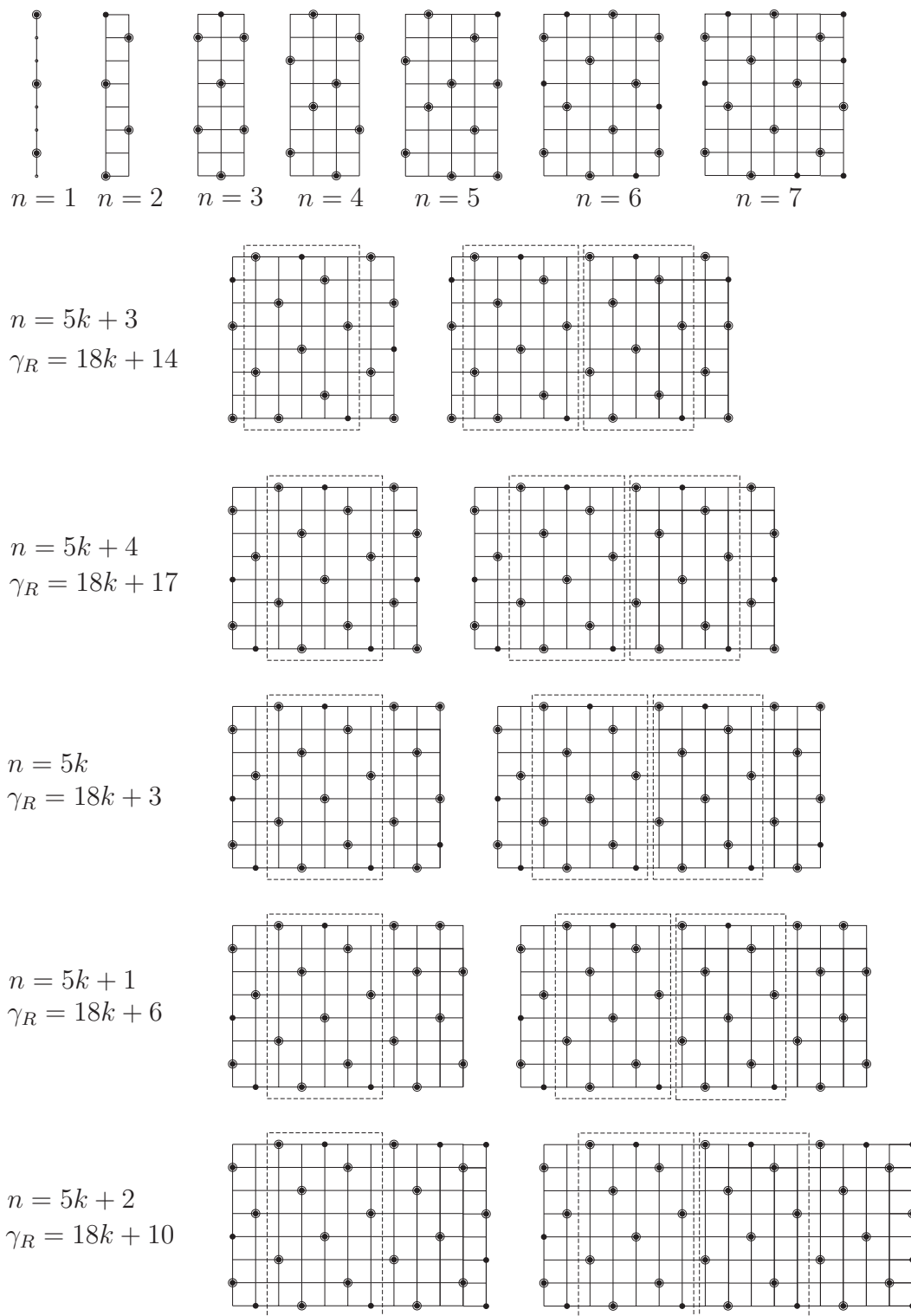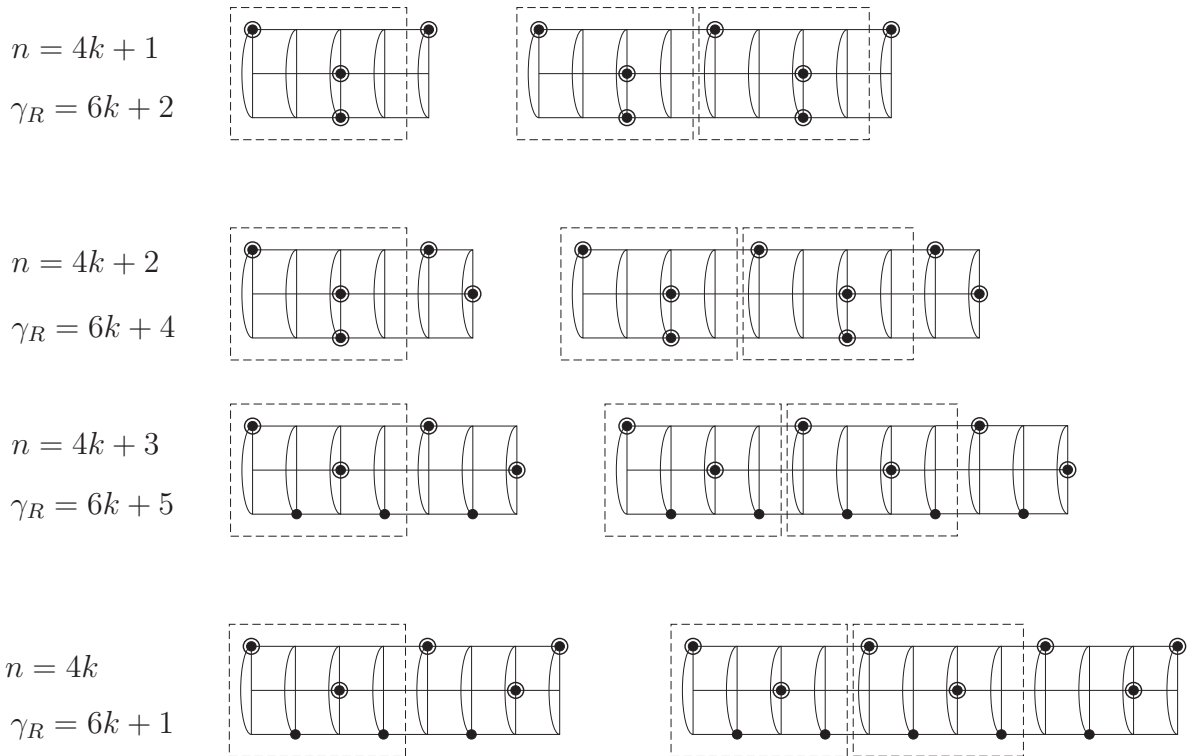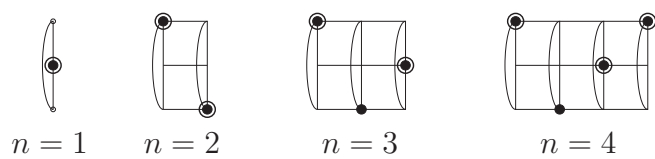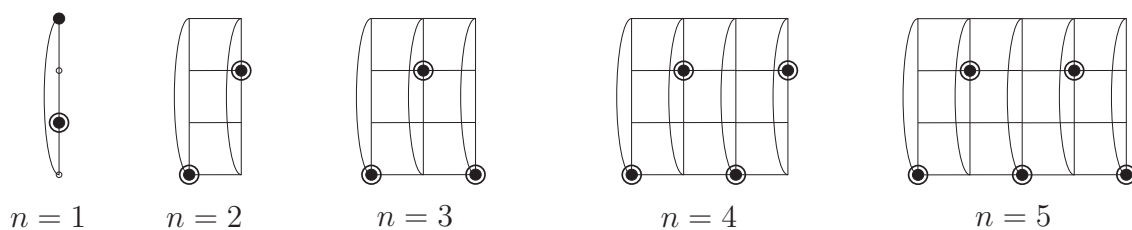
Figure 8: $P_n \square C_4$

Figure 9: $P_n \square C_5$
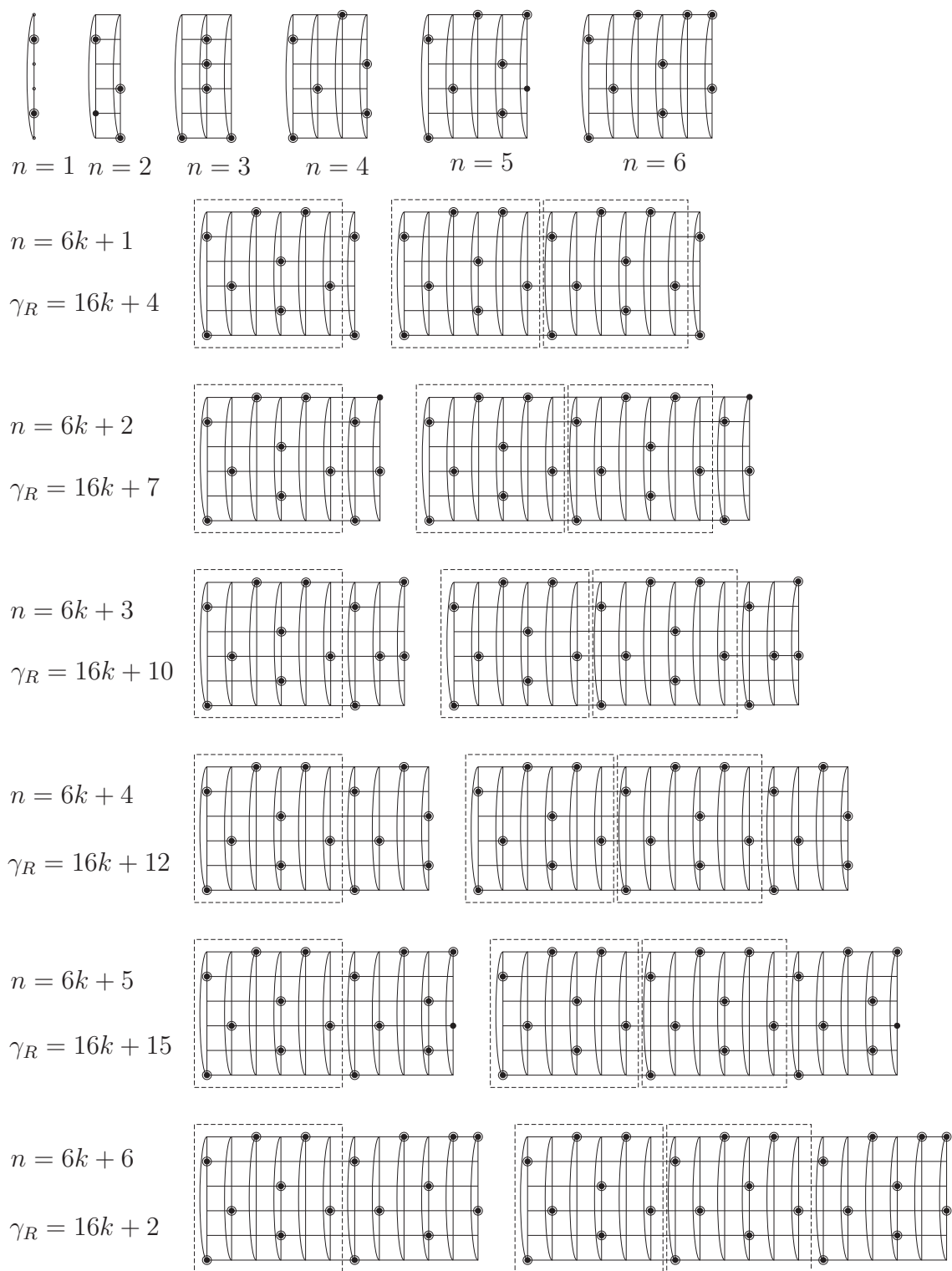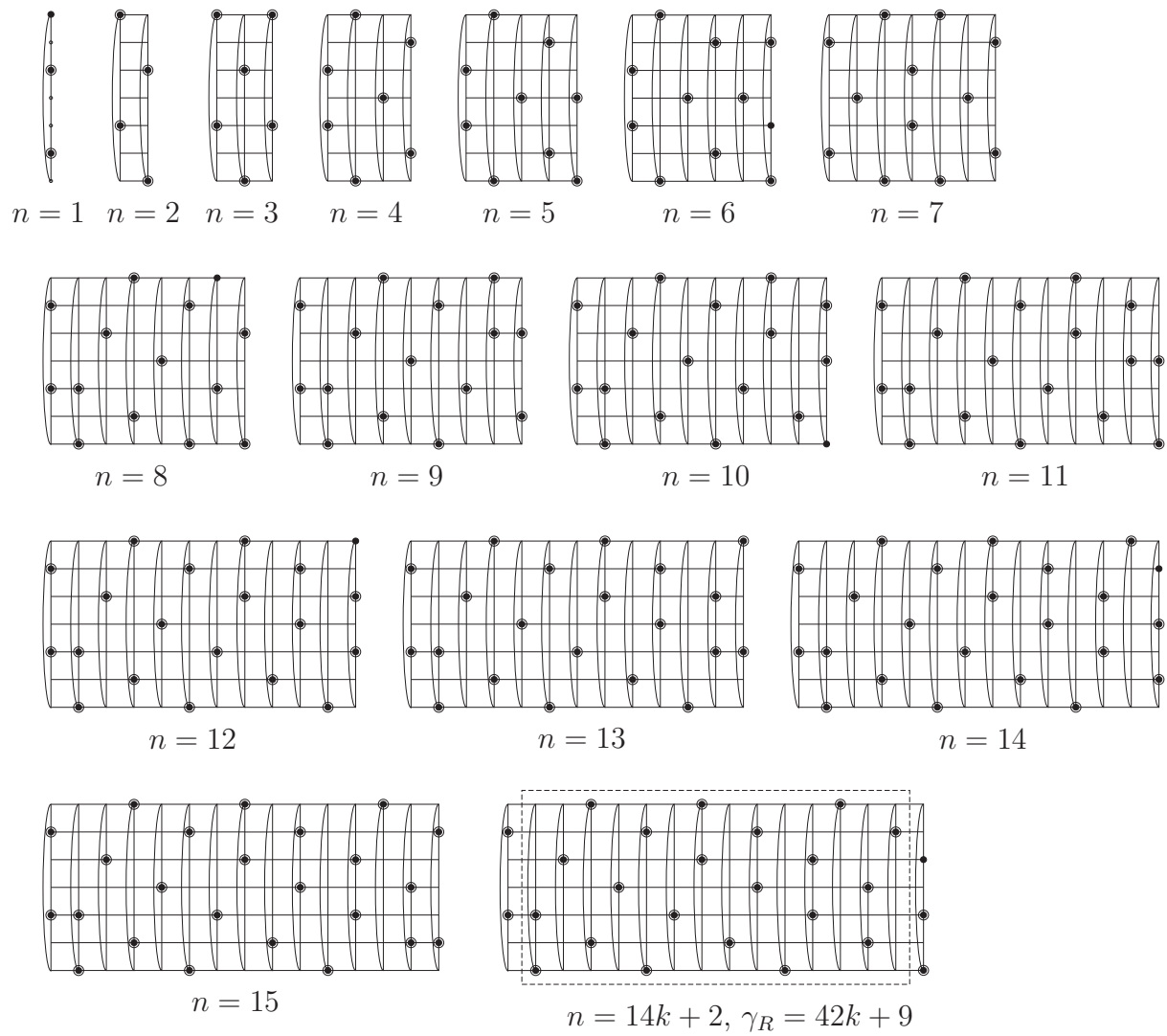
$n=1$ $n=2$ $n=3$ $n=4$ $n=5$ $n=6$

$n=6k+1$

$\gamma_R = 16k+4$

$n=6k+2$

$\gamma_R = 16k+7$

$n=6k+3$

$\gamma_R = 16k+10$

$n=6k+4$

$\gamma_R = 16k+12$

$n=6k+5$

$\gamma_R = 16k+15$

$n=6k+6$

$\gamma_R = 16k+2$

Figure 10: $P_n \square C_6$

$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$ $n = 7$

$n = 8$ $n = 9$ $n = 10$ $n = 11$

$n = 12$ $n = 13$ $n = 14$

$n = 15$

$n = 14k + 2$, $\gamma_R = 42k + 9$

Figure 11: $P_n \square C_7$

$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$ $n = 7$ $n = 8$

$n = 9$ $n = 10$ $n = 11$ $n = 12$

$n = 13$ $n = 14$ $n = 15$

$n = 16$ $n = 17$

$n = 18$ $n = 19$

$n = 20$ $n = 16k + 5$, $\gamma_R = 56k + 20$

Figure 12: $P_n \square C_8$

$n = 4k$
$\gamma_R = 4k$

$n = 4k + 1$
$\gamma_R = 4k + 2$

$n = 4k + 2$
$\gamma_R = 4k + 3$

$n = 4k + 3$
$\gamma_R = 4k + 4$

Figure 13: $C_n \square P_2$

$n = 3 \qquad n = 4$

$n = 4k + 1$
$\gamma_R = 6k + 2$

$n = 4k + 2$
$\gamma_R = 6k + 4$

$n = 4k + 3$
$\gamma_R = 6k + 6$

$n = 4k + 4$
$\gamma_R = 6k + 6$

Figure 14: $C_n \square P_3$

$n = 4k$

$\gamma_R = 8k$

$n = 4k + 1$

$\gamma_R = 8k + 2$

$n = 4k + 2$

$\gamma_R = 8k + 4$

$n = 4k + 3$

$\gamma_R = 8k + 6$

Figure 15: $C_n \square P_4$

$n = 3$ $\qquad$ $n = 4$ $\qquad$ $n = 5$ $\qquad$ $n = 6$

$n = 5k + 2$

$\gamma_R = 12k + 6$

$n = 5k + 3$

$\gamma_R = 12k + 8$

$n = 5k + 4$

$\gamma_R = 12k + 10$

$n = 5k + 5$

$\gamma_R = 12k + 12$

$n = 5k + 1$

$\gamma_R = 12k + 3$

Figure 16: $C_n \square P_5$

$n = 10k$

$\gamma_R = 24k$

$n = 10k + 4$

$\gamma_R = 24k + 8$

$n = 10k + 7$

$\gamma_R = 24k + 18$

$n = 10k + 8$

$\gamma_R = 24k + 20$
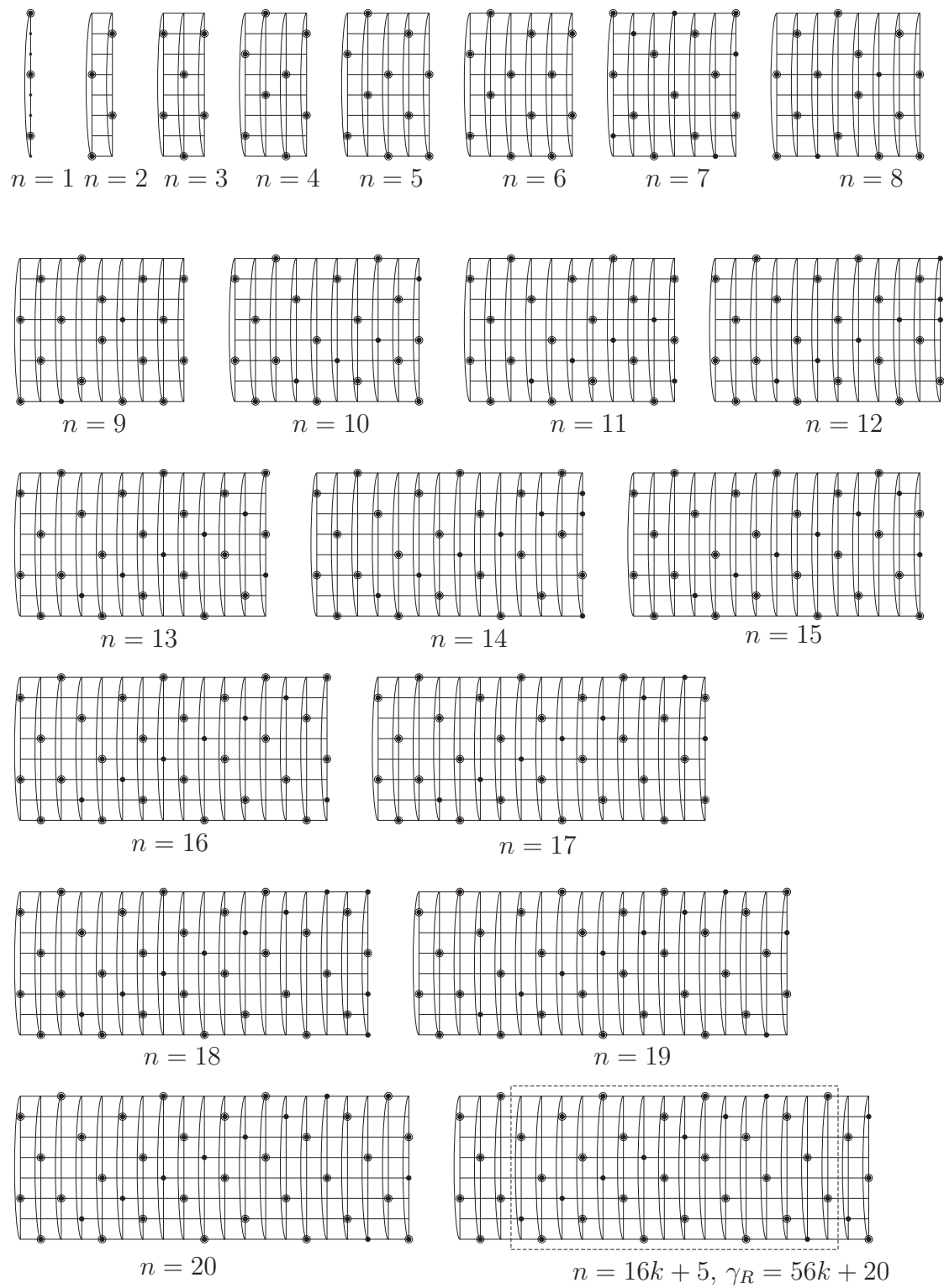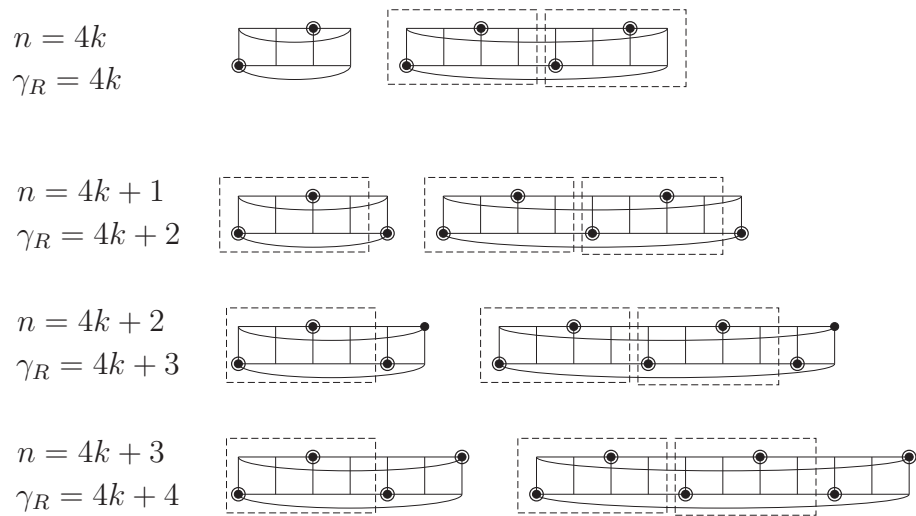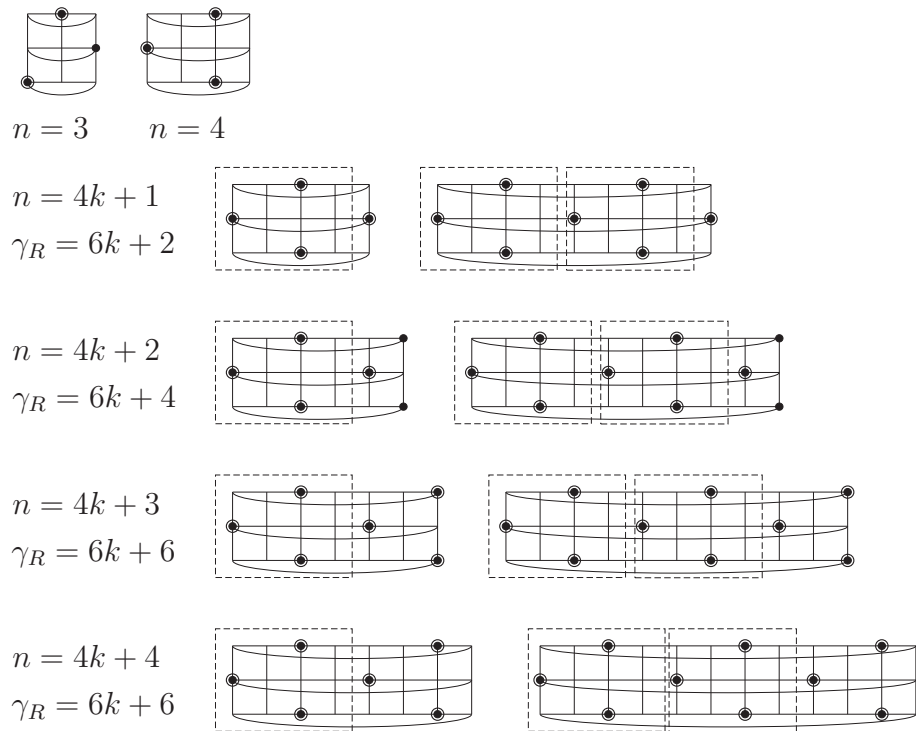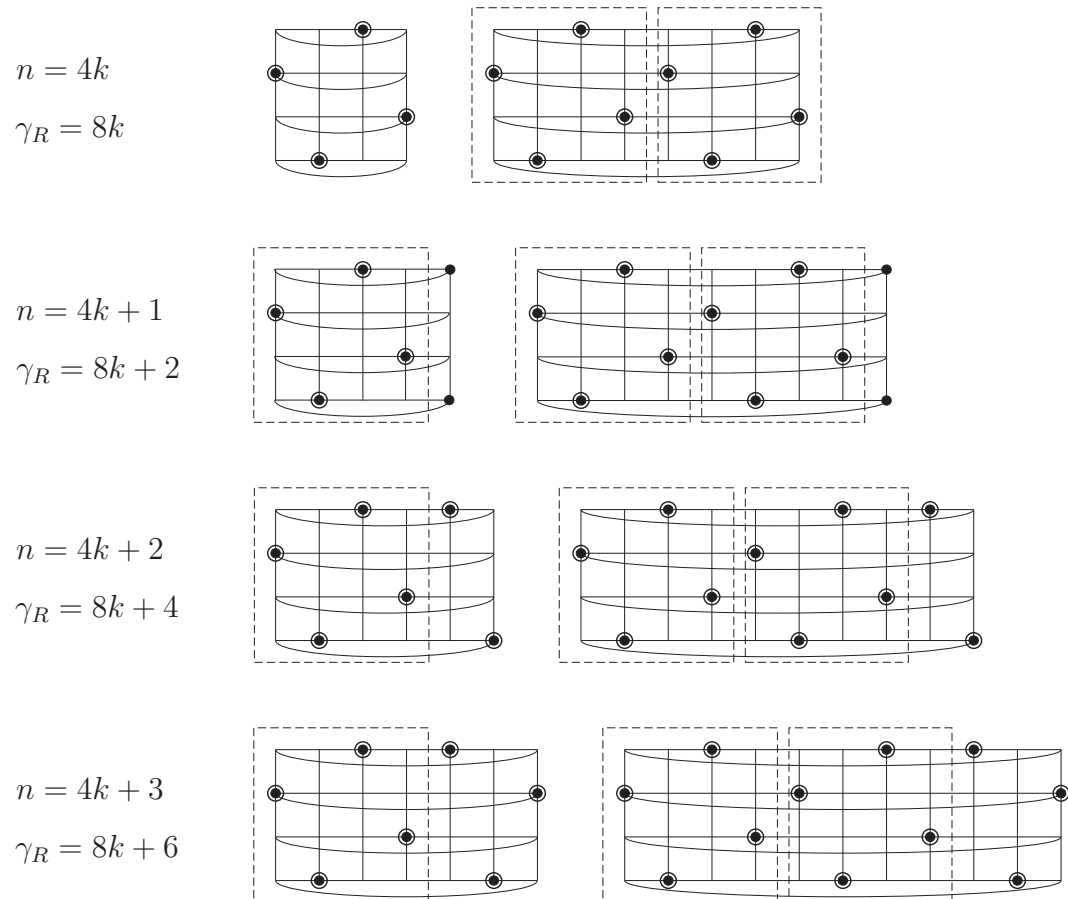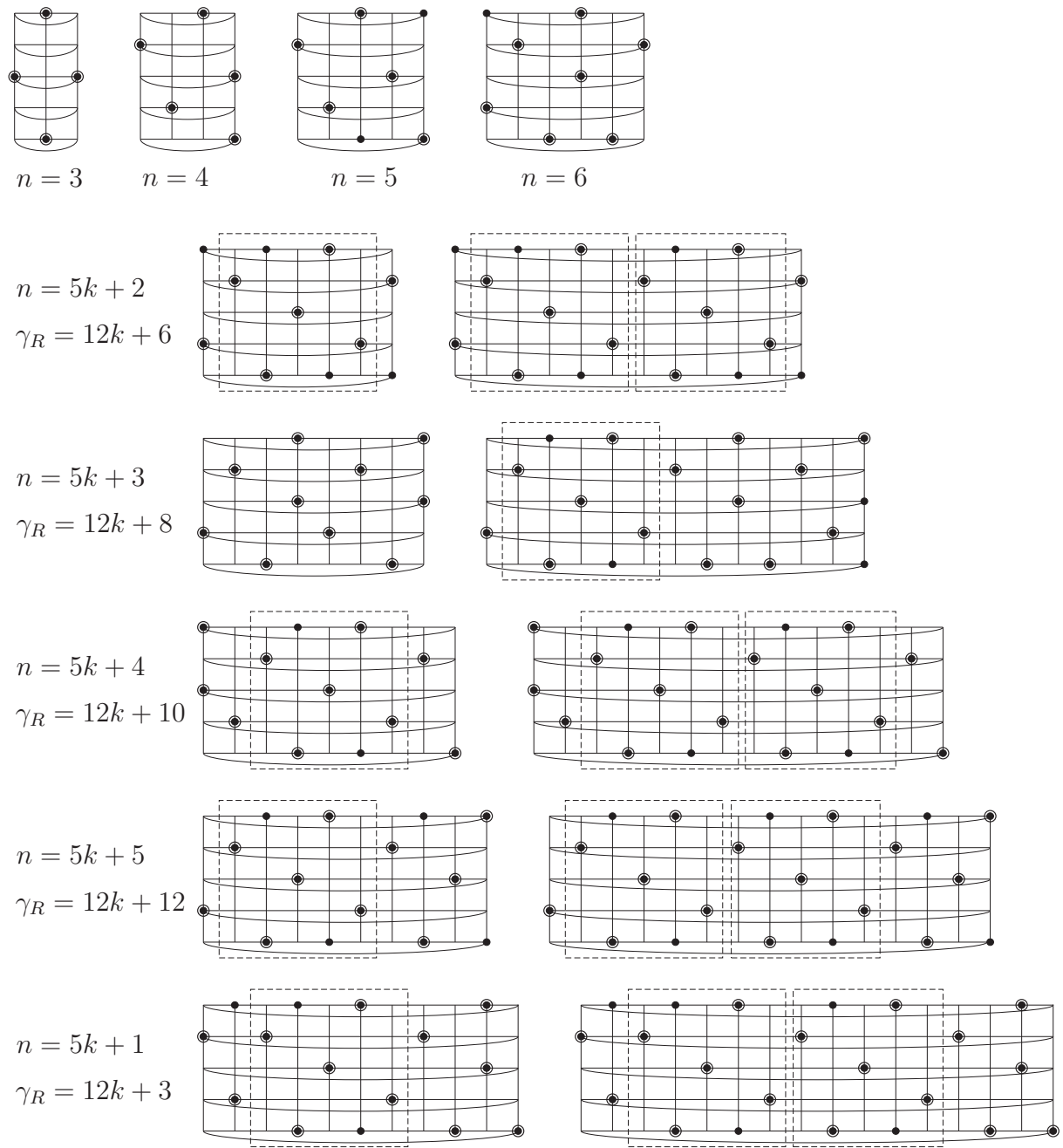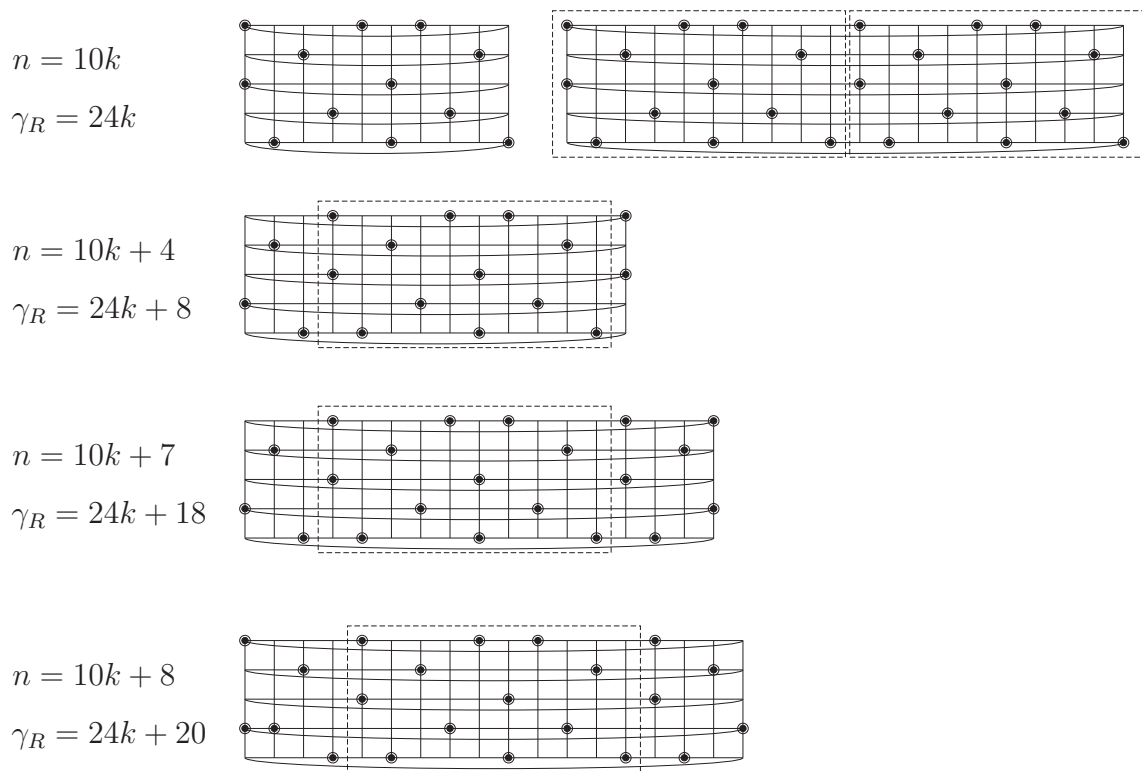
Figure 17: $C_{10k} \square P_5$, $C_{10k+4} \square P_5$, $C_{10k+7} \square P_5$ and $C_{10k+8} \square P_5$ are Roman graphs.

$n = 3$ $n = 4$

$n = 5k$
$\gamma_R = 14k$

$n = 5k + 1$
$\gamma_R = 14k + 4$

$n = 5k + 2$
$\gamma_R = 14k + 7$

$n = 5k + 3$
$\gamma_R = 14k + 10$

$n = 5k + 4$
$\gamma_R = 14k + 12$

Figure 18: $C_n \square P_6$

$n = 3$ $n = 6$

$n = 4k$
$\gamma_R = 6k$

$n = 4k + 1$
$\gamma_R = 6k + 2$

$n = 4k + 2$
$\gamma_R = 6k + 3$

$n = 4k + 3$
$\gamma_R = 6k + 5$

Figure 19: $C_n \square C_3$

$n = 3$ $n = 4$ $n = 5$ $n = 6$
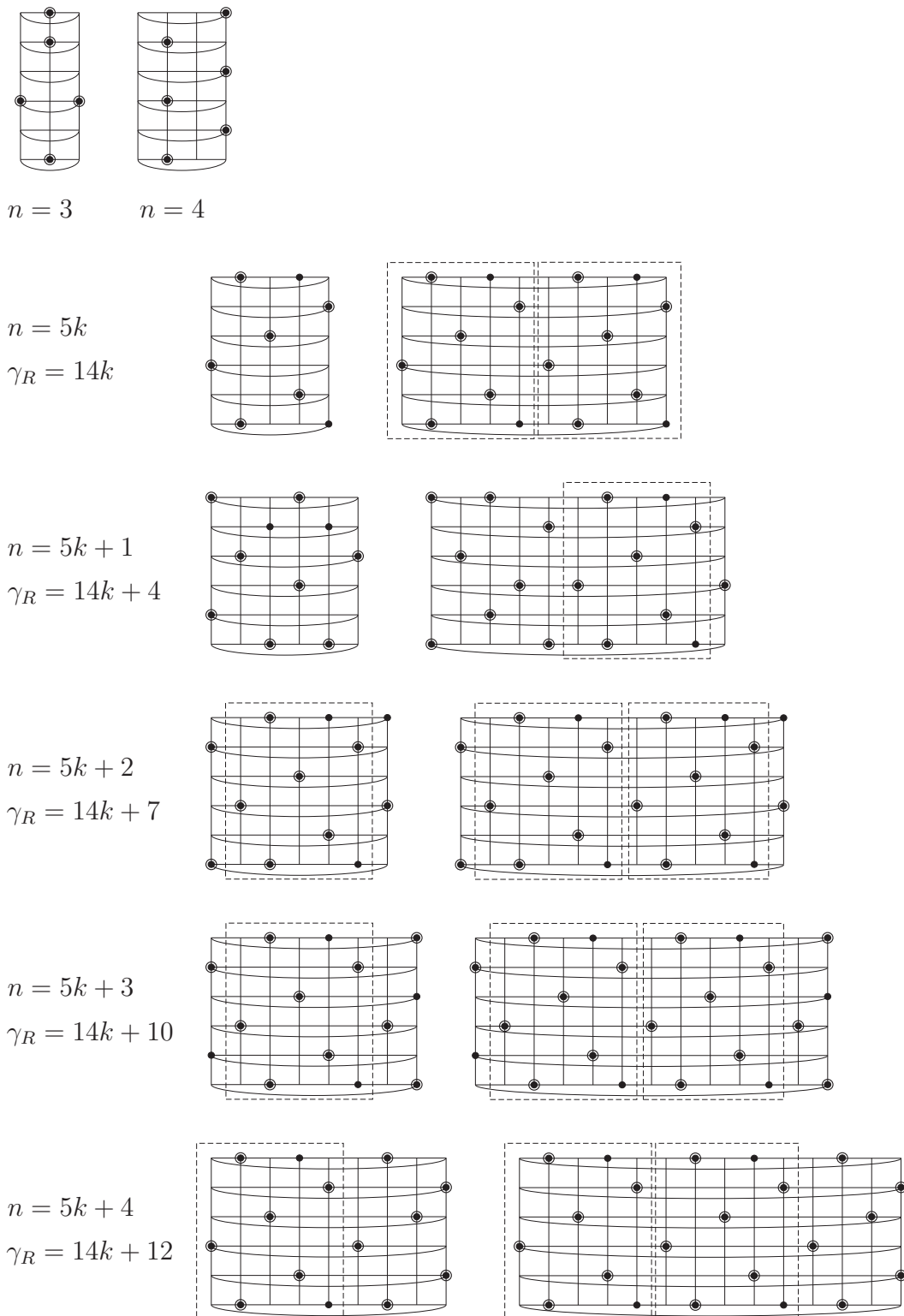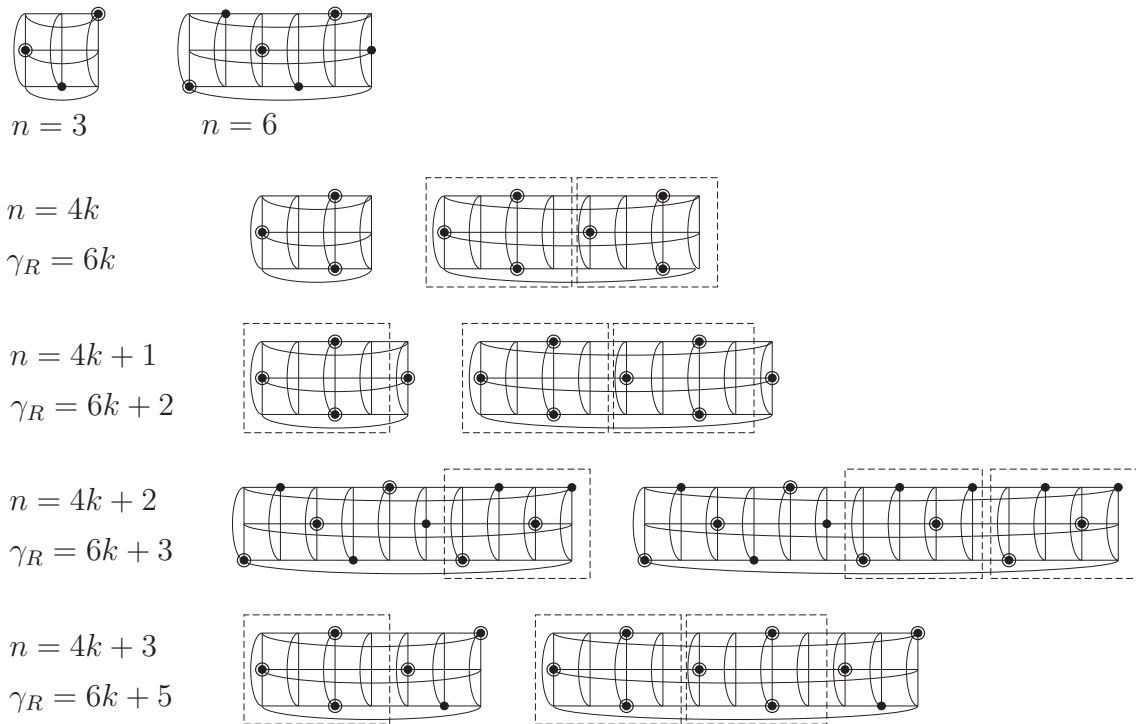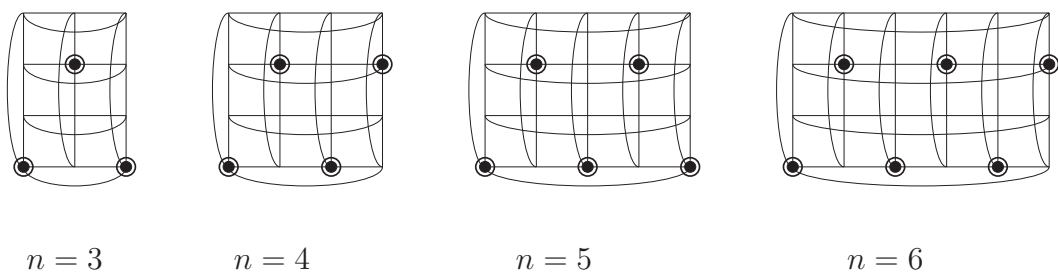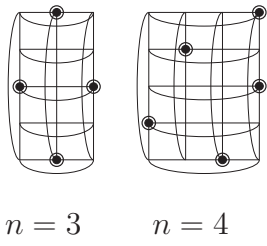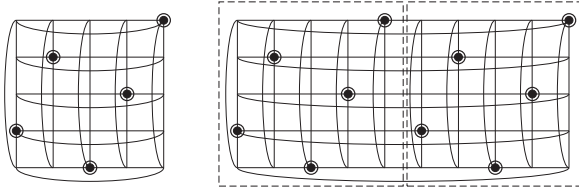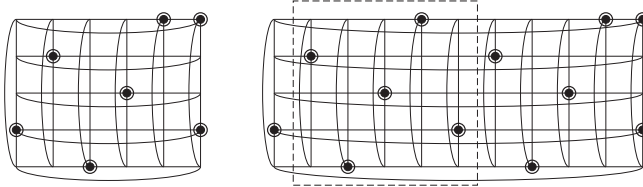
Figure 20: $C_n \square C_4$

$n = 3$ $n = 4$
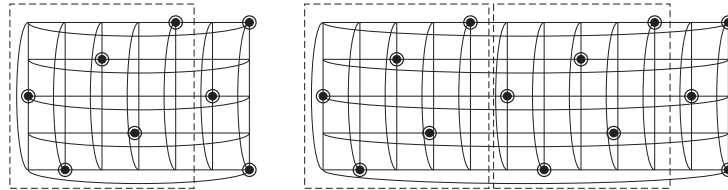
$n = 5k$
$\gamma_R = 10k$
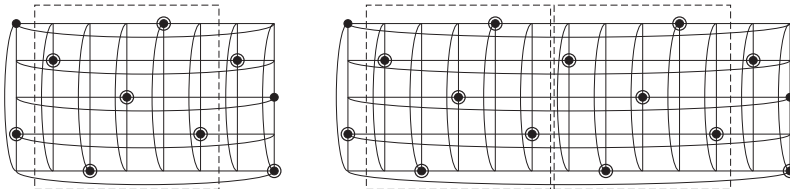
$n = 5k+1$
$\gamma_R = 10k+4$

$n = 5k+2$
$\gamma_R = 10k+6$

$n = 5k+3$
$\gamma_R = 10k+8$

$n = 5k+4$
$\gamma_R = 10k+10$

Figure 21: $C_n \square C_5$

$n = 3$ $n = 4$ $n = 5$ $n = 8$ $n = 13$

$n = 6k$
$\gamma_R = 16k$

$n = 6k + 4$
$\gamma_R = 16k + 12$

$n = 6k + 5$
$\gamma_R = 16k + 14$

$n = 18k + 1$
$\gamma_R = 48k + 4$

$n = 18k + 7$
$\gamma_R = 48k + 20$
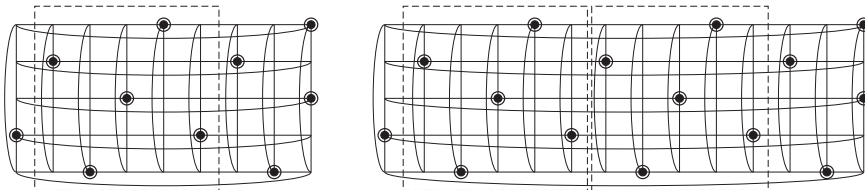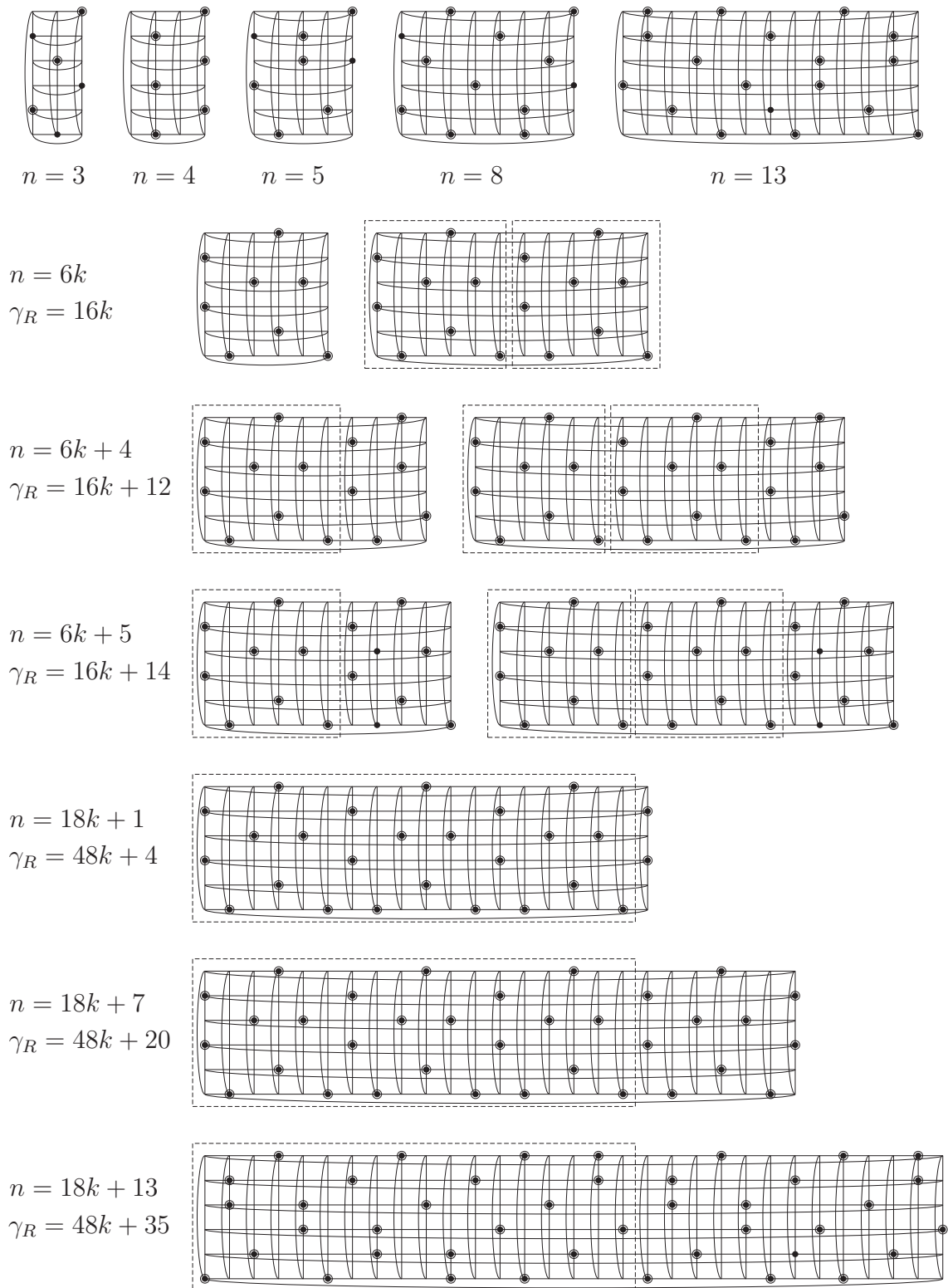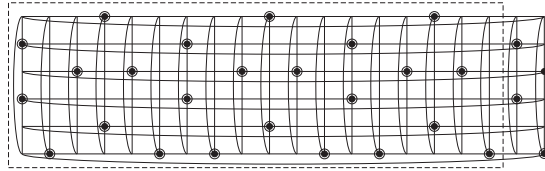
$n = 18k + 13$
$\gamma_R = 48k + 35$

Figure 22: $C_n \square C_6$
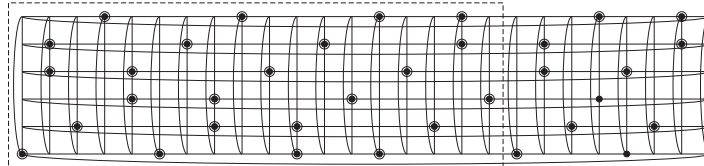
$n = 18k + 2$
$\gamma_R = 48k + 7$
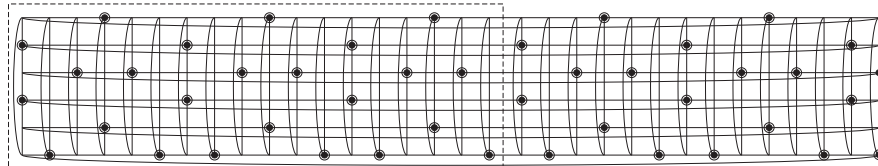
$n = 18k + 8$
$\gamma_R = 48k + 22$
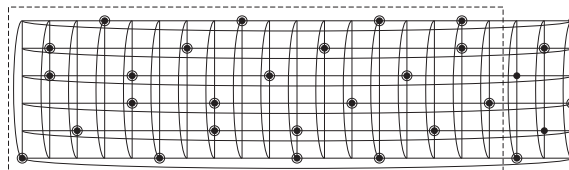
$n = 18k + 14$
$\gamma_R = 48k + 39$
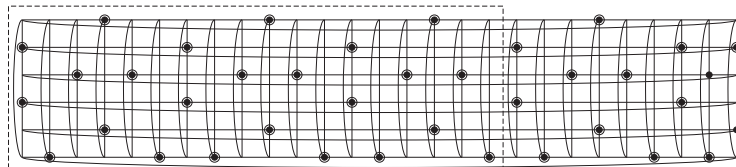
$n = 18k + 3$
$\gamma_R = 48k + 9$

$n = 18k + 9$
$\gamma_R = 48k + 26$

$n = 18k + 15$
$\gamma_R = 48k + 42$

Figure 23: $C_n \square C_6$