Generalized Galois numbers, inversions, lattice paths, Ferrers diagrams and limit theorems

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Abstract

Bliem and Kousidis recently considered a family of random variables whose distributions are given by the generalized Galois numbers (after normalization). We give probabilistic interpretations of these random variables, using inversions in random words, random lattice paths and random Ferrers diagrams, and use these to give new proofs of limit theorems as well as some further limit results.

1 Introduction

The *homogeneous multivariate Rogers–Szegö polynomial* in $m \geqslant 2$ variables is defined by

$$\tilde{H}_n(t_1,\dots,t_m) := \sum_{k_1+\dots+k_m=n} \binom{n}{k_1,\dots,k_m}_q t_1^{k_1}\cdots t_m^{k_m}, \tag{1}$$

where $\binom{n}{k_1,\dots,k_m}_q$ is the *q-multinomial coefficient* (or *Gaussian multinomial coefficient*)

$$\binom{n}{k_1,\dots,k_m}_q := \frac{[n]!_q}{[k_1]!_q\cdots[k_m]!_q} \qquad \text{for } n = k_1+\dots+k_m, \tag{2}$$

where $[k]!_q := [1]_q[2]_q\cdots[k]_q$ with $[\ell]_q := (1-q^\ell)/(1-q)$. Equivalently, one might consider the *inhomogeneous multivariate Rogers–Szegö polynomial*

$$H_n(t_1,\dots,t_{m-1}) := \tilde{H}_n(t_1,\dots,t_{m-1},1). \tag{3}$$

For these polynomials, see Rogers [13], Andrews [1] and Vinroot [17].

We concentrate here on the special value

$$G_n^{(m)}(q) = H_n(1,\dots,1) = \tilde{H}_n(1,\dots,1) = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\dots,k_m}_q, \tag{4}$$

studied in Vinroot [17] and Bliem and Kousidis [2]. This is a polynomial in q. In the special case $m = 2$, studied in e.g. Goldman and Rota [4], Nijenhuis, Solow and Wilf [11], Kac and Cheung [10, Chapter 7] and Hitzemann and Hochstättler [6], these numbers $G_n^{(2)}(q)$ are known as *Galois numbers*, and the numbers $G_n^{(m)}$ are therefore called *generalized Galois numbers* by [17] and [2]. Note that

$$G_n^{(m)}(1) = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\dots,k_m} = m^n, \tag{5}$$

by the multinomial theorem.

Bliem and Kousidis [2] noted that the polynomial $G_n^{(m)}(q)$ has non-negative coefficients, and thus

$$g_n^{(m)}(q) := \frac{G_n^{(m)}(q)}{G_n^{(m)}(1)} = m^{-n} G_n^{(m)}(q) \tag{6}$$

can be interpreted as the probability generating function of a random variable $G_{n,m}$. We let $\mathcal{G}_{n,m}$ denote the probability distribution with the probability generating function (6), and have thus $G_{n,m} \sim \mathcal{G}_{n,m}$. (We use, following [2], $G_{n,m}$ for an arbitrary random variable with this distribution. In the next sections we will construct specific random variables of this type.)

The purpose of the present paper is to provide some probabilistic interpretations of this random variable, see Sections 2–4, and to use these interpretations to give new, and perhaps simpler, proofs of the following results in [2]. We use $\xrightarrow{\mathrm{d}}$ for convergence in distribution and (later) $\overset{\mathrm{d}}{=}$ for equality in distribution. $N(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.

Theorem 1 ([2])**.** *The random variable $G_{n,m}$ has mean and variance*

$$\mathbb{E}\, G_{n,m} = \frac{n(n-1)}{4} \cdot \frac{m-1}{m}, \tag{7}$$

$$\operatorname{Var} G_{n,m} = \frac{n(n-1)(2n+5)}{72} \cdot \frac{m^2-1}{m^2}. \tag{8}$$

Theorem 2 ([2])**.** *If $m \to \infty$ with $n \geqslant 1$ fixed, then*

$$G_{n,m} \xrightarrow{\mathrm{d}} G_n, \tag{9}$$

where G_n is the number of inversions in a random permutation of $\{1,\dots,n\}$.

Theorem 3 ([2]). *If $n \to \infty$ with $m \geqslant 2$ fixed, then*

$$\frac{G_{n,m} - \mathbb{E}\, G_{n,m}}{\mathrm{Var}(G_{n,m})^{1/2}} \xrightarrow{\mathrm{d}} N(0,1); \tag{10}$$

equivalently,

$$\frac{G_{n,m} - \mathbb{E}\, G_{n,m}}{n^{3/2}} \xrightarrow{\mathrm{d}} N\Big(0, \frac{m^2-1}{36m^2}\Big). \tag{11}$$

Furthermore, we can also let both m and n tend to infinity; we show that there are no surprises in this case.

Theorem 4. *If $m, n \to \infty$, then*

$$\frac{G_{n,m} - \mathbb{E}\, G_{n,m}}{\mathrm{Var}(G_{n,m})^{1/2}} \xrightarrow{\mathrm{d}} N(0,1); \tag{12}$$

equivalently,

$$\frac{G_{n,m} - \mathbb{E}\, G_{n,m}}{n^{3/2}} \xrightarrow{\mathrm{d}} N\Big(0, \frac{1}{36}\Big). \tag{13}$$

Moreover, we show a local limit theorem strengthening Theorems 3 and 4.

Theorem 5. *If $n \to \infty$, then, with $\mu_{n,m} := \mathbb{E}\, G_{n,m}$ and $\sigma_{n,m}^2 := \mathrm{Var}\, G_{n,m}$ given by Theorem 1,*

$$\sigma_{n,m}\, \mathbb{P}(G_{n,m} = k) = \frac{1}{\sqrt{2\pi}} e^{-(k-\mu_{n,m})^2/2\sigma_{n,m}^2} + o(1), \tag{14}$$

uniformly in all $m \geqslant 2$ and $k \in \mathbb{Z}$.

Equivalently, we can in (14) *replace $\mu_{n,m}$ and $\sigma_{n,m}^2$ by the approximations $\bar\mu_{n,m} := \frac{m-1}{4m} n^2$ and $\bar\sigma_{n,m}^2 := \frac{m^2-1}{36m^2} n^3$.*

Proofs are given in Sections 5–6.

Remark 6. The name (generalized) Galois numbers comes from the following algebraic interpretation, see [4], [17], [10, Chapter 7], [15, Proposition 1.3.18] which, however, not will be important in the present paper.

If q is a prime power and V an n-dimensional vector space over the Galois field F_q with q elements, then it is not difficult to see that $\binom{n}{k_1,\dots,k_m}_q$ is the number of *flags* $\{0\} \subseteq V_1 \subseteq \cdots \subseteq V_m = V$, where V_i is a subspace of dimension $k_1 + \cdots + k_i$. Hence, $G_n^{(m)}(q)$ is the total number of such flags of fixed length m in $V = F_q^n$. In particular, the Galois number $G_n^{(2)}(q)$ is the number of subspaces of F_q^n.

2 Inversions

If $w = w_1 \cdots w_n$ is a word with letters from an ordered alphabet $\mathcal{A}$, then the number of *inversions* in w is the number of pairs (i, j) with $i < j$ and $w_i > w_j$; we denote this number by $\mathrm{Inv}(w)$. Using the notation $\mathbf{1}\{\mathcal{E}\}$ for the indicator of an event $\mathcal{E}$, we thus have

$$\mathrm{Inv}(w) = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\{w_i > w_j\}. \tag{15}$$

With the alphabet $\mathcal{A} = \{1, \dots, m\}$, it is well-known (and not difficult to see) that the q-multinomial coefficient $\binom{n}{n_1,\dots,n_m}_q$, where $n_1 + \cdots + n_m = n$, is the generating function of the number of inversions in words consisting of n_1 1's, ..., n_m m's, in the sense that if $a_{n_1,\dots,n_m}(\ell)$ is the number of such words with exactly ℓ inversions, then

$$\binom{n}{n_1, \dots, n_m}_q = \sum_{\ell=0}^{\infty} a_{n_1,\dots,n_m}(\ell) q^\ell, \tag{16}$$

see [1, Theorem 3.6].

Summing over all $n_1, \dots, n_m$ with $n_1 + \cdots + n_m = n$, we immediately obtain the following from (4) and (16).

Theorem 7. *$G_n^{(m)}(q)$ is the generating function of the number of inversions in words of length n in the alphabet $\{1, \dots, m\}$, in the sense that if $A_n^{(m)}(\ell)$ is the number of such words with exactly ℓ inversions, then*

$$G_n^{(m)}(q) = \sum_{\ell=0}^{\infty} A_n^{(m)}(\ell) q^\ell. \tag{17}$$

□

By the definition of the random variable $G_{n,m}$, (17) is equivalent to

$$\mathbb{P}(G_{n,m} = \ell) = A_n^{(m)}(\ell)/n^{-m}. \tag{18}$$

This can be formulated as follows, yielding our first construction of a random variable $G_{n,m}$.

Theorem 8. *Let $W_{n,m}$ be a uniformly random word of length n in the alphabet $\{1, \dots, m\}$. Then the number of inversions $\mathrm{Inv}(W_{n,m})$ has the distribution $\mathcal{G}_{n,m}$. In other words, $G_{n,m} \stackrel{\mathrm{d}}{=} \mathrm{Inv}(W_{n,m})$.* □

We can thus choose $G_{n,m} := \mathrm{Inv}(W_{n,m})$. (Recall that we have defined $G_{n,m}$ to be an arbitrary random variable with the desired distribution.)

If we write the random word $W_{n,m}$ as $X_1 \cdots X_n$, we have $X_1, \dots, X_n$ i.i.d. (independent and identically distributed) with the uniform distribution on $\{1, \dots, m\}$, and using (15), Theorem 8 may be reformulated as follows.

Corollary 9. *Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables, with every X_i uniformly distributed on $\{1,\dots,m\}$, and let*

$$V_{n,m} := \sum_{1\leqslant i<j\leqslant n} \mathbf{1}\{X_i > X_j\}. \tag{19}$$

Then $V_{n,m} \sim \mathcal{G}_{n,m}$. In other words, $G_{n,m} \stackrel{\mathrm{d}}{=} V_{n,m}$. □

Let $N_k := \#\{i \leqslant n : X_i = k\}$ be the number of occurences of the letter k in the random string $W_{n,m} = X_1 \cdots X_n$. Then $(N_1,\dots,N_m)$ has a multinomial distribution with $\mathbb{E}\, N_k = n/m$, and it is well known that if we keep m fixed, $n^{-1/2}(N_k - EN_k)_{k=1}^m \xrightarrow{\mathrm{d}} (Z_k)_{k=1}^m$ as $n\to\infty$, where $Z_1,\dots,Z_m$ are jointly normal with means $\mathbb{E}\, Z_k = 0$, variances $\operatorname{Var} Z_k = (m-1)/m^2$ and covariances $\operatorname{Cov}(Z_k,Z_l) = -1/m^2$ $(k\neq l)$. By Theorem 3, $V_{n,m} \stackrel{\mathrm{d}}{=} G_{n,m}$ has an asymptotic normal distribution, and this extends to joint asymptotic normality of $V_{n,m}$ and $N_1,\dots,N_m$.

Theorem 10. *For fixed m, as $n\to\infty$,*

$$\left(\frac{V - \mathbb{E}\, V_{n,m}}{n^{3/2}}, \frac{N_1 - EN_1}{n^{1/2}}, \dots, \frac{N_m - EN_m}{n^{1/2}}\right) \xrightarrow{\mathrm{d}} (Z^*, Z_1, \dots, Z_m),$$

where $Z^*, Z_1, \dots, Z_m$ are jointly normal with means 0, $\operatorname{Var} Z^* = (m^2-1)/36m^2$ as in (11), $Z_1,\dots,Z_m$ have the variances and covariances given above and Z^* is independent of $Z_1,\dots,Z_m$.

The proof is given in Section 5.

3 A U-statistic

Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ be independent random variables, with every X_i uniformly distributed on $\{1,\dots,m\}$ and every Y_i uniformly distributed on [0,1]. (Any common continuous distribution of Y_i would yield the same result.)

Fix $n \geqslant 1$. The values $Y_1,\dots,Y_n$ are a.s. distinct, and can thus be ordered as $Y_{\sigma(1)} < \cdots < Y_{\sigma(n)}$ for some (unique) permutation of $\{1,\dots,n\}$. Let $W_{n,m}$ be the word $X_{\sigma(1)}\cdots X_{\sigma(n)}$. Since $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ are independent, $W_{n,m}$ has the same distribution as $X_1\cdots X_n$, and is thus a uniformly random word in $\{1,\dots,m\}^n$. Consequently, Theorem 8 yields $\operatorname{Inv}(W_{n,m}) \sim \mathcal{G}_{n,m}$. Moreover, since $i<j \iff Y_{\sigma(i)} < Y_{\sigma(j)}$,

$$\begin{aligned}
\operatorname{Inv}(W_{n,m}) &= \sum_{1\leqslant i<j\leqslant n} \mathbf{1}\{X_{\sigma(i)} > X_{\sigma(j)}\} = \sum_{i,j=1}^{n} \mathbf{1}\{X_{\sigma(i)} > X_{\sigma(j)} \text{ and } i<j\} \\
&= \sum_{i,j=1}^{n} \mathbf{1}\{X_{\sigma(i)} > X_{\sigma(j)} \text{ and } Y_{\sigma(i)} < Y_{\sigma(j)}\} \\
&= \sum_{k,l=1}^{n} \mathbf{1}\{X_k > X_l\}\mathbf{1}\{Y_k < Y_l\}.
\end{aligned}$$

We have shown the following, yielding our second construction of $G_{n,m}$.

Theorem 11. *Let X_i and Y_i be as above, and define the random variable*

$$U_{n,m} := \sum_{i,j=1}^{n} \mathbf{1}\{X_i > X_j\}\mathbf{1}\{Y_i < Y_j\}. \tag{20}$$

Then $U_{n,m} \sim \mathcal{G}_{n,m}$. In other words, $G_{n,m} \stackrel{\mathrm{d}}{=} U_{n,m}$. □

Let $Z_i := (X_i, Y_i)$; this yields a sequence of i.i.d. random vectors taking values in $\mathcal{S} := \{1, \dots, m\} \times [0, 1]$. Define the functions $h, h^* : \mathcal{S}^2 \to \mathbb{R}$ by

$$h\big((x_1, y_1), (x_2, y_2)\big) := \mathbf{1}\{x_i > x_j\}\mathbf{1}\{y_i < y_j\}, \tag{21}$$

$$h^*\big((x_1, y_1), (x_2, y_2)\big) := h\big((x_1, y_1), (x_2, y_2)\big) + h\big((x_2, y_2), (x_1, y_1)\big). \tag{22}$$

Thus h^* is symmetric and (20) can be written

$$U_{n,m} = \sum_{i,j=1}^{n} h\big(Z_i, Z_j\big) = \sum_{1 \leqslant i < j \leqslant n} h^*\big(Z_i, Z_j\big), \tag{23}$$

which shows that $U_{n,m}$ is (for fixed m) a U-statistic [7].

4 Lattice paths and Ferrers diagrams

In this section we consider the special case $m = 2$. In this case, there is an alternative combinatorial description of the Gaussian binomial coefficients using using lattice paths instead of inversions, see Pólya [12]. Indeed, consider lattice paths in the first quadrant, starting at the origin and containing n unit steps East or North. There are 2^n such paths, and they may be encoded by the 2^n words of length n with the alphabet $\{\mathsf{E}, \mathsf{N}\}$. The area under each horizontal step equals the number of previous vertical steps, so by summing, we see that the area under the path equals the number of inversions in the corresponding word, where we use the ordering $\mathsf{E} < \mathsf{N}$.

Consequently, Theorem 8 yields the following.

Theorem 12. *Let $\theta(n)$ be the area under a uniformly random lattice path (of the type above) of length n. Then $\theta(n) \sim \mathcal{G}_{n,2}$. In other words, $G_{n,2} \stackrel{\mathrm{d}}{=} \theta(n)$.*

The random variable $\theta(n)$ was studied by Takács [16], who found its mean and variance and proved a central limit theorem and a local limit theorem (our Theorems 1, 3 and 5 for $m = 2$).

By symmetry, we may instead consider the area $\theta'(n)$ between the path and the y-axis. This area can be regarded as a Ferrers diagram; if the path ends at (s_1, s_2), then the height (number of non-empty rows) h and width w of the Ferrers diagram satisfy $h \leqslant s_2$ and $w \leqslant s_1$, and there is a bijection between all paths ending at (s_1, s_2) and all such Ferrers diagrams. (Note the bijection between such Ferrers diagrams with a given area N and the partitions of N into at most s_2 parts, each at most s_1; see [1, Theorem 3.5].)

Alternatively, by adding an extra row and column, we obtain a Ferrers diagram with height s_2+1 and width s_1+1; its right boundary consists of a path from $(-1,0)$ to (s_1, s_2+1), beginning with a horizontal step and ending with a vertical. Moreover, there is a bijection between all paths ending at (s_1, s_2) and all such Ferrers diagrams. We further see that the area of this Ferrers diagram equals $\theta' + s_1 + s_2 + 1$, where θ' is the area between the (original) path and the y-axis.

The *semiperimeter* of a Ferrers diagram equals its height plus width, and we thus have obtained a bijection between all Ferrers diagram with semiperimeter $n+2$ and all (north-east) lattice paths of length n. This bijection gives a correspondence between uniformly random Ferrers diagrams with semiperimeter $n+2$ and uniformly random lattice paths of length n, yielding the following theorem.

Theorem 13. *Let A_n be the area of a uniformly random Ferrers diagram with semiperimeter $n+2$. Then $A_n - n - 1 \sim \mathcal{G}_{n,2}$. In other words, $G_{n,2} \stackrel{\mathrm{d}}{=} A_n - n - 1$.*

Proof. If $\theta'(n)$ is the area between the corresponding random lattice path and the y-axis, then the arguments above show that

$$A_n = \theta'(n) + n + 1 \stackrel{\mathrm{d}}{=} \theta(n) + n + 1$$

and the result follows by Theorem 12. □

Corollary 14. *The random variable A_n has mean and variance*

$$\mathbb{E}\, A_n = \mathbb{E}\, G_{n,2} + n + 1 = \frac{n^2+7n+8}{8}, \tag{24}$$

$$\operatorname{Var} A_n = \operatorname{Var} G_{n,2} = \frac{n(n-1)(2n+5)}{96}. \tag{25}$$

Proof. By Theorems 13 and 1. □

Theorem 3 yields the central limit theorem

$$\frac{A_n - \mathbb{E}\, A_n}{\operatorname{Var}(A_n)^{1/2}} \xrightarrow{\mathrm{d}} N(0,1); \tag{26}$$

by (24)–(25), this can also be written as

$$\frac{A_n - n^2/8}{n^{3/2}} \xrightarrow{\mathrm{d}} N\Big(0, \frac{1}{48}\Big), \tag{27}$$

which was proved by other methods by Schwerdtfeger [14]. Furthermore, Schwerdtfeger [14] showed that if H_n is the height of the Ferrers diagram, then there is joint convergence of the normalised variables

$$\left(\frac{A_n - n^2/8}{\sqrt{n^3/48}}, \frac{H_n - n/2}{\sqrt{n/4}}\right) \xrightarrow{\mathrm{d}} (\zeta_1, \zeta_2), \tag{28}$$

where ζ_1, ζ_2 are independent standard normal variables. The asymptotic normality of H_n is immediate, since $H_n - 1$ is the y-coordinate of the endpoint of the corresponding lattice path, and thus $H_n - 1$ has the binomial distribution $\mathrm{Bi}(n, 1/2)$. The joint convergence follows by Theorem 10.

5 Proofs of Theorems 1–4 and 10

We will base most of the proofs on the representation in (20)–(23). (It is also possible to use (19), see Remark 17 and the proof of Theorem 10; (19) is simpler in some ways, but we prefer the symmetry in (20)–(23).)

We use the notations, with Z, h, h^* as in Section 3, see (21)–(22),

$$I_{ij} := h(Z_i, Z_j) = \mathbf{1}\{X_i > X_j\}\mathbf{1}\{Y_i < Y_j\}, \tag{29}$$

$$I^*_{ij} := h^*(Z_i, Z_j) = I_{ij} + I_{ji}. \tag{30}$$

Thus (23) can be written

$$G_{n,m} \overset{\mathrm{d}}{=} U_{n,m} = \sum_{1\leqslant i<j\leqslant n} I^*_{ij}. \tag{31}$$

Proof of Theorem 1. By symmetry and the independence of I_{ij} and I_{kl} when $\{i,j\}$ and $\{k,l\}$ are disjoint, (31) implies

$$\mathbb{E}\, G_{n,m} = \binom{n}{2} \mathbb{E}\, I^*_{12} = n(n-1)\, \mathbb{E}\, I_{12}, \tag{32}$$

$$\operatorname{Var} G_{n,m} = \binom{n}{2} \operatorname{Var} I^*_{12} + n(n-1)(n-2) \operatorname{Cov}\bigl(I^*_{12}, I^*_{13}\bigr). \tag{33}$$

Clearly,

$$\mathbb{E}\, I_{ij} = \mathbb{P}(X_i > X_j)\, \mathbb{P}(Y_i < Y_j) = \frac{\binom{m}{2}}{m^2} \cdot \frac{1}{2} = \frac{m-1}{4m} \tag{34}$$

and

$$\mathbb{E}\, I^*_{ij} = 2\, \mathbb{E}\, I_{ij} = \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}; \tag{35}$$

any of these yields (7) by (32).

Since I^*_{ij} is 0/1-valued, it follows from (35) also that

$$\operatorname{Var} I^*_{ij} = \mathbb{E}\, I^*_{ij}(1 - \mathbb{E}\, I^*_{ij}) = \frac{1}{4}\Bigl(1 - \frac{1}{m^2}\Bigr). \tag{36}$$

Furthermore, again using symmetry,

$$\begin{aligned}
\mathbb{E}\bigl(I^*_{12} I^*_{13}\bigr) &= 2\, \mathbb{E}\bigl(I_{12} I_{13}\bigr) + 2\, \mathbb{E}\bigl(I_{21} I_{13}\bigr) \\
&= 2\, \mathbb{P}\bigl(X_1 > X_2, X_3\bigr)\, \mathbb{P}\bigl(Y_1 < Y_2, Y_3\bigr) + 2\, \mathbb{P}\bigl(X_2 > X_1 > X_3\bigr)\, \mathbb{P}\bigl(Y_2 < Y_1 < Y_3\bigr) \\
&= 2\, \frac{\sum_{i=1}^m (i-1)^2}{m^3} \cdot \frac{1}{3} + 2 \frac{\binom{m}{3}}{m^3} \cdot \frac{1}{6} = \frac{m(m-1)(2m-1)}{9m^3} + \frac{m(m-1)(m-2)}{18m^3} \\
&= \frac{(m-1)(5m-4)}{18m^2}
\end{aligned}$$

and hence

$$\begin{aligned}
\operatorname{Cov}\bigl(I^*_{12}, I^*_{13}\bigr) &= \mathbb{E}\bigl(I^*_{12} I^*_{13}\bigr) - \mathbb{E}\bigl(I^*_{12}\bigr)^2 = \frac{(m-1)(5m-4)}{18m^2} - \frac{(m-1)^2}{4m^2} \\
&= \frac{(m-1)(m+1)}{36m^2}.
\end{aligned} \tag{37}$$

The variance formula (8) follows from (33), (36) and (37). □

Proof of Theorem 2. Consider the random word $W_{n,m} = X_1 \cdots X_n$ in Theorem 8. If we condition on the letters $X_1, \dots, X_m$ being distinct, then the number of inversions $\mathrm{Inv}(W_{n,m})$ has the same distribution as the number G_n of inversions in a random permutation. Hence, for any set $A \subset \mathbb{N}$,

$$\mathbb{P}\big(\mathrm{Inv}(W_{n,m}) \in A \mid X_1, \dots, X_n \text{ distinct}\big) = \mathbb{P}(G_n \in A)$$

and thus

$$\begin{aligned}\big|\mathbb{P}\big(\mathrm{Inv}(W_{n,m}) \in A\big) - \mathbb{P}(G_n \in A)\big| &\leqslant \mathbb{P}(X_1, \dots, X_n \text{ not distinct}) \\ &\leqslant \binom{n}{2} \mathbb{P}(X_1 = X_2) = \frac{\binom{n}{2}}{m} \to 0\end{aligned}$$

as $m \to \infty$, and thus $G_{n,m} \stackrel{\mathrm{d}}{=} \mathrm{Inv}(W_{n,m}) \xrightarrow{\mathrm{d}} G_n$. □

Remark 15. We have actually proved that the total variation distance $d_{\mathrm{TV}}(G_{n,m}, G_n) \leqslant \binom{n}{2}/m$. Moreover, the bound can be improved to

$$d_{\mathrm{TV}}(G_{n,m}, G_n) \leqslant \mathbb{P}(X_1, \dots, X_n \text{ not distinct}) = 1 - (m)_n/m^n,$$

where $(m)_n := m!/(m-n)!$.

Proof of Theorems 3 and 4. The two versions in each theorem are equivalent by (8), so it suffices to prove, for example, (11) and (13).

The central limit theorem Theorem 3 follows immediately from Hoeffding's central limit theorem for U-statistics [7] without any further calculations. Moreover, we shall see that the decomposition method used by Hoeffding [7] yields also Theorem 4; we therefore do the decomposition explicitly.

The idea is to decompose each term I^*_{ij} as

$$I^*_{ij} = \mu + \xi_i + \xi_j + \eta_{ij}, \tag{38}$$

where $\mu := \mathbb{E}\, I^*_{ij}$,

$$\xi_i := \mathbb{E}\big(I^*_{ij} - \mu \mid Z_i\big) = \mathbb{E}\big(I^*_{ij} \mid X_i, Y_i\big) - \mu \tag{39}$$

and η_{ij} is defined by (38). Then the random variables ξ_i ($1 \leqslant i \leqslant n$) and η_{ij} ($1 \leqslant i < j \leqslant n$) have mean 0 and are orthogonal (in L^2), so they are uncorrelated. In particular,

$$1 \geqslant \operatorname{Var} I^*_{ij} = \operatorname{Var} \xi_i + \operatorname{Var} \xi_j + \operatorname{Var} \eta_{ij}. \tag{40}$$

Moreover, $\xi_i = g(Z_i)$ for some function g, and thus the variables ξ_i are i.i.d.

By summing (38), we obtain by (31) a corresponding decomposition of $U_{n,m}$:

$$U_{n,m} = \binom{n}{2}\mu + (n-1)\sum_{i=1}^{n} \xi_i + \sum_{1 \leqslant i < j \leqslant n} \eta_{ij}. \tag{41}$$

Hence,

$$\frac{U_{n,m} - \mathbb{E}\, U_{n,m}}{n^{3/2}} = \frac{n-1}{n} n^{-1/2} \sum_{i=1}^{n} \xi_i + n^{-3/2} R, \tag{42}$$

where $R := \sum_{1 \leqslant i < j \leqslant n} \eta_{ij}$. Since the variables η_{ij} are uncorrelated, and $\operatorname{Var} \eta_{ij} \leqslant 1$ by (40), we have

$$\mathbb{E}\, R^2 = \operatorname{Var} R = \sum_{1 \leqslant i < j \leqslant n} \operatorname{Var} \eta_{ij} \leqslant \binom{n}{2} \leqslant n^2, \tag{43}$$

and thus $\mathbb{E}(n^{-3/2} R)^2 \to 0$. Hence, the last term in (42) is a small remainder term that can be ignored when $n \to \infty$. Furthermore, the decomposition (41) yields the variance decomposition

$$\begin{aligned} \operatorname{Var} U_{n,m} &= (n-1)^2 \sum_{i=1}^{n} \operatorname{Var} \xi_i + \sum_{1 \leqslant i < j \leqslant n} \operatorname{Var} \eta_{ij} \\ &= n(n-1)^2 \operatorname{Var} \xi_1 + \binom{n}{2} \operatorname{Var} \eta_{12} \\ &\sim n^3 \operatorname{Var} \xi_1 \end{aligned} \tag{44}$$

as $n \to \infty$, and thus by (8),

$$\operatorname{Var} \xi_1 = \frac{1}{36}\Big(1 - \frac{1}{m^2}\Big). \tag{45}$$

For fixed m (Theorem 3), the standard central limit theorem for sums of i.i.d. random variables now shows that

$$\frac{\sum_{i=1}^{n} \xi_i}{n^{1/2}} \xrightarrow{\mathrm{d}} N\Big(0, \frac{m^2-1}{36m^2}\Big), \tag{46}$$

and thus (11) follows from (42).

For $m \to \infty$ (Theorem 4), we have $\operatorname{Var} \xi_1 \to 1/36$ by (45); moreover, the random variables ξ_i are uniformly bounded (by 1), and thus the central limit theorem with e.g. Lyapounov's condition [5, Theorem 7.2.2] applies and shows that

$$\frac{\sum_{i=1}^{n} \xi_i}{n^{1/2}} \xrightarrow{\mathrm{d}} N\Big(0, \frac{1}{36}\Big), \tag{47}$$

and thus (13) follows from (42). □

Remark 16. It is interesting to do the decomposition (38) explicitly. Using the centred variables

$$X_i' := X_i - \mathbb{E}\, X_i = X_i - \frac{m+1}{2}, \tag{48}$$

$$Y_i' := Y_i - \mathbb{E}\, Y_i = Y_i - \frac{1}{2}, \tag{49}$$

we have by (29)

$$\mathbb{E}(I_{ij} \mid X_i, Y_i) = \frac{X_i - 1}{m}(1 - Y_i) = \frac{X_i' + (m-1)/2}{m}\Big(\frac{1}{2} - Y_i'\Big), \tag{50}$$

$$\mathbb{E}(I_{ji} \mid X_i, Y_i) = \frac{m - X_i}{m}Y_i = \frac{(m-1)/2 - X_i'}{m}\Big(Y_i' + \frac{1}{2}\Big), \tag{51}$$

and thus, using (30) and (35),

$$\xi_i := \mathbb{E}(I_{ij}^* \mid X_i, Y_i) - \mathbb{E}\, I_{ij}^* = -\frac{2}{m}X_i'Y_i'. \tag{52}$$

Hence, the decomposition is

$$I_{ij}^* = \frac{m-1}{2m} - \frac{2}{m}X_i'Y_i' - \frac{2}{m}X_j'Y_j' + \eta_{ij} \tag{53}$$

and

$$U_{n,m} = \binom{n}{2}\frac{m-1}{2m} - \frac{2(n-1)}{m}\sum_{i=1}^{n} X_i'Y_i' + R. \tag{54}$$

Note also that (45) follows from (52), and then (40) yields, using (36),

$$\operatorname{Var}\eta_{ij} = \operatorname{Var} I_{ij}^* - 2\operatorname{Var}\xi_i = \frac{1}{4}\Big(1 - \frac{1}{m^2}\Big) - \frac{2}{36}\Big(1 - \frac{1}{m^2}\Big) = \frac{7}{36}\Big(1 - \frac{1}{m^2}\Big), \tag{55}$$

which together with (45) and (44) yield another proof of (8).

Remark 17. It is also interesting to do the corresponding orthogonal decomposition of $V_{n,m}$ in (19). We have, similarly to (38),

$$\mathbf{1}\{X_i > X_j\} = \mu' + \xi_i' + \xi_j'' + \eta_{ij}', \tag{56}$$

where $\mu' := \mathbb{P}(X_i > X_j) = \frac{m-1}{2m}$, and, with X_i' as in (48),

$$\xi_i' := \mathbb{P}\big(X_i > X_j \mid X_i\big) - \mu' = \frac{X_i'}{m}, \tag{57}$$

$$\xi_j'' := \mathbb{P}\big(X_i > X_j \mid X_j\big) - \mu' = -\frac{X_j'}{m}, \tag{58}$$

and η_{ij}' is defined by (56). Summing we get,

$$\begin{aligned} V_{n,m} &= \mathbb{E}\, V_{n,m} + \sum_{i=1}^{n}(n-i)\xi_i' + \sum_{j=1}^{n}(j-1)\xi_j'' + \sum_{1\leqslant i<j\leqslant n}\eta_{ij}' \\ &= \mathbb{E}\, V_{n,m} + \frac{1}{m}\sum_{i=1}^{n}(n+1-2i)X_i' + \sum_{1\leqslant i<j\leqslant n}\eta_{ij}'. \end{aligned} \tag{59}$$

Straightforward calculations show that

$$\operatorname{Var} X_i' = \frac{1}{12}(m^2-1), \tag{60}$$

$$\operatorname{Var}(\mathbf{1}\{X_i > X_j\}) = \frac{1}{4}\Big(1-\frac{1}{m^2}\Big), \tag{61}$$

and, by (56),

$$\operatorname{Var}\eta_{ij}' = \operatorname{Var}(\mathbf{1}\{X_i > X_j\}) - \operatorname{Var}\xi_i' - \operatorname{Var}\xi_j'' = \frac{1}{12}\Big(1-\frac{1}{m^2}\Big). \tag{62}$$

Hence, (59) yields

$$\begin{aligned}\operatorname{Var} V_{n,m} &= \frac{1}{m^2}\sum_{i=1}^n (n+1-2i)^2 \operatorname{Var} X_i' + \sum_{1\leqslant i<j\leqslant n} \operatorname{Var}\eta_{ij}' \\ &= \frac{n(n-1)(n+1)}{36}\Big(1-\frac{1}{m^2}\Big) + \frac{n(n-1)}{24}\Big(1-\frac{1}{m^2}\Big),\end{aligned} \tag{63}$$

which gives yet another proof of (8).

We can also prove Theorems 3 and 4 using (59) instead of (41); again the final sum can be ignored since, using (62) and the fact that the η_{ij}' are uncorrelated,

$$\operatorname{Var}\Big(n^{-3/2}\sum_{i<j}\eta_{ij}'\Big) = n^{-3}\binom{n}{2}\frac{1}{12}\Big(1-\frac{1}{m^2}\Big) < \frac{1}{24n} \to 0 \tag{64}$$

as $n\to\infty$, cf. (43). The summands in $\sum_{i=1}^n (n+1-2i)X_i'$ are not identically distributed, but that does not matter since Lyapounov's condition holds. See [8, Corollary 11.20] for a general limit theorem for asymmetric sums like (19), and note that the argument in Section 3 is an instance of a general method to convert such sums into (symmetric) U-statistics by introducing the auxiliary variables Y_i, see [8, Remark 11.21].

In the case $m=2$, one can check that $\eta_{ij}' = -X_i'X_j'$ and thus

$$\sum_{1\leqslant i<j\leqslant n}\eta_{ij}' = -\frac{1}{2}\Big(\sum_{i=1}^n X_i'\Big)^2 + \frac{n}{2}, \tag{65}$$

which shows that the decomposition (59) then is essentially the same as the decomposition used by Takács [16].

Proof of Theorem 10. We use the decomposition (59) of $V_{n,m}$, and $N_k = \sum_{i=1}^n \mathbf{1}\{X_i = k\}$. The result follows by the central limit theorem with Lyapounov's condition applied to the random vector

$$\begin{aligned}&\Big(\frac{\sum_{i=1}^n (n+1-2i)X_i'}{n^{3/2}}, \frac{N_1-\mathbb{E}\,N_1}{n^{1/2}}, \dots, \frac{N_m-\mathbb{E}\,N_m}{n^{1/2}}\Big) \\ &\qquad = \sum_{i=1}^n \left(\frac{(n+1-2i)X_i'}{n^{3/2}}, \frac{\mathbf{1}\{X_i=1\}-1/m}{n^{1/2}}, \dots, \frac{\mathbf{1}\{X_i=m\}-1/m}{n^{1/2}}\right),\end{aligned}$$

together with (59) and (64); the variances and covariances are easily computed, noting that $\mathrm{Cov}\bigl(\sum_{i=1}^n (n+1-2i)X_i',\ \sum_{i=1}^n \mathbf{1}\{X_i = k\}\bigr) = 0$ for each k since $\sum_{i=1}^n (n+1-2i) = 0$. (This vector-valued central limit theorem follows, as is well-known, from the real-valued version [5, Theorem 7.2.2] by the Cramér–Wold device [5, Theorem 5.10.5].) □

6 Proof of Theorem 5

To prove the local limit theorem Theorem 5, we need estimates of the probability generating function $g_n^{(m)}(q) = m^{-n} G_n^{(m)}(q)$ for $q = e^{\mathrm{i}\theta}$ on the unit circle. We derive these estimates from the corresponding estimates of $\binom{n}{n_1,\dots,n_m}_q$ in [3] rather than from scratch. (We do not know whether the estimates below are the best possible.)

Consider a random word $W_{n,m}$ as in Section 2, let again $N_1, \dots, N_m$ be the number of occurrences of the different letters, and let $N^* := \max_{k\leqslant n} N_k$ and $N_* := n - N^*$. Similarly, for given $n_1, \dots, n_m$ with $n_1 + \dots + n_m = n$, let $n^* := \max_{k\leqslant n} n_k$ and $n_* := n - n^*$; let further

$$F_{n_1,\dots,n_m}(q) := \binom{n}{n_1,\dots,n_m}_q \Big/ \binom{n}{n_1,\dots,n_m}$$

be the probability generating function of the number of inversions in a random word consisting of n_1 1's, …, n_m m's, cf. (16). Thus $F_{n_1,\dots,n_m}(q)$ is the probability generating function of $V_{n,m} = \mathrm{Inv}(W_{n,m})$ conditioned on $N_k = n_k$, $k = 1, \dots, m$.

Lemma 18. *There exists $c > 0$ such that for all $m \geqslant 2$, $n \geqslant 2$ and real $\theta \in [-\pi, \pi]$,*

$$\bigl|g_n^{(m)}(e^{\mathrm{i}\theta})\bigr| \leqslant \begin{cases} e^{-cn^3\theta^2}, & 0 \leqslant |\theta| \leqslant 1/n, \\ e^{-cn}, & 1/n \leqslant |\theta| \leqslant \pi. \end{cases} \tag{66}$$

Proof. We assume in the proof for simplicity that n is large enough; this case is enough for our application in Theorem 5. It is easy (but not very interesting) to complete the proof by verifying the estimates (66) for each fixed $n \geqslant 2$ and some c (that now might depend on n); we omit the details but mention that the case when m is large follows using Theorem 2. We let $c_1, c_2, \dots$ denote some positive constants whose values are not important.

By [3, Lemma 4.1] there exists $\tau \in (0,1)$ such that if $|\theta| \leqslant \tau/n$, then for any $n_1, \dots, n_m$ with $n_1 + \dots + n_m = n$,

$$\bigl|F_{n_1,\dots,n_m}(e^{\mathrm{i}\theta})\bigr| \leqslant e^{-\sigma^2\theta^2/4},$$

where σ^2 depends on $n_1, \dots, n_m$ and by [3, Lemma 3.1] $\sigma^2 \geqslant n^2 n_*/36$. Furthermore, by [3, Lemma 4.4] there exists $c_1 > 0$ such that if $\tau/n \leqslant |\theta| \leqslant \pi$, then

$$\bigl|F_{n_1,\dots,n_m}(e^{\mathrm{i}\theta})\bigr| \leqslant e^{-c_1 n_*}.$$

Hence, if $n^* \leqslant 3n/4$ so that $n_* \geqslant n/4$ we have the estimates

$$\bigl|F_{n_1,\dots,n_m}(e^{\mathrm{i}\theta})\bigr| \leqslant e^{-c_2 n^3\theta^2}, \qquad |\theta| \leqslant \tau/n, \tag{67}$$

and

$$\bigl|F_{n_1,\dots,n_m}(e^{\mathrm{i}\theta})\bigr| \leqslant e^{-c_3 n}, \qquad \tau/n \leqslant |\theta| \leqslant \pi. \tag{68}$$

We return to our string $W_{n,m}$ with random numbers $N_1, \dots, N_m$ of different letters. We can, for any $m \geqslant 2$, partition $\{1, \dots, m\}$ into three sets with at most $m/2$ elements each, and thus

$$\mathbb{P}(N^* > 3n/4) \leqslant 3\,\mathbb{P}\bigl(\mathrm{Bi}(n, 1/2) > 3n/4\bigr) \leqslant 3e^{-c_4 n} \tag{69}$$

by Chernoff's inequality, see e.g. [9, Theorem 2.1].

When $|\theta| \leqslant \tau/n$, which implies $n^3\theta^2 = O(n)$, we obtain by (67) and (69),

$$\begin{aligned}
\bigl|g_n^{(m)}(e^{\mathrm{i}\theta})\bigr| &= \bigl|\mathbb{E}\, e^{\mathrm{i}\theta V_{n,m}}\bigr| \\
&= \Bigl|\mathbb{E}\bigl(e^{\mathrm{i}\theta V_{n,m}} \mid N^* \leqslant 3n/4\bigr)\,\mathbb{P}(N^* \leqslant 3n/4) \\
&\qquad + \mathbb{E}\bigl(e^{\mathrm{i}\theta V_{n,m}} \mid N^* > 3n/4\bigr)\,\mathbb{P}(N^* > 3n/4)\Bigr| \\
&\leqslant e^{-c_2 n^3\theta^2}\,\mathbb{P}(N^* \leqslant 3n/4) + \mathbb{P}(N^* > 3n/4) \\
&\leqslant e^{-c_2 n^3\theta^2} + 3e^{-c_4 n} \\
&\leqslant 4e^{-c_5 n^3\theta^2}.
\end{aligned} \tag{70}$$

This verifies (66) with some $c > 0$ for $c_6 n^{-3/2} \leqslant |\theta| \leqslant \tau/n$.

For $|\theta| < c_6 n^{-3/2}$, we first note that $\mathbb{P}(N^* \leqslant 3n/4) \geqslant c_7 > 0$ for all $m, n \geqslant 2$; this holds for all large n by (69) (and is easily seen for each fixed n). Hence, by the calculations in (70),

$$\begin{aligned}
1 - \bigl|g_n^{(m)}(e^{\mathrm{i}\theta})\bigr| &\geqslant 1 - \mathbb{P}(N^* > 3n/4) - \mathbb{P}(N^* \leqslant 3n/4)e^{-c_2 n^3\theta^2} \\
&= \mathbb{P}(N^* \leqslant 3n/4)\bigl(1 - e^{-c_2 n^3\theta^2}\bigr) \geqslant c_7 c_8 n^3\theta^2,
\end{aligned}$$

verifying (66) in this case too (for $c \leqslant c_7 c_8$).

Finally, for $\tau/n \leqslant |\theta| \leqslant \pi$, we obtain by arguing as in (70), now using (68) and (69),

$$\bigl|g_n^{(m)}(e^{\mathrm{i}\theta})\bigr| \leqslant e^{-c_3 n}\,\mathbb{P}(N^* \leqslant 3n/4) + \mathbb{P}(N^* > 3n/4) \leqslant e^{-c_3 n} + 3e^{-c_4 n} \leqslant e^{-c_9 n},$$

provided n is large enough. This completes the proof (for large n) for the cases $\tau/n \leqslant |\theta| \leqslant 1/n$ and $1/n \leqslant |\theta| \leqslant \pi$. □

Proof of Theorem 5. Consider any sequence $m = m(n) \geqslant 2$. We will show that (14) holds uniformly in k for any such sequence $m(n)$; this is equivalent to the asserted uniform convergence for all $m \geqslant 2$.

Denote the characteristic function of $G_{n,m}$ by $\varphi_n(\theta)$, and recall that it is given by $\varphi_n(\theta) = g_n^{(m)}(e^{\mathrm{i}\theta})$, see (6). It follows from Theorems 3 and 4 that

$$\frac{G_{n,m} - \mu_{n,m}}{\sigma_{n,m}} \xrightarrow{\mathrm{d}} N(0,1) \tag{71}$$

as $n \to \infty$. (To see this we may by considering subsequences assume that $m(n)$ converges to either a finite limit or to ∞; then (71) is (10) or (12).) Thus, by the continuity theorem, for any fixed $\theta \in \mathbb{R}$,

$$e^{-\mathrm{i}\theta\mu_{n,m}/\sigma_{n,m}}\varphi_n(\theta/\sigma_{n,m}) \to e^{-\theta^2/2}. \tag{72}$$

Let

$$r_n(\theta) := e^{-\mathrm{i}\theta\mu_{n,m}/\sigma_{n,m}}\varphi_n(\theta/\sigma_{n,m})\mathbf{1}\{|\theta| \leqslant \pi\sigma_{n,m}\} - e^{-\theta^2/2}, \tag{73}$$

and note that $r_n(\theta) \to 0$ as $n \to \infty$ for each fixed θ by (72) since $\sigma_{n,m} \to \infty$ by (8).

By Fourier inversion we have

$$\begin{aligned}
\sigma_{n,m}\,\mathbb{P}(G_{n,m} = k) &= \frac{\sigma_{n,m}}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}kt}\varphi(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi\sigma_{n,m}}^{\pi\sigma_{n,m}} e^{-\mathrm{i}k\theta/\sigma_{n,m}}\varphi(\theta/\sigma_{n,m})\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}(\mu_{n,m}-k)\theta/\sigma_{n,m}}\Big(r_n(\theta) + e^{-\theta^2/2}\Big)\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}(\mu_{n,m}-k)\theta/\sigma_{n,m}} r_n(\theta)\,\mathrm{d}\theta + \frac{1}{\sqrt{2\pi}} e^{-(\mu_{n,m}-k)^2/2\sigma_{n,m}^2},
\end{aligned}$$

and thus, for all $k \in \mathbb{Z}$,

$$\left|\sigma_{n,m}\,\mathbb{P}(G_{n,m} = k) - \frac{1}{\sqrt{2\pi}} e^{-(\mu_{n,m}-k)^2/2\sigma_{n,m}^2}\right| \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|r_n(\theta)\big|\,\mathrm{d}\theta.$$

The result (14) follows since

$$\int_{-\infty}^{\infty} \big|r_n(\theta)\big|\,\mathrm{d}\theta \to 0$$

as $n \to \infty$ by dominated convergence, using Lemma 18; note that if $|\theta| \leqslant \pi\sigma_{n,m}$, then $|\theta| \leqslant n^{3/2}$ since $\pi^2\sigma_{n,m}^2 < n^3$ by (8), and hence (66) yields

$$\big|\varphi_n(\theta/\sigma_{n,m})\big| = \big|g_n^{(m)}(e^{\mathrm{i}\theta/\sigma_{n,m}})\big| \leqslant e^{-cn^3\theta^2/\sigma_{n,m}^2} + e^{-cn} \leqslant e^{-c\theta^2} + e^{-c\theta^{2/3}};$$

hence, for all $n \geqslant 2$ and $\theta \in \mathbb{R}$,

$$\big|r_n(\theta)\big| \leqslant 2e^{-c\theta^2} + e^{-c\theta^{2/3}}.$$

The version with $\bar\mu_{n,m}$ and $\bar\sigma_{n,m}^2$ follows in exactly the same way, starting with

$$\frac{G_{n,m} - \bar\mu_{n,m}}{\bar\sigma_{n,m}} \xrightarrow{\mathrm{d}} N(0,1), \tag{74}$$

which is equivalent to (71) since $\bar\sigma_{n,m}^2 \sim \sigma_{n,m}^2$ and $\bar\mu_{n,m} = \mu_{n,m} + o(\sigma_{n,m})$ as $n \to \infty$ by Theorem 1. □

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