Turán $\boldsymbol{H}$-densities for 3-graphs

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Abstract

Given an r-graph H on h vertices, and a family $\mathcal{F}$ of forbidden subgraphs, we define $\mathrm{ex}_H(n, \mathcal{F})$ to be the maximum number of induced copies of H in an $\mathcal{F}$-free r-graph on n vertices. Then the *Turán H-density* of $\mathcal{F}$ is the limit

$$\pi_H(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}_H(n, \mathcal{F})/\binom{n}{h}.$$

This generalises the notions of *Turán density* (when H is an r-edge), and *inducibility* (when $\mathcal{F}$ is empty). Although problems of this kind have received some attention, very few results are known.

We use Razborov's semi-definite method to investigate Turán H-densities for 3-graphs. In particular, we show that

$$\pi_{K_4^-}(K_4) = 16/27,$$

with Turán's construction being optimal. We prove a result in a similar flavour for K_5 and make a general conjecture on the value of $\pi_{K_t^-}(K_t)$. We also establish that

$$\pi_{4.2}(\emptyset) = 3/4,$$

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where 4.2 denotes the 3-graph on 4 vertices with exactly 2 edges. The lower bound in this case comes from a random geometric construction strikingly different from previous known extremal examples in 3-graph theory. We give a number of other results and conjectures for 3-graphs, and in addition consider the inducibility of certain directed graphs. Let $\vec{S}_k$ be the *out-star* on k vertices; i.e. the star on k vertices with all $k-1$ edges oriented away from the centre. We show that

$$\pi_{\vec{S}_3}(\emptyset) = 2\sqrt{3} - 3,$$

with an iterated blow-up construction being extremal. This is related to a conjecture of Mubayi and Rödl on the Turán density of the 3-graph C_5. We also determine $\pi_{\vec{S}_k}(\emptyset)$ when $k = 4, 5$, and conjecture its value for general k.

Keywords: Turán problems, extremal hypergraph theory, flag algebras

1 Introduction

1.1 Basic notation and definitions

Given $n \in \mathbb{N}$, write $[n]$ for the integer interval $\{1, 2, \ldots, n\}$. Let $r \in \mathbb{N}$. An *r-graph* or *r-uniform hypergraph* G is a pair $G = (V, E)$, where $V = V(G)$ is a set of *vertices* and $E = E(G) \subseteq V^{(r)} = \{A \subseteq V : |A| = r\}$ is a set of *r-edges*. We shall often write $x_1 x_2 \cdots x_r$ as a short-hand for the r-edge $\{x_1, x_2, \ldots, x_r\}$.

Given a family of r-graphs $\mathcal{F}$, we say that G is $\mathcal{F}$-free if it contains no member of $\mathcal{F}$ as a subgraph. A classical aim of extremal hypergraph theory is to determine the maximum number of r-edges that an $\mathcal{F}$-free r-graph on n vertices may contain. We call the corresponding function of n the *Turán number* of $\mathcal{F}$, and denote it by

$$\mathrm{ex}(n, \mathcal{F}) = \max\{|E(G)| \ : \ G \text{ is } \mathcal{F}\text{-free}, \ |V(G)| = n\}.$$

In this paper we shall be concerned with the following generalisation of the Turán number. Given an r-graph H on h vertices, and an r-graph G on $n \geqslant h$ vertices, let $e_H(G)$ denote the number of h-sets from $V(G)$ that induce a copy of H in G. (So for example if H is an r-edge, then $e_H(G)$ counts the number of edges in G.) Then, given a family of forbidden r-graphs $\mathcal{F}$, we define the *Turán H-number* of $\mathcal{F}$, denoted $\mathrm{ex}_H(n, \mathcal{F})$, to be the maximum number of induced copies of H that an $\mathcal{F}$-free r-graph on n vertices may contain:

$$\mathrm{ex}_H(n, \mathcal{F}) = \max\{e_H(G) : \ G \text{ is } \mathcal{F}\text{-free}, \ |V(G)| = n\}.$$

In general, the Turán H-number is, like the usual Turán number, hard to determine, and we are interested instead in the asymptotic proportion of h-vertex subsets that induce a copy of H. The following is well-known.

Proposition 1. *Let $\mathcal{F}$ be a family of r-graphs and let H be an r-graph on h vertices. Then the limit*

$$\pi_H(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}_H(n,\mathcal{F})/\binom{n}{h}$$

exists.

Proof. For $n \geqslant h$, it follows by averaging over n-vertex subsets that

$$\mathrm{ex}_H(n+1,\mathcal{F})/\binom{n+1}{h} \leqslant \mathrm{ex}_H(n,\mathcal{F})/\binom{n}{h}.$$

Thus the sequence $\mathrm{ex}_H(n,\mathcal{F})/\binom{n}{h}$ is nonincreasing, and because it is bounded below (e.g. by 0), it is convergent. □

We call $\pi_H(\mathcal{F})$ the *Turán H-density* of $\mathcal{F}$. In the case where H is the r-graph on r vertices with a single edge, we recover the classical *Turán density*, $\pi(\mathcal{F})$.

It is easy to see that Proposition 1 and the definitions of $\mathrm{ex}_H(n,\mathcal{F})$ and $\pi_H(\mathcal{F})$ when H and $\mathcal{F}$ consist of r-graphs could just as well have been made in the setting of directed r-graphs. We let our definitions carry over *mutatis mutandis*.

In this paper, we shall mainly investigate 3-graphs, although we shall make a digression into directed 2-graphs in Section 3.

1.2 Previous work on inducibility

When $\mathcal{F} = \emptyset$, $\pi_H(\emptyset)$ is known as the *inducibility* of H. The inducibility of 2-graphs was first investigated by Pippenger and Golumbic [28] and later by Exoo [9]. Motivated by certain questions in Ramsey Theory, Exoo proved some general bounds on $\pi_H(\emptyset)$ as well as giving some constructions for small H with $|V(H)| \leqslant 4$. Bollobás, Nara and Tachibana [5] then proved that $\pi_{K_{t,t}}(\emptyset) = (2t)!/2^t(t!)^2$, where $K_{t,t}$ is the balanced complete bipartite graph on $2t$ vertices, $K_{t,t} = ([2t], \{\{ij\} : \ i \leqslant t < j\})$. What is more, they determined $\mathrm{ex}_{K_{t,t}}(n,\emptyset)$ exactly, with the optimal construction a balanced complete bipartite graph. More generally, Brown and Sidorenko [6] showed that if H is complete bipartite then the graphs attaining the Turán H-number may be chosen to be themselves complete bipartite.

Given a graph H and an integer $b \geqslant 1$, the *(balanced) b-blow-up* of H, denoted $H(b)$, is the graph on $b\,|V(H)|$ vertices obtained by taking for every vertex $x \in V(H)$ a set of b vertices $x_1, x_2, \dots, x_b$ and putting an edge between x_i and y_j if and only if $xy \in E(H)$. Bollobás, Egawa, Harris and Jin [4] proved that for all $t \in \mathbb{N}$ and all b sufficiently large, the Turán $K_t(b)$-number $\mathrm{ex}_{K_t(b)}(n,\emptyset)$ is attained by balanced blow-ups of K_t. This was recently generalised in an asymptotic sense by Hatami, Hirst and Norine [17] who proved that for any graph H and for all b sufficiently large, the Turán $H(b)$-density is given by considering the 'limit' of balanced blow-ups of H. Their proof relied on the use of weighted graphs.

Finally, several H-density results for small H were obtained this year by Grzesik [16], Hatami, Hladký, Král, Norine and Razborov [18], Hirst [20] and Sperfeld [33], all using the semi-definite method of Razborov [29]. Grzesik [16], and independently Hatami, Hladký, Král, Norine and Razborov [18], proved an old conjecture of Erdős [25] that the number of (induced) copies of the 5-cycle $C_5^{(2)} = ([5], \{12, 23, 34, 45, 51\})$ in a triangle-free graph on n vertices is at most $(n/5)^5$. This bound is attained by a balanced blow-up of $C_5^{(2)}$, thus establishing that

$$\pi_{C_5^{(2)}}(K_3) = 24/625.$$

To describe the other two sets of results, we need to make some more definitions. Let

$$K_{1,1,2} = ([4], \{12, 13, 14, 23, 24\}),\ \text{paw} = ([4], \{12, 23, 31, 14\})$$

and

$$\vec{C}_3 = ([3], \{\vec{12}, \vec{23}, \vec{31}\}),\ \vec{K}_2 \sqcup E_1 = ([3], \{\vec{12}\}).$$

Then Hirst showed that

$$\pi_{K_{1,1,2}}(\emptyset) = 72/125,\ \pi_{\text{paw}}(\emptyset) = 3/8,$$

with extremal configurations a balanced blow-up of K_5 and the complement of a balanced blow-up of $([4], \{12, 34\})$ respectively. Sperfeld proved

$$\pi_{\vec{C}_3}(\emptyset) = 1/4,\ \pi_{\vec{K}_2 \sqcup E_1}(\emptyset) = 3/4,$$

with extremal configurations a random tournament on n vertices and the disjoint union of two tournaments on $n/2$ vertices respectively.

1.3 Flag algebras and Flagmatic

Similarly to the works cited above [16, 18, 20, 33], the upper bounds on Turán H-densities we present in this paper have been obtained using the semi-definite method of Razborov [29]. A by-product of the theory of flag algebras, the semi-definite method gives us a systematic way of proving linear inequalities between subgraph densities. It has recently been used in a variety of contexts and has yielded many new results and improved bounds. (See e.g. [2, 3, 10, 16, 18, 19, 20, 23, 30, 31, 33].)

While it is clearly a powerful and useful tool in extremal combinatorics, the semi-definite method requires its users to overcome two barriers. First of all, a presentation of the method is usually given in the language of flag algebras, quantum graphs or graphons, which, while not impenetrable, is certainly forbidding at first. Second, the method involves numerous small computations, the enumeration of large graph families and optimisation of the entries of large positive semi-definite matrices; none of which can practically be done by hand. The assistance of a computer program is therefore necessary to use the semi-definite method in any nontrivial fashion.

In our earlier paper [10], we sought to remove these two obstacles by giving an elementary presentation of the semi-definite method from the point of view of extremal combinatorics, stripping it away from the more general framework of flag algebras, and by releasing 'Flagmatic', an open-source implementation of Razborov's semi-definite method.

Additionally, in an effort to avoid having large matrices and lists of graphs cluttering the main body of the paper, we have used Flagmatic to produce certificates of our results. These certificates, along with Flagmatic, can be downloaded from our website:

http://flagmatic.org/

The certificates are also given in the ancillary files section of our arXiv submission. The certificates are in a straight-forward human-readable format, which is documented in our previous paper [10]. The website also contains an independent checker program called `inspect_certificate.py`, which can be used to examine the certificates and help verify our proofs.

We shall not repeat here our introduction to the semi-definite method, nor our discussion of certificates and checker programs, but refer the reader back to [10] for details and use Flagmatic as a 'black box' for the remainder of this paper.

Finally, let us note that some information on extremal constructions can sometimes be extracted from proofs via the semi-definite method. We address this, and in particular the issue of stability, in a forthcoming paper [11].

1.4 Contents and structure of the paper

Let us define formally the 3-graphs that we study in this paper. First of all, we have the complete 3-graph on 4 vertices, K_4, also known as the *tetrahedron*. We shall also be interested in K_4^-, the unique (up to isomorphism) 3-graph on 4 vertices with exactly 3 edges, and in the *(strong) 5-cycle*, $C_5 = ([5], \{123, 234, 345, 451, 512\})$. (Note that this differs from the 2-graph $C_5^{(2)}$ introduced in the section 1.2.) Let also K_t denote the complete 3-graph on t vertices and K_t^- the 3-graph obtained from K_t by deleting a 3-edge, and let H_6 be the 3-graph obtained from C_5 by adding a new vertex labelled '6' to the vertex set and adding the following five edges: $136, 356, 526, 246, 416$.

A 3-graph is said to have *independent neighbourhoods* if for any pair of distinct vertices x, y, the joint neighbourhood of x, y,

$$\Gamma_{xy} = \{z : \; xyz \text{ is an edge}\}$$

is an independent set. Having independent neighbourhoods is easily seen to be equivalent to not containing the graph $F_{3,2} = ([5], \{123, 124, 125, 345\})$ as a subgraph.

Finally, following the notation used by Flagmatic, we write $m.k$ for the collection of all 3-graphs on m vertices spanning exactly k edges, up to isomorphism. For example, $4.3 = \{K_4^-\}$.

Our exact results for Turán H-densities of 3-graphs are listed in the following table:

Result	Extremal construction
$\pi_{K_4^-}(K_4) = 16/27$	Turán's construction: balanced blow-up of $([3], \{112, 223, 331, 123\})$.
$\pi_{4.2}(\emptyset) = 3/4$	Random geometric construction; see Theorem 28.
$\pi_{4.2}(C_5, F_{3,2}) = 9/16$	Balanced blow-up of K_4.
$\pi_{4.2}(K_4^-, F_{3,2}) = 5/9$	Balanced blow-up of H_6.
$\pi_{4.2}(K_4^-, C_5, F_{3,2}) = 4/9$	Balanced blow-up of a 3-edge.
$\pi_{K_4}(F_{3,2}) = 3/32$	Balanced blow-up of K_4.
$\pi_{K_4^-}(F_{3,2}) = 27/64$	Unbalanced blow-up of $([2], \{112\})$.
$\pi_{5.6}(\emptyset) = 20/27$	Balanced blow-up of the 3-graph $([3], \{112, 221, 223, 332, 113, 331\})$.
$\pi_{5.7}(\emptyset) = 20/27$	Balanced blow-up of the 3-graph $([3], \{111, 222, 333, 112, 223, 331, 123\})$.
$\pi_{5.9}(\emptyset) = 5/8$	Balanced complete bipartite 3-graph.

In addition, we prove two inducibility results for directed graphs. We define the *out-star* of order k to be the directed graph

$$\vec{S}_k = ([k], \{\vec{1i} : \ i \in [k] \setminus \{1\}\}).$$

We prove that

$$\pi_{\vec{S}_3}(\emptyset) = 2\sqrt{3} - 3,$$

with the extremal construction being an unbalanced blow-up of $\vec{S}_2$, iterated inside the part corresponding to the vertex labelled 2. (Here 'iterated' just means: repeat the construction inside the vertices that were allocated to part 2 after each iteration of the construction, until you run out of vertices.) Sperfeld [33] previously gave bounds for this problem.

This result is interesting to us for two reasons: first of all, this directed 2-graph problem has a somewhat close and unexpected relation to the Turán problem of maximising the number of 3-edges in a C_5-free 3-graph. Second, we believe this is the first 'simple' instance for which it can be shown that an iterated blow-up construction is extremal. (We elaborate on this in Section 3.)

While it is not directly relevant to 3-graphs, which are the main focus of this paper, we also determine $\pi_{\vec{S}_k}(\emptyset)$ for $k = 4, 5$ and make a conjecture regarding the value of $\pi_{\vec{S}_k}(\emptyset)$ for general k.

Our paper is structured as follows: in Section 2 we present our 3-graph results. Section 2.1 deals with the case where we forbid K_4 and other complete graphs, while Sections 2.2, 2.3 and 2.4 are concerned with the cases where we forbid C_5, K_4^- and both C_5 and K_4^- respectively. In Section 2.5 we consider 3-graphs with the independent neighbourhood property, and Section 2.6 gathers our results on inducibilities of 3-graphs, in particular our proof that $\pi_{4.2}(\emptyset) = 3/4$. Finally, in Section 3 we move on to consider directed 2-graphs and discuss the relation between $\pi_{\vec{S}_3}(\emptyset)$ and a conjecture of Mubayi and Rödl regarding the Turán density of the 3-graph C_5.

As previously mentioned, the certificates for all the results are available on the Flagmatic website, and in the ancillary files of our arXiv submission. Each certificate has a unique filename, which is given in the following table:

Result	Certificate	Result	Certificate
Theorem 4	`k4max43.js`	Theorem 28	`max42.js`
Proposition 7	`c5max43.js`	Proposition 31	`max43.js` and `41max43.js`
Proposition 9	`c5max42.js`	Theorem 32	`max56.js`
Theorem 11	`c5f32max42.js`	Theorem 33	`max57.js`
Proposition 13	`k4-max42.js`	Theorem 34	`max59.js`
Theorem 15	`k4-f32max42.js`	Proposition 36	`maxf32.js`
Proposition 18	`k4-c5max42.js`	Proposition 38	`maxc5.js`
Theorem 20	`k4-c5f32max42.js`	Theorem 42	`maxs3.js`
Theorem 23	`f32max43.js`	Theorem 44	`maxs4.js`
Theorem 24	`f32max44.js`	Theorem 46	`maxs5.js`
Proposition 25	`f32max42.js` and `f32max41.js`		

2 Main results

2.1 Forbidding K_4

The problem of determining the Turán density of the complete 3-graph on 4 vertices, K_4, has been open for more than sixty years. Turán conjectured that the answer is 5/9, with the lower bound coming from a balanced blow-up of $([3], \{112, 223, 331, 123\})$.

Conjecture 2 (Turán)**.**

$$\pi(K_4) = 5/9.$$

Many other non-isomorphic K_4-free constructions with asymptotic edge-density 5/9 have since been found [7, 8, 13, 22], so that if Turán's conjecture is true, there is no stable extremal configuration and a proof is likely to be very hard.

Razborov observed that Turán's original construction is the only one known in which no 4-set spans exactly one 3-edge. Adding in this restriction, he found that he could use the semi-definite method to prove a weaker form of Turán's conjecture:

Theorem 3 (Razborov [30])**.**

$$\pi(K_4,\ \text{induced } 4.1) = 5/9.$$

What is more, Pikhurko [26] showed that Turán's construction is the unique, stable extremal configuration for this problem. We can show that in fact what Turán's construction does is to maximise the K_4^--density in K_4-free 3-graphs; this can be thought of as the most natural weakening of Turán's conjecture.

Theorem 4.

$$\pi_{K_4^-}(K_4) = 16/27$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from Turán's construction, a balanced blow-up of $([3], \{112, 223, 331, 123\})$.

Calculations of induced densities in blow-up configurations recur frequently in this paper. While these are straightforward, let us give all the details here as an example. We want to compute the induced density of K_4^- in Turán's construction on n vertices, T_n. Let us suppose that our n vertices are equitably partitioned as $V_1 \sqcup V_2 \sqcup V_3$, with part V_i corresponding to vertex i in $([3], \{112, 223, 331, 123\})$. Which 4-sets induce a copy of K_4^- in T_n ? There are two kinds: firstly, one could pick three vertices in V_i and one vertex in V_{i+1} (taking addition modulo 3). Secondly one could pick two vertices in part V_i and one vertex in each of the two remaining parts V_{i+1} anbd V_{i+2}. It is easy to see that these are the only possibilities for spanning a K_4^-. The induced density of K_4^- is therefore

$$\begin{aligned} d_{K_4^-}(T_n) &= \left(\sum_i \binom{|V_i|}{3} |V_{i+1}| + \binom{|V_i|}{2} |V_{i+1}||V_{i+2}| \right) / \binom{n}{4} \\ &= \left(3 \left(\frac{n}{3}\right)^4 \left(\frac{1}{6} + \frac{1}{2} + O(n^{-1}) \right) \right) / \binom{n}{4} \\ &= \frac{16}{27} + O(n^{-1}). \end{aligned}$$

Taking the limit as $n \to \infty$ we get that the asymptotically maximal induced density of K_4^- in a K_4-free 3-graph is at least $16/27$, as claimed. $\square$

In addition, by essentially mimicking Pikhurko's argument, it is possible to show that any K_4-free 3-graph with K_4^--density 'close' to $16/27$ is 'close' to Turán's construction in the *edit distance*. That is, one can make it into a copy of Turán's construction by changing 'few' edges. We address this, and the more general issue of obtaining stability from proofs via the semi-definite method, in a forthcoming note [11].

Having established that $\pi_{K_4^-}(K_4) = 16/27$, can we say anything about $\pi_{K_5^-}(K_5)$? In Section 2.4 we give a result, Theorem 34, that implies $\pi_{K_5^-}(K_5) = 5/8$, with the lower bound coming from a complete balanced bipartite 3-graph. More generally, we believe we know what the value of $\pi_{K_t^-}(K_t)$ should be.

Define a sequence $(H_t)_{t \geqslant 2}$ of degenerate 3-graphs on t vertices as follows. Let

$$H_2 = ([2], \{111, 222, 112, 221\}),$$

and

$$H_3 = ([3], \{111, 222, 333, 112, 223, 331\}).$$

Now for $t \geqslant 4$, define H_t by adding vertices $t-1$ and t to H_{t-2}, together with the edges

$$(t-1)(t-1)(t-1),\ ttt,\ (t-1)(t-1)t,\ (t-1)tt.$$

Then let $G_t(n)$ denote the complement of a balanced blow-up of H_{t-1} on n vertices. This construction is due to Keevash and Mubayi, and is well-known (see for example Keevash [21]) to be K_t-free.

Conjecture 5. $G_t(n)$ is the unique (up to isomorphism) 3-graph with $\mathrm{ex}(n, K_t)$ edges and $\mathrm{ex}_{K_t^-}(n, K_t)$ induced copies of K_t^-.

It is easy to work out that $G_t(n)$ has edge-density

$$1 - \frac{4}{(t-1)^2} + o(1),$$

and a slightly more involved calculation shows that its K_t^--density is

$$\frac{t!}{(t-1)^{t-1}} \frac{2^{(t-1)/2}}{3} + o(1)$$

if t is odd, and

$$\frac{t!}{(t-1)^t} \frac{(5t-8)}{3} 2^{(t-6)/2} + o(1)$$

if t is even. Note that for $t = 4$ and $t = 5$ this agrees with Theorems 4 and 34 respectively.

2.2 Forbidding C_5

Mubayi and Rödl [24] studied the Turán density problem for C_5, and came up with the following ingenious construction. Partition the vertex set into two parts A and B with $|A| \approx \sqrt{3}\,|B|$, and add all edges that have two vertices in A and one vertex in B, and then iterate inside B. This can be described succinctly as an unbalanced blow-up of the (degenerate) 3-graph $([2], \{112\})$, iterated inside part 2. We leave it as an exercise for the reader to verify that this is indeed C_5-free. Mubayi and Rödl conjectured that this construction is best possible, and recent applications of the semi-definite method [10, 30] have provided strong evidence in that direction.

Conjecture 6 (Mubayi, Rödl [24])**.**

$$\pi(C_5) = 2\sqrt{3} - 3.$$

Observe now that Mubayi and Rödl's construction avoids K_4 as well as C_5. More generally any construction following the same pattern (i.e. any unbalanced blow-up of $([2], \{112\})$ iterated inside part 2) is $\{K_4, C_5\}$-free. Thus if their construction (which corresponds to one particular assignment of weights to parts 1 and 2) does maximise the edge-density in a C_5-free 3-graph, it is natural to expect that the same pattern of construction (albeit with a different assignment of weights) will also maximise the K_4^--density over C_5-free 3-graphs. This does appear to be the case:

Proposition 7.

$$0.423570 < \alpha \leqslant \pi_{K_4^-}(C_5) < 0.423592,$$

where α is the maximum value of

$$f(x) = \frac{4x(1-x)^3}{1-x^4},$$

in the interval $[0,1]$, which, by solving a cubic equation, can be computed explicitly to be

$$\alpha = 4 - 6\left((\sqrt{2}+1)^{1/3} - (\sqrt{2}-1)^{1/3}\right).$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a blow-up of $([2], \{112\})$, with proportion x of the vertices placed inside part 2, iterated inside part 2. The function $f(x)$ then calculates exactly the asymptotic density of K_4^- in such a construction. The sign of the derivative of f is determined by the product of a cubic and a linear factor. Performing the required calculus, the maximum of f can then be determined in closed form. □

Note that the maximum of f occurs at a cubic irrational, and not at a quadratic irrational as happens when we maximise the number of 3-edges. What is more, we place proportion approximately 0.366025 (i.e. a little more than 1/3) of the vertices inside part B when maximising the edge-density; and this drops down to approximately 0.253077 (i.e. a little more than 1/4) when maximising the K_4^- density. This is to be expected; in the first case we want an average 3-set to have about one vertex in part B, while in the latter case we want an average 4-set to have about one vertex in part B.

We conjecture that the lower bound in Proposition 7 is tight:

Conjecture 8.

$$\pi_{K_4^-}(C_5) = 4 - 6\left((\sqrt{2}+1)^{1/3} - (\sqrt{2}-1)^{1/3}\right).$$

Given the difference in the proportion of vertices assigned to part 2 between the case where we are maximising the number of edges and the case where we are maximising the number of copies of K_4^- in a C_5-free 3-graph, one could expect that the way to maximise the number of copies of 4.2—that is, of 4-sets spanning exactly 2 edges—would also be to take a blow-up of $([2], \{112\})$, iterated inside part 2, with a suitable proportion of vertices (say a little over 1/2) being assigned to part 2 at each stage of the iteration. This yields an asymptotic density of only

$$\max_{x \in [0,1]} \frac{6x^2(1-x)^2}{1-x^4},$$

which is approximately 0.404653. However, it turns out we can do much better using a different construction:

Proposition 9.

$$0.571428 < 4/7 \leqslant \pi_{4.2}(C_5) < 0.583852.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). For the lower bound, consider a balanced blow-up of K_4, iterated inside each part. □

We believe that the lower bound in Proposition 9 is tight:

Conjecture 10.

$$\pi_{4.2}(C_5) = 4/7.$$

While the upper bound we can obtain is still some way off 4/7, the following exact result gives us rather more confidence about Conjecture 10:

Theorem 11.

$$\pi_{4.2}(C_5, F_{3,2}) = 9/16.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). For a lower bound construction, take a balanced blow-up of K_4. □

In a sense Theorem 11 tells us that if we do not allow ourselves to use iterated blow-up constructions, then a blow-up of K_4 is the best we can do. We therefore expect that one cannot do better than an iterated blow-up of K_4 when this restriction is lifted.

2.3 Forbidding K_4^-

The 3-graph on four vertices with three edges, K_4^-, is the smallest 3-graph with non-trivial Turán density, both in terms of the number of vertices and the number of edges. Disproving an earlier conjecture of Turán, Frankl and Füredi [12] showed that $\pi(K_4^-) \geqslant 2/7$ by considering a balanced blow-up of H_6, iterated inside each of its 6 parts. Using his semi-definite method, Razborov [30] proved upper bounds for $\pi(K_4^-)$ quite close to this value (and small improvements were subsequently given in [2] and [10]), leading to the natural conjecture that the construction of Frankl and Füredi is in fact best possible:

Conjecture 12 (Frankl-Füredi, Razborov)**.**

$$\pi(K_4^-) = 2/7.$$

Should the conjecture be true, one would expect that an iterated blow-up of H_6 also maximises the number of induced copies of 4.2. As in the previous subsection, the semi-definite method is not quite able to close the gap; again we refer the reader to [10] for a discussion of why iterated blow-up constructions might be 'hard' for the method.

Proposition 13.

$$0.558139 < 24/43 \leqslant \pi_{4.2}(K_4^-) < 0.558378$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a balanced iterated blow-up of H_6. □

We believe that the lower bound is tight:

Conjecture 14.

$$\pi_{4.2}(K_4^-) = 24/43.$$

As before, restricting the setting to that of 3-graphs with independent neighbourhoods helps quite a lot, both for the original Turán problem and for the Turán 4.2-density problem. In [10] it was proved that $\pi(K_4^-, F_{3,2}) = 5/18$. The extremal construction, a balanced blow-up of H_6, is also extremal for the following problem.

Theorem 15.

$$\pi_{4.2}(K_4^-, F_{3,2}) = 5/9.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a balanced blow-up of H_6. □

2.4 Forbidding K_4^- and C_5

In [10], we considered the problem of forbidding both K_4^- and C_5. We have a lower bound of $\pi(K_4^-, C_5) \geqslant 1/4$ by considering a balanced blow-up of a 3-edge, with the construction iterated inside each of the 3 parts; and we gave an upper bound of $\pi(K_4^-, C_5) < 0.251073$ using the semi-definite method, leading us to conjecture that the lower bound is tight:

Conjecture 16 ([10]).

$$\pi(K_4^-, C_5) = 1/4.$$

Another construction yielding the same lower bound is as follows: let H_7 be the 6-regular 3-graph on 7 vertices

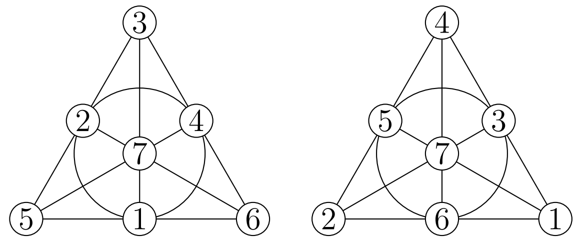
$$H_7 = ([7], \{124, 137, 156, 235, 267, 346, 457, 653, 647, 621, 542, 517, 431, 327\}).$$

This can be thought of as the unique (up to isomorphism) 3-graph G on 7 vertices such that for every vertex $x \in V(G)$, the link-graph $G_x = (V(G) \setminus \{x\}, \{yz :\ xyz \in E(G)\})$ is the 6-cycle. Alternatively, H_7 can be obtained as the union of two edge-disjoint copies of the Fano plane on the same vertex set

$$F_1 = ([7], \{124, 137, 156, 235, 267, 346, 457\}) \text{ and}$$
$$F_2 = ([7], \{653, 647, 621, 542, 517, 431, 327\}),$$

as depicted in Figure 1. (This elegant perspective is due to Füredi.)

It is an easy exercise to check that a balanced blow-up of H_7 with the construction iterated inside each of the 7 parts is both C_5-free and K_4^--free. (Alternatively, see [10] for details.) This also gives us a lower bound of 1/4 on $\pi(K_4^-, C_5)$. When we require independent neighbourhoods, iterated blow-ups are prohibited, and it turns out that a non-iterated blow-up of H_7 does better than a blow-up of a 3-edge (which gives edge-density 2/9):



F_1 F_2

Figure 1: Füredi's double Fano construction.

Theorem 17 ([10]).

$$\pi(K_4^-, C_5, F_{3,2}) = 12/49,$$

with the lower bound attained by a balanced blow-up of H_7.

Let us now turn to the problem of maximising the number of copies of 4.2 in a (C_5, K_4^-)-free 3-graph. As we are forbidding K_4^- (which is the same as forbidding 4.3), one might expect the problem of maximising the density of 4-sets spanning 2 edges to be essentially equivalent to the problem of maximising the number of edges (in the sense that the same pattern is extremal in both cases). However, the extremal behaviour of the two problems is different. An iterated blow-up of H_7 yields a lower bound of 20/57 (≈ 0.350877) for $\pi_{4.2}(K_4^-, C_5)$, but an iterated blow-up of a 3-edge does much better:

Proposition 18.

$$0.461538 < 6/13 \leqslant \pi_{4.2}(K_4^-, C_5) < 0.461645.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a balanced iterated blow-up of a 3-edge. □

We make the inevitable conjecture that the lower bound in Proposition 18 is tight:

Conjecture 19.

$$\pi_{4.2}(K_4^-, C_5) = 6/13.$$

Besides the relative proximity of the upper and lower bounds in Proposition 18, further motivation for Conjecture 19 can be found in the following exact result.

Theorem 20.

$$\pi_{4.2}(K_4^-, C_5, F_{3,2}) = 4/9.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a balanced blow-up of a 3-edge. □

By contrast, a balanced blow-up of H_7 only gives a lower bound of 120/343. Thus when $\{K_4^-, C_5, F_{3,2}\}$ is forbidden, the construction that maximises the density of the most dense 3-graph on four vertices that is allowed does not follow the same pattern as the construction that maximises the edge-density. This difference is not superficial: not only are the two constructions not isomorphic, but there is no homomorphism from H_7 into (a blow-up of) a 3-edge.

Indeed, label the 3 parts of the blow-up G of a 3-edge A, B and C, and suppose $f : V(H_7) \to A \sqcup B \sqcup C$ is a homomorphism. Since 137 is an edge of H_7, it must then be that 1, 3 and 7 are each mapped to different parts A, B, C; without loss of generality we may assume that $f(1) \in A$, $f(3) \in B$ and $f(7) \in C$. Since 134 is also an edge of H_7 we must also have $f(4) \in A$. But then 467 is an edge of H_7 with $f(4), f(7) \in C$, and so cannot be mapped by f to an edge of G, contradicting our assumption that f is a homomorphism.

This structural difference between the problems of maximising the number of 3-edges and of maximising the number of copies of 4.2 in a K_4^--free 3-graph is a somewhat surprising phenomenon. We ask whether this is due solely to the fact that we are forbidding C_5 and $F_{3,2}$ on top of K_4^-:

Question 21. Let m and $2 \leqslant t \leqslant \binom{m}{3}$ be integers. Does there exist for every $n \in \mathbb{N}$ an $m.t$-free 3-graph on n vertices that has both the maximum number of edges and the maximum number of copies of $m.(t-1)$ possible in an $m.t$-free graph?

Of course this question is most interesting when $t = \binom{m}{3}$; here $m.t$ and $m.(t-1)$ consist of just K_t and K_t^- respectively. In this case we believe the answer to Question 21 is 'yes', which is, in a weaker form, our Conjecture 21 from Section 2.1.

2.5 Independent neighbourhoods

We have now seen several examples of how restricting the setting to 3-graphs with independent neighbourhoods can render Turán problems significantly more tractable to the semi-definite method; we refer the reader to [10] for a heuristic discussion of why this might be so. In this subsection, we study Turán H-density problems in $F_{3,2}$-free 3-graphs for their own sake. The Turán density problem for $F_{3,2}$ was solved by Füredi, Pikhurko and Simonovits:

Theorem 22 (Füredi, Pikhurko, Simonovits [14])**.**

$$\pi(F_{3,2}) = 4/9.$$

In fact, they showed rather more: the unique, stable extremal configuration is an unbalanced blow-up of $([2], \{112\})$, with the size of the two parts chosen so as to maximise the number of edges, so that roughly 2/3 of the vertices are assigned to part 1 and 1/3 to part 2 [15]. Note that this configuration is K_4-free. We would therefore expect the same pattern to maximise the induced density of K_4^- in an $F_{3,2}$-free graph. This does turn out to be the case, with only the obvious change in the proportion of vertices assigned to each part necessary to obtain an extremal construction: we now want a random 4-set to have exactly three vertices in part 1 and one in part 2, rather than a random 3-set to have two vertices in part 1 and one in part 2.

Theorem 23.

$$\pi_{K_4^-}(F_{3,2}) = 27/64.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a blow-up of $([2], \{112\})$, with three quarters of the vertices assigned to part 1 and the rest to part 2. $\square$

As the above construction is C_5-free, Theorem 23 also implies

$$\pi_{K_4^-}(C_5, F_{3,2}) = 27/64,$$

providing us with an analogue for K_4^- of Theorem 11 from Section 2.2.

The next 3-graph whose density in $F_{3,2}$-free 3-graphs we investigate is K_4. Observing that K_5 is not $F_{3,2}$-free, one is naturally led to guess that the K_4-density is maximised by taking a balanced blow-up of K_4. This does indeed turn out to be the case:

Theorem 24.

$$\pi_{K_4}(F_{3,2}) = 3/32.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from a balanced blow-up of K_4. $\square$

Thus we are left with two 3-graphs on 4 vertices whose density in $F_{3,2}$-free 3-graphs we would like to maximise. However, we have been unable to obtain sharp results:

Proposition 25.

$$\begin{array}{rcccl} 4/9 & \leqslant & \pi_{4.1}(F_{3,2}) & < & 0.514719, \\ 9/16 & \leqslant & \pi_{4.2}(F_{3,2}) & < & 0.627732. \end{array}$$

Proof. The upper bounds are from flag algebra calculations using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bounds are from balanced blow-ups of $([3], \{112, 223, 331\})$ and K_4 respectively. $\square$

2.6 Inducibility

In this subsection, we study $\pi_H(\emptyset)$ for small 3-graphs H. The quantity $\pi_H(\emptyset)$ is often called the *inducibility* of H. Let $\bar{G}$ denote the *complement* of a 3-graph G; that is, the graph containing all edges not present in G. A graph G is said to be *self-complementary* if G and $\bar{G}$ are isomorphic.

It is easy to see that the H-density of a 3-graph G is equal to the $\bar{H}$-density of $\bar{G}$. Two immediate consequences of this are:

Lemma 26. *For any 3-graph H,*

$$\pi_H(\emptyset) = \pi_{\bar{H}}(\emptyset).$$

Lemma 27. *If H is self-complementary, then either there are at least two extremal constructions, or the extremal construction is itself self-complementary.*

We first study $\pi_H(\emptyset)$ for the 3-graphs H with $|V(H)| = 4$. Clearly we have $\pi_{K_4}(\emptyset) = \pi_{\bar{K}_4}(\emptyset) = 1$, so this leaves us only two values to determine, $\pi_{4.2}(\emptyset)$ and $\pi_{K_4^-}(\emptyset)$ (which by Lemma 26 is the same as $\pi_{4.1}(\emptyset)$).

Theorem 28.

$$\pi_{4.2}(\emptyset) = 3/4.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is from the following random geometric construction.

First of all, place n vertices on the boundary of the unit disc, spaced at equal intervals. Each pair of vertices (x, y) defines a chord of the unit circle. Consider the division of the unit disc into (mostly polygonal) regions given by these chords. We independently assign each region a value 0 or 1 with equal probability. Then, for each triple of vertices (x, y, z), we add the 3-edge xyz if and only if the sum of the values of the regions contained inside the triangle xyz is odd. This gives us our construction.

We shall now prove that with positive probability, at least 3/4 of the 4-sets of vertices induce the graph 4.2. Let us begin with two observations.

First of all, let R be any collection of regions. Then the probability that the sum of their values is odd is exactly 1/2. (So in particular, our construction has 3-edge density 1/2.) Second, if R and R' are two disjoint collections of regions, the parity of the sum of the values of the regions in R is independent from the parity of the sum of the values of the regions in R'.

From now on, let us speak of the parity of a collection of regions as a shorthand for the parity of the sum of the values of the regions it contains. Consider a 4-set of vertices $S = \{a, b, c, d\}$. We may assume without loss of generality that when traversing the unit circle clockwise from a, the vertices b, c and d are met in that order, as depicted in Figure 2. So a, b, c, d are the vertices of a convex quadrilateral. Let e be the intersection point of the diagonals ac and bd, and let R_1, R_2, R_3 and R_4 denote the triangles abe, bce, cde and ade.

Now, if zero or four of the R_i have odd parity, then the 4-set $S = \{a, b, c, d\}$ spans no edges in our construction. If one or three of the R_i has odd parity, then S spans two edges; this happens with probability 1/2. If two of the R_i have odd parity, there are two cases to consider: either the two R_i with odd parity are adjacent to each other—i.e. their boundaries intersect in a nontrivial line segment—in which case S spans two edges, or they are opposite one another, in which case S spans four edges. The former case occurs with probability 1/4.

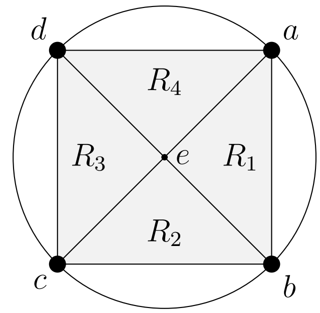


Figure 2: From the proof of Theorem 28.

Therefore the probability that S spans two edges is $3/4$. Since the choice of S was arbitrary, it follows that with positive probability our construction gives a 4.2-density of at least $3/4$, whence we are done. $\square$

Observe that in the lower bound construction we have just given, all 4-sets span an even number of 3-edges. Call any 3-graph with this property a *four-span-even* 3-graph. We can construct four-span-even 3-graphs from 2-graphs, a fact we can use to give an alternative description of the construction in Theorem 28.

Given a 2-graph G_2, let $G_3 = f(G_2)$ be the 3-graph on the same vertex set $V(G_3) = V(G_2)$ taking as its 3-edges the 3-sets spanning an odd number of 2-edges in G_2.

Proposition 29. *The operator f is a surjection from the set of 2-graphs to the set of four-span-even 3-graphs.*

Proof. First of all let us establish the easy fact that f maps 2-graphs to four-span-even graphs. Let G_2 be a 2-graph and let $G_3 = f(G_2)$. Consider a 4-set of vertices $S \subseteq V(G_2)$. Given $X \subseteq S$, write $e(X)$ for the number of edges of G_2 contained in X. Let also t be the number of 3-subsets of S spanning an odd number of 2-edges of G_2. Then,

$$\begin{aligned} t &\cong \sum_{X \in S^{(3)}} e(X) \mod 2 \\ &\cong 2e(S) \mod 2 \\ &\cong 0 \mod 2, \end{aligned}$$

so that $G_3 = f(G_2)$ has the four-span-even property as claimed.

Now let us show that every four-span-even 3-graph lies in the image of f.

First of all, observe that if G_3 is a four-span-even 3-graph then either G_3 contains no edges or G_3 is connected. Indeed, suppose G_3 contains at least one 3-edge $\{abc\}$. Then, by the four-span-even property, for every vertex $d \notin \{a, b, c\}$ there exists at least one 3-edge containing d and two of a, b, c.

As the 2-graph on n vertices with no edges is mapped by f to the 3-graph on n vertices with no edge, we may thus restrict our attention to four-span-even 3-graphs with at least one edge. We now prove by induction on the number of vertices that every n-vertex four-span-even 3-graph with at least one 3-edge is the image under f of some 2-graph.

The base case $n = 3$ is trivial: we have only one 3-graph to consider, namely the 3-edge $([3], \{123\})$, and both of the 2-graphs $([3], \{12\})$ and $([3], \{12, 13, 23\})$ are mapped to it by f. For the inductive step, let G_3 be a four-span-even 3-graph on $n+1 > 3$ vertices with at least one 3-edge. Let x be a vertex of G_3 such that $G_3' = G_3 - \{x\}$ contains at least one 3-edge. By the inductive hypothesis there is a 2-graph G_2' on n vertices such that $f(G_2') = G_3'$.

Now let us define an auxiliary 2-graph H on $V(G_3')$ by setting $\{ab\}$ to be an edge if *either* $\{ab\} \in E(G_2')$ and $\{xab\} \notin E(G_3)$ *or* $\{ab\} \notin E(G_2')$ and $\{xab\} \in E(G_3)$. (We can think of H as the graph telling us which pairs a, b need to receive an odd number of 2-edges from x to extend G_2' to a 2-graph mapping to G_3 under f.)

Claim 30. *The auxiliary 2-graph H is complete bipartite.*

Proof. Suppose H contains a triangle, i.e. three vertices a, b, c with ab, ac, bc all edges of H. Then consider the 4-set $S = \{abcx\}$. By the definitions of H and G_2', if $\{a, b, c\}$ spans two or three 2-edges of G_2' then S spans exactly one 3-edge in G_3, while if $\{a, b, c\}$ spans zero or one 2-edge of G_2' then S spans exactly three 3-edges in G_3. This contradicts the fact that G_3 has the four-span-even property.

Similarly, suppose H contains three vertices a, b, c spanning a single edge ab. Consider the 4-set $S = \{abcx\}$. Again by the definitions of H and G_2', if $\{a, b, c\}$ spans three 2-edges of G_2' then S spans exactly three 3-edges in G_3, if $\{a, b, c\}$ spans no 2-edge in G_2' then S spans exactly one 3-edge in G_3, and if $\{a, b, c\}$ spans one or two 2-edges of G_2' then S spans exactly one or exactly three 3-edges in G_3, contradicting the four-span-even property.

Thus every 3-set in H spans either no or two edges exactly. It immediately follows that H is complete bipartite. $\square$

Now choose one of the (at most) two vertex classes in H, and call it C. Let G_2 be the 2-graph on $n + 1$ vertices defined by $V(G_2) = V(G_2') \cup \{x\}$ and $E(G_2) = E(G_2') \cup \{xc : c \in C\}$. Since H is complete bipartite, all pairs of vertices in G_2' which needed to receive an odd number of 2-edges from x (i.e. all edges of H) receive exactly one such edge, while all pair of vertices which needed to receive an even number of 2-edges from x (i.e. all non-edges of H) receive either zero or two of them, as required. Thus G_2 is an extension of G_2' mapping to G_3 under f. This concludes the proof of the inductive step. $\square$

This operator f gives us another way of viewing the lower-bound construction in Theorem 28. Consider a random 2-graph G_2 on $[n]$, with each edge being present with probability $p = 1/2$. Now let $G_3 = f(G_2)$. By a straightforward case analysis we get that the probability a given 4-set from $[n]$ contains exactly two edges of G_3 is

$$\begin{aligned}\mathbb{P}(G_3[S] \cong 4.2) &= 6p^5(1-p) + 12p^4(1-p)^2 + 12p^3(1-p)^3 + 12p^2(1-p)^2 + 6p(1-p)^5 \\ &= 48 \cdot 2^{-6} = \frac{3}{4}.\end{aligned}$$

By linearity of expectation, the expected density of 4.2 in G_3 is thus $3/4$, from which it follows that there exist arbitrarily large 3-graph with 4.2 density greater or equal to $3/4$.

Note that this lower bound construction is quite different from previously known 3-graph constructions. Of those that have appeared in the literature, it resembles most the geometric construction of Frankl and Füredi [12], which it in some sense generalises. This construction also features vertices on the unit circle, where 3-edges are added whenever the corresponding triangle contains the origin in its interior. (This construction achieves a 4.2-density of $1/2$.)

Let us now consider the inducibility of K_4^-. Here by contrast we do not believe we have a good lower bound. We get a similar upper bound if we forbid 4-sets of vertices from spanning exactly one edge.

Proposition 31.

$$0.592592 < 16/27 \leqslant \pi_{K_4^-}(\emptyset) < 0.651912.$$

Also,

$$16/27 \leqslant \pi_{K_4^-}(\text{induced } 4.1) \leqslant 0.650930.$$

Proof. The upper bounds are from flag algebra calculations using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound in both cases is from Turán's construction: a balanced blow-up of $([3], \{123, 112, 223, 331\})$. □

It seems likely that both $\pi_{K_4^-}(\emptyset)$ and $\pi_{K_4^-}(\text{induced } 4.1)$ take values close to 0.65. Since Turán's construction has no induced copies of 4.1 and is (by Theorem 4) a K_4-free 3-graph maximising the K_4^--density, this would indicate that the actual extremal construction(s) for the inducibility of K_4^- have strictly positive K_4-density.

Turning to 5-vertex graphs, we are able to obtain a few more exact results.

Theorem 32.

$$\pi_{5.4}(\emptyset) = \pi_{5.6}(\emptyset) = 20/27.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound (for 5.6) is from a balanced blow-up of $([3], \{112, 221, 223, 332, 113, 331\})$. (This is just a balanced tripartition with all 3-edges meeting a part in two vertices exactly.) □

Theorem 33.

$$\pi_{5.3}(\emptyset) = \pi_{5.7}(\emptyset) = 20/27.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound (for 5.7) is from a balanced blow-up of $([3], \{111, 222, 333, 123, 112, 223, 331\})$. (This is just Turán's construction with all three parts made complete.) □

Theorem 34.

$$\pi_{5.1}(\emptyset) = \pi_{5.9}(\emptyset) = 5/8.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is obtained by taking a complete balanced bipartite 3-graph. □

In the forthcoming note [11], we prove that the complete balanced bipartite 3-graph is in fact the stable extremum for the inducibility of 5.9. This relates Theorem 34 to a conjecture of Turán on the Turán density of K_5, the complete 3-graph on 5 vertices.

Conjecture 35 (Turán)**.**

$$\pi(K_5) = 3/4.$$

One of the constructions attaining the bound is given by taking a balanced complete bipartite 3-graph. Many other non-isomorphic constructions are known [32]. However, what Theorem 34 shows is that the complete bipartite 3-graph is, out of all of these, the one which maximises the number of induced copies of K_5^-, that is of 5-sets spanning all but one of the possible 3-edges. This is a direct analogue of our earlier result Theorem 4.

We close this section on 3-graphs by giving upper bounds on the inducibility of two other 3-graphs on 5 vertices.

Proposition 36.

$$0.349325 < \alpha < \pi_{F_{3,2}}(\emptyset) < 0.349465,$$

where α *is the maximum of*

$$\frac{10x^2(1-x)^3}{1-x^5}$$

in the interval $[0, 1]$.

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound is obtained by taking a unbalanced blow-up of $([2], \{112, 222\})$, iterated inside part 1, where a proportion α of the vertices are assigned to part 1 at each stage. □

We believe that the lower bound construction given above is extremal:

Conjecture 37.

$$\pi_{F_{3,2}}(\emptyset) = \max_{x \in [0,1]} \frac{10x^2(1-x)^3}{1-x^5}.$$

Finally, we note that the random geometric construction given in Theorem 28, which is extremal for the inducibility of the self-complementary graph 4.2, also gives a reasonably good lower bound on the inducibility of the self-complementary graph C_5:

Proposition 38.

$$0.1875 = 3/16 \leqslant \pi_{C_5}(\emptyset) < 0.198845.$$

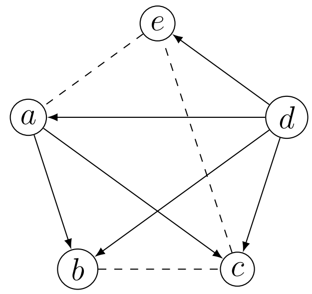


Figure 3: From the proof of Proposition 39.

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound comes from considering the random geometric construction we introduced in the proof of Theorem 28. As the vertices are scattered on the unit circle, any five of them define a convex pentagon. Drawing in the diagonals divides this pentagon into 11 disjoint polygonal regions. The result then follows from a rather tedious case analysis. Alternatively we can use the random 2-graph viewpoint introduced after the proof of Theorem 28 to do this. □

3 A digression into directed graphs

3.1 The out-star of order 3

We define the *out-star* of order k to be the directed graph

$$\vec{S}_k = ([k], \{\vec{1i} : \ i \in [k] \setminus \{1\}\}).$$

That is, the star with $k-1$ edges oriented away from the centre. In this subsection, we shall be interested in particular in $\vec{S}_3$ and its relation to the 3-graph C_5, the strong cycle on 5 vertices.

Given a directed graph D on n vertices, let us define a 3-graph $G(D)$ on the same vertex set by setting xyz to be a 3-edge whenever the 3-set $\{x, y, z\}$ induces a copy of $\vec{S}_3$ in D.

Proposition 39. *$G(D)$ is a (C_5, K_4)-free 3-graph.*

Proof. Let us first show that $G(D)$ is K_4 free. Suppose $\{a, b, c, d\}$ is a 4-set of vertices in $G(D)$ that spans a K_4. Without loss of generality, we may assume that $\vec{ab}, \vec{ac}$ are in $E(D)$. Therefore neither $\vec{bc}, \vec{cb}$ are in $E(D)$. Since $\{a, b, d\}$ also spans a 3-edge in $G(D)$, it follows that $\vec{ad} \in E(D)$ and $\vec{bd}, \vec{db} \notin E(D)$. But then $\{b, d, c\}$ spans at most one edge of D, and hence cannot be a 3-edge of $G(D)$, a contradiction.

Now suppose $\{a, b, c, d, e\}$ is a 5-set of vertices that spans a C_5 in $G(D)$, with edges abc, bcd, cde, dea and eab. Since abc is an edge, $\{a, b, c\}$ must induce a copy of $\vec{S}_3$ in D.

First of all, suppose we have $\vec{ab}, \vec{ac}$ in $E(D)$, and $\vec{bc}, \vec{cb}$ not in $E(D)$, as depicted in Figure 3. As $bcd \in E(G(D))$, $\{b, c, d\}$ must span a copy of $\vec{S}_3$, and we must have $\vec{db}, \vec{dc} \in E(D)$. Similarly, as $cde \in E(G(D))$ we have $\vec{de} \in E(D)$ and $\vec{ec}, \vec{ce} \notin E(D)$. Again as $dea \in E(G(D))$ we must have $\vec{da} \in E(D)$ and $\vec{ae}, \vec{ea} \notin E(D)$. But then $\{e, a, b\}$ cannot induce a copy of $\vec{S}_3$ in D, and hence eab cannot be a 3-edge of $G(D)$, a contradiction.

By symmetry, this argument also rules out the possibility of having $\vec{ca}, \vec{cb}$ both in $E(D)$ and $\vec{ab}, \vec{ba} \notin E(D)$. This leaves us with one last possibility, namely that both $\vec{ba}, \vec{bc}$ are in $E(D)$ and neither of $\vec{ac}, \vec{ca}$ is in $E(D)$. Since bcd is an edge of $G(D)$, this implies that $\vec{bd}$ is in $E(D)$ while neither of $\vec{cd}, \vec{dc}$ is. But this also leads to a contradiction by our previous argument, with bcd now playing the role of abc. Thus $G(D)$ must be C_5-free, as claimed. □

In fact more is true: the proof of the second part of Proposition 39 generalises to show that, for all integers $t \geqslant 3$ with t congruent to 1 or 2 modulo 3, $G(D)$ contains no copy of the strong t-cycle

$$C_t = ([t], \{123,\ 234,\ \dots\ , (t-2)(t-1)t,\ (t-1)t1,\ t12\}.$$

An interesting question is whether some kind of converse is true. Note that an immediate consequence of Proposition 39 is the following:

Corollary 40.

$$\pi_{\vec{S}_3}(\emptyset) \leqslant \pi(K_4, C_5, C_7).$$

It is easy to check that the conjectured extremal 3-graph construction of Mubayi and Rödl for the $\pi(C_5)$ problem is both K_4-free and C_t-free for all $t \geqslant 3$, where t is congruent to 1 or 2 modulo 3. We ask therefore the following question:

Question 41. Does there exist, for every $\varepsilon > 0$, a $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that if G is a C_5-free 3-graph on $n > N$ vertices with at least $(2\sqrt{3} - 3 - \delta)\binom{n}{3}$ edges, then there is a directed graph D on n vertices such that the 3-graphs G and $G(D)$ differ on at most $\varepsilon\binom{n}{3}$ edges?

An affirmative answer to Question 41 would, by our next result, automatically imply Conjecture 6:

Theorem 42.

$$\pi_{\vec{S}_3}(\emptyset) = 2\sqrt{3} - 3.$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound comes from an unbalanced blow-up of the directed graph

$$\vec{S}_2 = ([2], \{\vec{12}\}),$$

and iterating the construction inside part 1, setting at each stage of the construction a proportion $(\sqrt{3} - 1)/2$ of the vertices in part 1 and the remaining $(3 - \sqrt{3})/2$ proportion of the vertices in part 2. □

Denote the lower bound construction in Theorem 42 by D; then $G(D)$ is exactly the C_5-free construction of Mubayi and Rödl described in Section 2.2. It is an interesting question as to why exactly it is that Flagmatic can give us exact bounds on the $\vec{S}_3$-density problem for directed graphs, but not for the Turán density problem for the 3-graph C_5.

In a forthcoming note [11], we use the directed graph removal lemma of Alon and Shapira [1] to prove that the construction D is stable for this problem.

Theorem 42 is, to the best of our knowledge, the first known irrational inducibility. But perhaps more significantly, it is the first 'simple' problem for which an iterated blowup construction can be shown to be extremal. Pikhurko [27] has shown the far stronger result that every iterated blowup construction for 3-graphs is the unique extremal configuration for *some* Turán density problem. However his proof works by a kind of compactness argument, and does not give explicit families of suitable forbidden 3-graphs, but rather proves that such families exist.

3.2 Other directed graphs

Let us now consider $\vec{S}_4$. As in the previous subsection, given a directed graph D we define a 3-graph G on the same vertex set by letting xyz be a 3-edge if the 3-set $\{x, y, z\}$ induces a copy of the out-star of order 3, $\vec{S}_3$. Then the number of copies of K_4^- in $G(D)$ is exactly the number of copies of $\vec{S}_4$ in D, whence we have:

Proposition 43.

$$\pi_{\vec{S}_4}(\emptyset) \leqslant \pi_{K_4^-}(C_5).$$

Proof. By Proposition 39, for every directed graph D, $G(D)$ is C_5-free. The claimed inequality follows directly from our remark that copies of K_4^- in $G(D)$ correspond exactly to copies of $\vec{S}_4$ in D. □

We conjectured in Section 2.2 that

$$\pi_{K_4^-}(C_5) = 4 - 6\left((\sqrt{2}+1)^{1/3} - (\sqrt{2}-1)^{1/3}\right),$$

or, more helpfully, the maximum of

$$\frac{4x(1-x)^3}{1-x^4}$$

for $x \in [0, 1]$, which is attained at the unique real root of $3t^3 + 3t^2 + 3t - 1$. We have been unable to prove this using the semi-definite method, but, just as in the previous subsection, the directed graph problem proves to be more tractable, allowing us to show:

Theorem 44.

$$\pi_{\vec{S}_4}(\emptyset) = \frac{4p(1-p)^3}{1-p^4},$$

where p is the real root of $3t^3 + 3t^2 + 3t - 1$.

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound comes from taking an unbalanced blow-up of $\vec{S}_2$ and iterating the construction inside part 1, setting at each stage of the construction a proportion p of the vertices in part 1 and the remaining $1-p$ proportion of the vertices in part 2. $\square$

As in Theorem 42, call D our lower bound construction for Theorem 44. Then $G(D)$ coincides exactly with our lower bound construction in Section 2.2 for $\pi_{K_4^-}(C_5)$, which we conjectured to be optimal.

So what about $\pi_{\vec{S}_k}(\emptyset)$ for general k? Given Theorems 42 and 44 it is natural to guess that in general an unbalanced blowup of $\vec{S}_2$ iterated inside part 1 should be best possible. As we have shown, this is true for the cases $k=3$ and $k=4$, and we conjecture that this remains true for general k:

Conjecture 45. For every $k \geqslant 3$,

$$\pi_{\vec{S}_k}(\emptyset) = \alpha_k,$$

where

$$\alpha_k = \max_{x \in [0,1]} \frac{kx(1-x)^{k-1}}{1-x^k}.$$

With a little bit of calculus, we can describe α_k more precisely; the maximum of

$$\frac{kx(1-x)^{k-1}}{1-x^k}$$

occurs when $x = x_k$, where x_k is the unique positive root of the polynomial

$$(k-1)(t+t^2+\cdots+t^{k-1})-1.$$

Note that $x_k \in [0, 1/(k-1)]$ and $x_k \to 1/(k-1)$ as $k \to \infty$, as we would expect from our construction. Thus also $\alpha_k \to 1/e$ as $k \to \infty$.

With the aid of Flagmatic, we are able to establish the case $k=5$ of Conjecture 45:

Theorem 46.

$$\pi_{\vec{S}_5}(\emptyset) = \alpha_5,$$

where

$$\alpha_5 = \max_{x \in [0,1]} \frac{5x(1-x)^4}{1-x^5},$$

Proof. The upper bound is from a flag algebra calculation using Flagmatic (see Section 1.3 for how to obtain a certificate). The lower bound comes from taking an unbalanced blow-up of $\vec{S}_2$ and iterating the construction inside part 1, setting at each stage of the construction a proportion x_5 of the vertices in part 1 and the remaining $1-x_5$ proportion of the vertices in part 2, where x_5 is the value of x in $[0,1]$ at which the function $x \mapsto 5x(1-x)^4/(1-x^5)$ attains its maximum, i.e. the unique positive root of $4(x+x^2+\cdots+x^4)-1$. $\square$

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References

[1] N. Alon and A. Shapira, Testing subgraphs in directed graphs, *J. Comput. System Sci.*, **69** (2004), 353–382.

[2] R. Baber and J. Talbot, Hypergraphs do jump, *Combin. Probab. Comput.*, **20** (2010), 161–171.

[3] R. Baber and J. Talbot, New Turán densities for 3-graphs, preprint, 2011. http://arxiv.org/abs/1110.4287

[4] B. Bollobás, Y. Egawa, A. Harris, and G. P. Jin, The maximal number of induced r-partite subgraphs, *Graphs Combin.*, **11** (1995), 1–19.

[5] B. Bollobás, C. Nara, and S. Tachibana, The maximal number of induced complete bipartite graphs, *Discrete Math.*, **62** (1986), 271–275.

[6] J. I. Brown and A. Sidorenko. The inducibility of complete bipartite graphs, *J. Graph Theory*, **18** (1994), 629–645.

[7] W.G. Brown, On an open problem of Paul Turán concerning 3-graphs, *Studies in Pure Mathematics*, Birkhäuser, 1983, 91–93.

[8] D. G. Fon-der-Flaass, A method for constructing $(3,4)$-graphs, *Mat. Zametki*, **44** (1988), 546–550.

[9] G. Exoo, Dense packings of induced subgraphs, *Ars Combin.*, **22** (1986), 5–10.

[10] V. Falgas-Ravry and E. R. Vaughan, On applications of Razborov's flag algebra calculus to extremal 3-graph theory, preprint, 2011. http://arxiv.org/abs/1110.1623

[11] V. Falgas-Ravry and E. R. Vaughan, Stability and the semi-definite method, manuscript, 2011.

[12] P. Frankl and Z. Füredi, An exact result for 3-graphs, *Discrete Math.*, **50** (1984), 323–328.

[13] A. Frohmader, More constructions for Turán's (3,4)-conjecture, *Electron. J. Combin.*, **15** (2008), R137.

[14] Z. Füredi, O. Pikhurko, and M. Simonovits, The Turán density of the hypergraph $\{abc, ade, bde, cde\}$, *Electron. J. Combin.*, **10** (2003), R18.

[15] Z. Füredi, A. Pikhurko, and M. Simonovits, On Triple Systems with Independent Neighborhoods, *Combin. Probab. Comput.*, **14** (2005), 795–813.

[16] A. Grzesik, On the maximum number of C_5's in a triangle-free graph, preprint, 2011. http://arxiv.org/abs/1102.0962

[17] H. Hatami, J. Hirst, and S. Norine, The inducibility of blow-up graphs, preprint, 2011. http://arxiv.org/abs/1108.5699

[18] H. Hatami, J. Hladký, D. Král, S. Norine, and A. Razborov, On the number of pentagons in triangle-free graphs, preprint, 2011. http://arxiv.org/abs/1102.1634

[19] H. Hatami, J. Hladký, D. Král, S. Norine, and A. Razborov, Non-three-colorable common graphs exist, preprint, 2011. http://arxiv.org/abs/1105.0307

[20] J. Hirst, The Inducibility of Graphs on Four Vertices, preprint, 2011. http://arxiv.org/abs/1109.1592

[21] P. Keevash, Hypergraph Turán Problems, in *Surveys in combinatorics 2011*, Cambridge, 2011.

[22] A. Kostochka, A class of constructions for Turán's (3,4) problem, *Combinatorica*, **2** (1982), 187–192.

[23] D. Král, L. Mach, and J.-S. Sereni, A new lower bound based on Gromov's method of selecting heavily covered points, preprint, 2011. http://arxiv.org/abs/1108.0297

[24] D. Mubayi and V. Rödl, On the Turán number of triple systems, *J. Combin. Theory Ser. A*, **100** (2002), 136–152.

[25] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics (Cambridge, 1983)*, Academic Press, 1984, 1–17.

[26] O. Pikhurko, The minimum size of 3-graphs without a 4-set spanning no or exactly three edges, *European J. Combin.* **32** (2011), 1142–1155.

[27] O. Pikhurko, On possible Túran densities, preprint, 2012.

[28] N. Pippenger and M. C. Golumbic, The inducibility of graphs, *J. Combinatorial Theory Ser. B*, **19** (1975), 189–203.

[29] A. Razborov, Flag Algebras, *J. Symbolic Logic*, **72** (2007), 1239–1282.

[30] A. Razborov, On 3-hypergraphs with forbidden 4-vertex configurations, *SIAM J. Discrete Math.*, **24** (2010), 946–963.

[31] A. Razborov. On the Caccetta-Häggkvist conjecture with forbidden subgraphs, preprint, 2011. http://arxiv.org/abs/1107.2247

[32] A. Sidorenko, What we know and what we do not know about Turán numbers, *Graphs Combin.*, **11** (1995), 179–199.

[33] K. Sperfeld. The inducibility of small oriented graphs, preprint, 2011. http://arxiv.org/abs/1111.4813