An enumeration of flags in finite vector spaces

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Abstract

By counting flags in finite vector spaces, we obtain a q-multinomial analog of a recursion for q-binomial coefficients proved by Nijenhuis, Solow, and Wilf. We use the identity to give a combinatorial proof of a known recurrence for the generalized Galois numbers.

1 Introduction

For a parameter $q \neq 1$, and a positive integer n, let $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$, and $(q)_0 = 1$. For non-negative integers n and k, with $n \geqslant k$, the *q-binomial coefficient* or *Gaussian polynomial*, denoted $\binom{n}{k}_q$, is defined as $\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$.

The *Rogers-Szegö polynomial* in a single variable, denoted $H_n(t)$, is defined as

$$H_n(t) = \sum_{k=0}^{n} \binom{n}{k}_q t^k.$$

The Rogers-Szegö polynomials first appeared in papers of Rogers [16, 17] which led up to the famous Rogers-Ramanujan identities, and later were independently studied by Szegö [19]. They are important in combinatorial number theory ([1, Ex. 3.3–3.9] and [5, Sec. 20]), symmetric function theory [20], and are key examples of orthogonal polynomials [2]. They also have applications in mathematical physics [11, 13].

The Rogers-Szegö polynomials satisfy the recursion (see [1, p. 49])

$$H_{n+1}(t) = (1+t)H_n(t) + t(q^n - 1)H_{n-1}(t). \tag{1.1}$$

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Letting $t = 1$, we have $H_n(1) = \sum_{k=0}^{n} \binom{n}{k}_q$, which, when q is the power of a prime, is the total number of subspaces of an n-dimensional vector space over a field with q elements. The numbers $G_n = H_n(1)$ are the *Galois numbers*, and from (1.1), satisfy the recursion

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1}. \tag{1.2}$$

The Galois numbers were studied from the point of view of finite vector spaces by Goldman and Rota [6], and have been studied extensively elsewhere, for example, in [15, 9]. In particular, Nijenhuis, Solow, and Wilf [15] give a bijective proof of the recursion (1.2) using finite vector spaces, by proving, for integers $n \geqslant k \geqslant 1$,

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n - 1)\binom{n-1}{k-1}_q. \tag{1.3}$$

For non-negative integers $k_1, k_2, \ldots, k_m$ such that $k_1 + \cdots + k_m = n$, we define the q-multinomial coefficient of length m as

$$\binom{n}{k_1, k_2, \ldots, k_m}_q = \frac{(q)_n}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_m}},$$

so that $\binom{n}{k}_q = \binom{n}{k, n-k}_q$. If $\underline{k}$ denotes the m-tuple $(k_1, \ldots, k_m)$, write the corresponding q-multinomial coefficient as $\binom{n}{k_1,\ldots,k_m}_q = \binom{n}{\underline{k}}_q$. For a subset $J \subseteq \{1, \ldots, m\}$, let $\underline{e}_J$ denote the m-tuple $(e_1, \ldots, e_m)$, where

$$e_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

For example, if $m = 3$, $J = \{1, 3\}$, and $\underline{k} = (k_1, k_2, k_3)$, then $\binom{n}{\underline{k}-\underline{e}_J}_q = \binom{n}{k_1-1, k_2, k_3-1}_q$. The main result of this paper, which is obtained in Section 2, Theorem 2.1, is a combinatorial proof through enumerating flags in finite vectors spaces of the following generalization of the identity (1.3). For $m \geqslant 2$, and any $k_1, \ldots, k_m > 0$ such that $k_1 + \cdots + k_m = n + 1$, we have

$$\binom{n+1}{k_1, \ldots, k_m}_q = \sum_{J \subseteq \{1,\ldots,m\}, |J|>0} (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q.$$

In Section 3, we prove a recursion which generalizes (1.2). In particular, the generalized Galois number $G_n^{(m)}$ is defined as

$$G_n^{(m)} = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m}_q,$$

which, in the case that q is the power of a prime, enumerates the total number of flags of length $m - 1$ of an n-dimensional $\mathbb{F}_q$-vector space. Quite recently, the asymptotic statistics of these generalized Galois numbers have been studied by Bliem and Kousidis [3] and Kousidis [12].

Directly following from Theorem 2.1, we prove in Theorem 3.1 that, for $n \geqslant m-1$,

$$G_{n+1}^{(m)} = \sum_{i=0}^{m-1} \binom{m}{i+1} (-1)^i \frac{(q)_n}{(q)_{n-i}} G_{n-i}^{(m)},$$

which also follows from a known recurrence for the multivariate Rogers-Szegö polynomials.

2 Flags in finite vector spaces

In this section, let q be the power of a prime, and let $\mathbb{F}_q$ denote a finite field with q elements. If V is an n-dimensional vector space over $\mathbb{F}_q$, then the q-binomial coefficient $\binom{n}{k}_q$ is the number of k-dimensional subspaces of V (see [10, Thm. 7.1] or [18, Prop. 1.3.18]). So, the Galois number

$$G_n = H_n(1) = \sum_{k=0}^{n} \binom{n}{k}_q,$$

is the total number of subspaces of an n-dimensional vector space over $\mathbb{F}_q$.

Now consider the q-multinomial coefficient in terms of vector spaces over $\mathbb{F}_q$. It follows from the definition of a q-multinomial coefficient and the fact that $\binom{n}{k}_q = \binom{n}{n-k}_q$ that we have

$$\begin{aligned}\binom{n}{k_1, k_2, \ldots, k_m}_q &= \binom{n}{k_1}_q \binom{n-k_1}{k_2}_q \cdots \binom{n-k_1-\cdots-k_{m-2}}{k_{m-1}}_q \\ &= \binom{n}{n-k_1}_q \binom{n-k_1}{n-k_1-k_2}_q \cdots \binom{n-k_1-\cdots-k_{m-2}}{n-k_1-\cdots-k_{m-2}-k_{m-1}}_q.\end{aligned}$$

So, if V is an n-dimensional vector space over $\mathbb{F}_q$, the q-multinomial coefficient $\binom{n}{k_1,\ldots,k_m}_q$ is equal to the number of ways to choose an $(n-k_1)$-dimensional subspace W_1 of V, an $(n-k_1-k_2)$-dimensional subspace W_2 of W_1, and so on, until finally we choose an $(n-k_1-\cdots-k_{m-1})$-dimensional subspace W_{m-1} of some $(n-k_1-\cdots-k_{m-2})$-dimensional subspace W_{m-2} (see also [14, Sec. 1.5]). That is,

$$W_{m-1} \subseteq W_{m-2} \subseteq \cdots \subseteq W_2 \subseteq W_1$$

is a *flag* of subspaces of V of length $m-1$, where dim $W_i = n - \sum_{j=1}^{i} k_j$.

We now turn to a bijective proof of the identity (1.3), that for integers $n \geqslant k \geqslant 1$,

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q + (q^n-1)\binom{n-1}{k-1}_q.$$

While the bijective interpretation of this identity which we give now is different from the proof given by Nijenhuis, Solow, and Wilf in [15], it is the interpretation which is most helpful for the proof of our main result. Fix V to be an $(n+1)$-dimensional $\mathbb{F}_q$-vector space. There are $\binom{n+1}{k}_q$ ways to choose a k-dimensional subspace W of V. Fix a basis $\{v_1, v_2, \ldots, v_{n+1}\}$ of V. Any k-dimensional subspace W can be written as $\mathrm{span}(W', v)$ where W' is a $(k-1)$-dimensional subspace of $V' = \mathrm{span}(v_1, \ldots, v_n)$, for some v. We may choose W in three distinct ways. If $v \in V'$, then W is a subspace of V', for which there are $\binom{n}{k}_q$ choices. Call this a *type 1* subspace of V. If $v_{n+1} \in W$, then we may take $v = v_{n+1}$, and W is determined by W', for which there are $\binom{n}{k-1}_q$ choices. We call this a *type 2* subspace of V. Finally, if both $W \not\subset V'$ and $v_{n+1} \notin W$, then we call W a *type 3* subspace of V, and it follows from (1.3) (and can be shown directly, as well) that there are $(q^n - 1)\binom{n-1}{k-1}_q$ choices for W.

We may now prove our main result.

Theorem 2.1. *For $m \geqslant 2$, and any $k_1, \ldots, k_m > 0$ such that $k_1 + \cdots + k_m = n+1$, we have*

$$\binom{n+1}{k_1, \ldots, k_m}_q = \sum_{J \subseteq \{1,\ldots,m\}, |J|>0} (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k} - \underline{e}_J}_q$$

Proof. Fix V to be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$. Fix a basis of each subspace U of V, so that we may speak of subspaces of type 1, 2, or 3 of each subspace U with respect to this fixed basis. Consider a flag F of subspaces of $V = W_0$, $W_{m-1} \subset \cdots \subset W_2 \subset W_1$, such that if we define k_i for $1 \leqslant i \leqslant m$ by $\sum_{j=1}^{i} k_j = n + 1 - \dim W_i$, then each $k_i > 0$. The total number of such flags is $\binom{n+1}{k_1,\ldots,k_m}_q$. Consider now a labeling of such flags in the following way. Given a flag F as above, define

$$r = \min\{1 \leqslant j \leqslant m \mid W_j \text{ is a type 1 subspace of } W_{j-1}\},$$

and

$$J = \{r\} \cup \{1 \leqslant j \leqslant r-1 \mid W_j \text{ is a type 3 subspace of } W_{j-1}\}.$$

Define the flag F to be a *type J flag* of V. That is, for any nonempty $J \subseteq \{1, \ldots, m\}$, we may speak of flags of type J of V. We shall prove that

$$(-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k} - \underline{e}_J}_q \tag{2.1}$$

is the number of type J flags of length $m-1$ of the $\mathbb{F}_q$-space V. Once this claim is proven, we will have accounted for all $2^m - 1$ terms on the right-side of the desired result of Theorem 2.1, and all possible ways to choose our flag.

We prove the claim by induction on m, where the base case of $m = 2$ follows from (1.3) and its interpretation in terms of subspaces of types 1, 2, and 3, as given above. We must consider each possible nonempty $J \subseteq \{1, \ldots, m\}$, and show that in each case, the quantity (2.1) counts the number of type J flags. So, consider a flag of subspaces $W_{m-1} \subset \cdots \subset W_2 \subset W_1$ of V, where $\dim W_i = n + 1 - \sum_{j=1}^{i} k_j$.

First, if $J = \{1\}$, then the number of ways to choose W_1 to be a type 1 subspace of V of dimension $n+1-k_1$ is $\binom{n}{n+1-k_1}_q$, while the number of ways to choose the remaining length

$m-2$ flag $W_{m-1} \subset \cdots \subset W_2$ of W_1 is exactly $\binom{n+1-k_1}{k_2,\ldots,k_m}_q$. Thus, the total number of ways to choose our flag of type J with $J = \{1\}$ is $\binom{n}{n+1-k_1}_q \binom{n+1-k_1}{k_2,\ldots,k_m}_q = \binom{n}{k_1-1,k_2,\ldots,k_m}_q$, which is exactly the expression (2.1) for $J = \{1\}$, as claimed. So, we now suppose $J \neq \{1\}$, so if r is the maximum element of J, we have $r > 1$. We consider the cases of whether $1 \in J$ or $1 \notin J$ separately.

Suppose that $1 \notin J$. Then, we must choose our flag so that W_1 is a type 2 subspace of V, of which there are $\binom{n}{n-k_1}_q$ such subspaces. Now, if we define $I = J-1 = \{j-1 \mid j \in J\}$, so that $I \subset \{1, \ldots, m-1\}$ and $|I| = |J|$, we must choose the rest of our type J flag of V by choosing a type I flag of W_1 of length $m-2$. If we let $\underline{k}' = (k_2, \ldots, k_m)$, then by our induction hypothesis, the number of type I flags of length $m-2$ of the $(n+1-k_1)$-dimensional space W_1 is

$$(-1)^{|I|-1} \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q.$$

So, the total number of ways to choose the type J flag of length $m-1$ in V is

$$\binom{n}{n-k_1}_q (-1)^{|I|-1} \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q.$$

A direct computation yields

$$\binom{n}{n-k_1}_q \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} = \frac{(q)_n}{(q)_{n-|I|+1}} \binom{n+1-|I|}{n-k_1-|I|+1}_q,$$

and further note that

$$\binom{n+1-|I|}{n-k_1-|I|+1}_q \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q = \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q,$$

where $\underline{k} = (k_1, \ldots, k_m)$. Together, these give

$$\binom{n}{n-k_1}_q (-1)^{|I|-1} \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q$$
$$= (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q,$$

giving the claim that when $1 \notin J$, the number of type J subspaces of length $m-1$ of V is given by (2.1).

Finally, suppose that $1 \in J$, and $J \neq \{1\}$. So, we must choose our flag so that W_1 is a type 3 subspace of V, and there are $(q^n-1)\binom{n-1}{n-k_1}_q$ such subspaces. If we let $I = (J-1) \setminus \{0\}$ (so that now $|J| = |I|+1$), then we must choose the rest of our flag as a type I flag of length $m-2$ of W_1. Letting again $\underline{k}' = (k_2, \ldots, k_m)$, then by our induction hypothesis, the total number of flags of type J of length $m-1$ of V is given by

$$(q^n-1)\binom{n-1}{n-k_1}_q (-1)^{|I|-1} \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q.$$

A computation gives

$$(q^n-1)\binom{n-1}{n-k_1}_q \frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}} = (-1)\frac{(q)_n}{(q)_{n-|I|}}\binom{n-|I|}{n-k_1-|I|+1}_q,$$

and also note

$$\binom{n-|I|}{n-k_1-|I|+1}_q \binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q = \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q,$$

where $\underline{k}=(k_1,\dots,k_m)$, since $|I|=|J|-1$. We finally obtain that

$$\begin{aligned}(q^n-1)\binom{n-1}{n-k_1}_q (-1)^{|I|-1}\frac{(q)_{n-k_1}}{(q)_{n-k_1-|I|+1}}&\binom{n+1-k_1-|I|}{\underline{k}'-\underline{e}_I}_q \\ &= (-1)^{|J|-1}\frac{(q)_n}{(q)_{n-|J|+1}}\binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q,\end{aligned}$$

is the the number of type J subspaces of length $m-1$ of V, as claimed. □

3 Generalized Galois numbers

Define the homogeneous Rogers-Szegö polynomial in m variables for $m \geqslant 2$, denoted $\tilde{H}_n(t_1,t_2,\dots,t_m)$, by

$$\tilde{H}_n(t_1,t_2,\dots,t_m) = \sum_{k_1+\cdots+k_m=n}\binom{n}{k_1,\dots,k_m}_q t_1^{k_1}\cdots t_m^{k_m},$$

and define the Rogers-Szegö polynomial in $m-1$ variables, denoted $H_n(t_1,\dots,t_{m-1})$, by

$$H_n(t_1,\dots t_{m-1}) = \tilde{H}(t_1,\dots,t_{m-1},1).$$

The homogeneous multivariate Rogers-Szegö polynomials were first defined by Rogers [16] in terms of their generating function, and several of their properties are given by Fine [5, Section 21]. The definition of the multivariate Rogers-Szegö polynomial H_n is given by Andrews in [1, Chap. 3, Ex. 17], along with a generating function, although there is little other study of these polynomials elsewhere in the literature (however, there is a non-symmetric version of a bivariate Rogers-Szegö polynomial [4]).

The multivariate Rogers-Szegö polynomials satisfy a recursion which generalizes (1.1), although it seems not to be very well-known, as the only proof and reference to it that the author has found is in the physics literature, in papers of Hikami [7, 8]. For any finite set of variables X, let $\mathbf{e}_i(X)$ denote the ith elementary symmetric polynomial in the variables X. Then the Rogers-Szegö polynomials in $m-1$ variables satisfy the following recursion:

$$H_{n+1}(t_1,\dots,t_{m-1}) = \sum_{i=0}^{m-1}\mathbf{e}_{i+1}(t_1,\dots,t_{m-1},1)(-1)^i\frac{(q)_n}{(q)_{n-i}}H_{n-i}(t_1,\dots,t_{m-1}). \qquad (3.1)$$

The sum of all q-multinomial coefficients of length m, or the generalized Galois number $G_n^{(m)}$, is then

$$H_n(1,1,\ldots,1) = G_n^{(m)} = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m}_q.$$

From the discussion at the beginning of Section 2, when q is the power of a prime, $G_n^{(m)}$ is exactly the total number of flags of subspaces of length $m-1$ in an n-dimensional $\mathbb{F}_q$-vector space.

Since the number of terms in the elementary symmetric polynomial $\mathbf{e}_{i+1}(t_1,\ldots,t_{m-1},1)$ is $\binom{m}{i+1}$, then the following, our last result, follows directly from the formal identity (3.1) proved by Hikami. However, we give a proof which follows directly from Theorem 2.1, and is thus a bijective proof through the enumeration of flags in a finite vector space.

Theorem 3.1. *The generalized Galois numbers satisfy the recursion, for $n \geqslant m-1$,*

$$G_{n+1}^{(m)} = \sum_{i=0}^{m-1} \binom{m}{i+1} (-1)^i \frac{(q)_n}{(q)_{n-i}} G_{n-i}^{(m)}.$$

Proof. For convenience, whenever any $k_i < 0$, we define the q-multinomial coefficient $\binom{n}{k_1,k_2,\ldots,k_m}_q = 0$. Granting this, we have Theorem 2.1 holds for all $k_i \geqslant 0$. We now begin with the definition of $G_{n+1}^{(m)}$ as the sum of all q-multinomial coefficients, and we directly apply Theorem 2.1 to rewrite the sum, as follows:

$$\begin{aligned}
G_{n+1}^{(m)} &= \sum_{k_1+\cdots k_m=n+1} \binom{n+1}{k_1,\ldots,k_m}_q \\
&= \sum_{k_1+\cdots k_m=n+1} \sum_{\substack{J\subseteq\{1,\ldots,m\}\\ |J|>0}} (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q \\
&= \sum_{\substack{J\subseteq\{1,\ldots m\}\\ |J|>0}} \sum_{\substack{\underline{k}=(k_1,\ldots,k_m)\\ k_1+\cdots+k_m=n+1}} (-1)^{|J|-1} \frac{(q)_n}{(q)_{n-|J|+1}} \binom{n+1-|J|}{\underline{k}-\underline{e}_J}_q \\
&= \sum_{i=0}^{m-1} \sum_{\substack{J\subseteq\{1,\ldots,m\}\\ |J|=i+1}} \sum_{\substack{\underline{k}=(k_1,\ldots,k_m)\\ k_1+\cdots+k_m=n+1}} (-1)^i \frac{(q)_n}{(q)_{n-i}} \binom{n-i}{\underline{k}-\underline{e}_J}_q \\
&= \sum_{i=0}^{m-1} \binom{m}{i+1} \sum_{\substack{k'_1+\cdots+k'_m=n-i\\ \underline{k}'=(k'_1,\ldots,k'_m)}} (-1)^i \frac{(q)_n}{(q)_{n-i}} \binom{n-i}{\underline{k}'}_q \\
&= \sum_{i=0}^{m-1} \binom{m}{i+1} (-1)^i \frac{(q)_n}{(q)_{n-i}} G_{n-i}^{(m)},
\end{aligned}$$

where the next-to-last equality follows from the fact that each index $\underline{k}'$ may be obtained from an index $\underline{k}$ from any of the $\binom{m}{i+1}$ subsets J of size $i+1$. □

By a very similar argument, we may see that in fact the recursion for the multinomial Rogers-Szegö polynomials in (3.1) also follows from Theorem 2.1.

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