Minimal Covers of the Archimedean Tilings, Part 1

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Abstract

We discuss representations of non-finite polyhedra as quotients of regular polytopes. We provide some structural results about the minimal regular covers of non-finite polyhedra and about the stabilizer subgroups of their flags under the flag action of the automorphism group of the covering polytope. As motivating examples we discuss the minimal regular covers of the Archimedean tilings, and we construct explicit minimal regular covers for three of them.

Keywords: Abstract polytope, uniform tiling, Archimedean tiling, quotient polytope, regular cover, string C-group.

1 Introduction

Symmetric maps on surfaces and the automorphism groups of symmetric maps have been studied since the early 20th century, though mostly in the case where the surfaces considered are compact (see [Bra27], [CM80]). Tilings of the Euclidean or hyperbolic plane are examples of maps on non-compact surfaces and require somewhat different methods

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of investigation. In the late 20th century, maps whose vertex-figures are polygons were seen to be abstract polyhedra (rank 3 polytopes) [MS02, Section 6B]. While much work has been done on the study of regular abstract polytopes (the primary reference work on the subject being [MS02]), the study of less constrained polytopes is still largely in its infancy. A particularly important tool for the study of non-regular abstract polytopes is the seminal work of Michael Hartley [Har99a], in which he demonstrates that any abstract polytope may be realized as the quotient of some regular abstract polytope. The current work is part of an attempt to better understand the structure of these quotients both geometrically, as covering maps, and algebraically via the group actions induced by Hartley's theory of regular covers. In [HW10], Hartley and Williams described explicit representations for the sporadic Archimedean solids, as well as developing general techniques for generating such covers. Further, they identified the smallest possible, or *minimal*, regular covers that may be obtained in each case via the type of quotient operation described in [Har99a] were identified.

More generally, in the context of the study of maps, much is known about the regular covers of maps and the closely related monodromy groups of maps (e.g., [Orb07, Wil76, Wil94, Wil02]). In the context of abstract polyhedra, that is, 3-polytopes, it is straightforward to extend these ideas to demonstrate that the minimal regular cover of a polyhedron is itself a polyhedron (i.e., its monodromy group satisfies the intersection condition; details will be provided in the forthcoming [MPW]). Thus in the case of polyhedra, minimal regular covers both exist and are unique. However, for abstract polytopes of rank 4 or higher, minimal covers are not, in general, unique, nor do the monodromy groups of these polytopes necessarily satisfy the intersection condition. In particular, there is an abstract 4-polytope (the *Tomotope*) that is known to admit infinitely many distinct minimal regular covers [MPW12].

Contrary to the case when the maps are finite, there has been little attention given to regular covers of infinite maps. Natural candidates to consider are the Archimedean tilings, that is, vertex-transitive tessellations of the plane where all tiles are convex regular polygons. Tilings with symmetry properties have been the subject of intense interest and investigation, especially since the publication of Grünbaum and Shephard's *Tilings and Patterns* [GS87].

In [PW11], we provided representations of the Archimedean tilings as quotients of regular hyperbolic tilings using a general representation theory developed by Hartley in [Har99a] (more basic information about such representations is available in [Har99b] and [HW10]). The goal of the current work is to identify *minimal* regular covers for the Archimedean tilings, in the sense that any other regular cover of a tiling is also a cover of the minimal one. More precisely, the Archimidean tilings are quotients of regular polyhedra obtained using the type of quotient described in [Har99a], such that any other cover of that type also covers the minimal cover. This paper provides a partial solution to this problem, determining minimal covers for the Archimedean tilings (3.6.3.6), (4.8.8), and (3.12.12). These three tilings share the property that the number of flag-orbits is at most 3 and that the generators of some relevant groups, like the stabilizer of all flags in some flag-orbit for (3.6.3.6) and (4.8.8), act on each flag either like the identity or like a half-turn. Finding the minimal regular covers of the remaining five tessellations demands other techniques in addition to the ones used for the three described in this paper, and we shall determine them in Part 2.

In Section 2 we discuss the definition and structure of abstract polytopes, the correspondence of regular abstract polytopes with string C-groups, and provide formal descriptions of regular covers in the context of abstract polyhedra. In Section 3 we discuss some of our prior work on representing Archimedean tilings as quotients of hyperbolic tilings, as well as presenting explicit details on the procedure used to generate those quotients. We also correct some deficiencies we discovered in the theoretical framework provided in [PW11]. In Section 4 we discuss the ways in which the covering maps of Archimedean tilings are different from the covering maps of finite polyhedra, and we provide a detailed discussion of the sizes of the fibers in these covers. Section 5 includes a general procedure for identifying a regular minimal cover of a uniform tiling, as well as explicit descriptions for the minimal regular covers for the Archimedean tilings (3.6.3.6), (4.8.8) and (3.12.12). Section 6 contains a discussion of concluding remarks, a discussion of the evidence for the existence of nontrivial minimal covers for the remaining Archimedean tilings, and open questions.

2 Basic Concepts, Including a Discussion of String C-Groups

Following [MS02, §2A], we define an *abstract d-polytope* $\mathcal{P}$ to be a partially ordered set whose elements are called *faces*, with partial order denoted by $\leqslant$, that satisfies the following four properties. It contains a minimum face F_{-1} and a maximum face F_d. All maximal totally ordered subsets of $\mathcal{P}$, called the *flags* of $\mathcal{P}$, include F_{-1} and F_d and contain precisely $d+2$; the set of all flags of $\mathcal{P}$ is denoted $\mathcal{F}(\mathcal{P})$. Consequently, $\leqslant$ induces a strictly increasing rank function such that the ranks of F_{-1} and F_d are -1 and d respectively. Finally, $\mathcal{P}$ is strongly connected and satisfies the "diamond condition" (see [MS02, Section 2A] for details).

In the present paper we are interested only in *abstract polyhedra*, that is, abstract polytopes of rank 3; however Lemma 7 of §4 applies to abstract polytopes of general rank. Throughout the remainder of this paper we will use "polyhedra" to mean either the geometric objects or abstract polyhedra, as appropriate. The *vertices* and *edges* of an abstract polyhedron are its faces of rank 0 and 1 respectively. In this context there is little possibility of confusion if we refer to the rank 2 faces simply as *faces*. We define a *section* F/G of a polytope to be the collection of all faces H such that $G \leqslant H \leqslant F$, that is $F/G := \{H \in \mathcal{P} \mid G \leqslant H \leqslant F\}$. The *vertex-figure* at a vertex v is the section $\{F \in \mathcal{P} \,|\, v \leqslant F\}$. In the case of polyhedra, the diamond condition requires that every edge contains precisely two vertices and is contained in precisely two faces, and the section determined by a face f and a vertex v contains either exactly two edges or is the empty set. As a consequence of the diamond condition, given $i \in \{0, 1, 2\}$ and a flag Ψ, there exists a unique flag Ψ^i that coincides with Ψ in all faces except in the face of rank i.

The flag Ψ^i is called the *i-adjacent flag* of Ψ. The requirement of strong connectivity for polyhedra implies that each face and each vertex-figure is isomorphic to a polygon, that is, a cycle in the graph theoretic sense. The *degree* of a vertex v is the number of edges containing v, and the *co-degree* of a face f is the number of edges contained in f. Whenever every vertex of a polyhedron $\mathcal{P}$ has the same degree p, and every face of $\mathcal{P}$ has the same co-degree q we say that $\mathcal{P}$ is *equivelar* and has *Schläfli type* $\{p, q\}$.

An *automorphism* of a polyhedron $\mathcal{P}$ is an order preserving bijection of its elements. We say that a polyhedron is *regular* if its automorphism group $\Gamma(\mathcal{P})$ is transitive on the set of flags of $\mathcal{P}$, which we will denote by $\mathcal{F}(\mathcal{P})$. The Platonic solids and the familiar regular tessellations by triangle (3^6), squares (4^4) and hexagons (6^3) are examples of abstract regular polyhedra.

A *string C-group* G of rank 3 is a group with distinguished involutory generators ρ_0, ρ_1, ρ_2, where $(\rho_0\rho_2)^2 = \varepsilon$, the identity in G, and $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle = \langle \rho_1 \rangle$ (this is called the *intersection condition*). The automorphism group of an abstract regular polyhedron $\mathcal{P}$ is always a string C-group of rank 3. Given an arbitrarily chosen *base flag* Φ of $\mathcal{P}$, ρ_i is the (unique) automorphism mapping Φ to the i-adjacent flag Φ^i. Furthermore, any string C-group of rank 3 is the automorphism group of an abstract regular polyhedron [MS02], so, up to isomorphism, there is a one-to-one correspondence between the string C-goups of rank 3 and the abstract regular polyhedra. Thus, in the study of regular abstract polyhedra we may either work with the polyhedron as a poset or with its automorphism group. In this paper the automorphism group of a regular polyhedron $\mathcal{P}$ will be denoted by $\Gamma(\mathcal{P})$, or simply by Γ whenever there is no possibility of confusion.

For any polyhedron $\mathcal{P}$ we define permutations r_0, r_1, r_2 on $\mathcal{F}(\mathcal{P})$ by

$$\Psi r_i := \Psi^i,$$

for every flag Ψ of $\mathcal{P}$ and $i = 0, 1, 2$ (note that these are *not* automorphisms of $\mathcal{P}$). The group $\mathrm{Mon}(\mathcal{P}) := \langle r_0, r_1, r_2 \rangle$ will be referred to as the *monodromy group* of $\mathcal{P}$ (see [HOW09], but note that this definition differs from the definition in [Zvo98], where the author only considers words with even length in the generators r_i).

The *flag action* of a string C-group $\Gamma = \langle \rho_0, \rho_1, \rho_2 \rangle$ on $\mathcal{P}$ is the contravariant group homomorphism $\Gamma \to \mathrm{Mon}(\mathcal{P})$ defined by $\rho_i \mapsto r_i$, provided such a homomorphism exists. In this context, if $\alpha = \alpha' \rho_i$ for some $\alpha, \alpha' \in \Gamma$ then $\Psi^\alpha = (\Psi^{\alpha'}) r_i = (\Psi^{\alpha'})^i$. Note that, by definition of automorphism, the action of each r_i (and thus the flag action) commutes with the automorphisms of any given polyhedron. That is,

$$(\Psi r_i)\alpha = (\Psi \alpha) r_i \tag{1}$$

for $i = 0, 1, 2$ and $\alpha \in \Gamma(\mathcal{P})$, and more generally,

$$(\Psi w)\alpha = (\Psi \alpha) w$$

for every $w \in \mathrm{Mon}(\mathcal{P}), \alpha \in \Gamma(\mathcal{P})$.

Whenever we have a flag action from $\Gamma(\mathcal{P})$ to $\mathrm{Mon}(\mathcal{Q})$, where $\mathcal{Q}$ is an Archimedean tiling, and an element $\gamma \in \Gamma(\mathcal{P})$ preserving flag orbits, for any flag of $\mathcal{Q}$ there exists a unique isometry I_γ of $\mathbb{R}^2$ mapping a given flag Φ to $\Phi\gamma$. Furthermore, if $\gamma, \delta \in \Gamma(\mathcal{P})$, and I_γ, I_δ are the corresponding isometries with respect to a given flag Φ, then $\Phi\gamma\delta = \Phi I_\delta I_\gamma$. In other words, there is a contravariant group homomorphism from $\Gamma(\mathcal{P})$ to $\mathrm{Mon}(\mathcal{Q})$. Note that this group homomorphism depends on the choice of Φ; however, the group homomorphisms corresponding to two flags in the same orbit under $\Gamma(\mathcal{Q})$ are isomorphic (in the sense that if $\Psi = \Phi\phi$ for some $\phi \in \Gamma(\mathcal{Q})$, and I_γ, J_γ are the isometries corresponding to $\gamma \in \Gamma(\mathcal{P})$ with respect to Φ and Ψ respectively, then $I_\gamma = \phi^{-1} J_\gamma \phi$). In the upcoming sections we will use such a group homomorphism, where the place of Γ is occupied by an arbitrary subgroup of Γ generated by $w_1, \ldots, w_k$. That is,

Proposition 1. *Let $\mathcal{Q}$ be an Archimedean tiling, $\mathcal{P}$ a regular cover of $\mathcal{Q}$, Φ any flag of $\mathcal{Q}$, $\Delta = \langle \delta_1, \ldots, \delta_s \rangle \leqslant \Gamma(\mathcal{P})$, and for $i = 1, \ldots, s$ let h_i denote the unique isometry of $\mathbb{R}^2$ mapping a given flag Φ to $\Phi\delta_i$. Then there is a contravariant group homomorphism from Δ to $\langle h_1, \ldots, h_s \rangle$. Moreover, this homomorphism is an isomorphism if and only if every element in Δ acts nontrivially on Φ.*

We say that the regular polytope $\mathcal{P}$ is a *cover* of $\mathcal{Q}$, denoted by $\mathcal{P} \searrow \mathcal{Q}$, if $\mathcal{Q}$ admits a flag action from $\Gamma(\mathcal{P})$. (This implies the notion of covering described in [MS02, p. 43].) For example, the (universal) polyhedron with automorphism group isomorphic to the *Coxeter group* $[\infty, \infty] := \langle \rho_0, \rho_1, \rho_2 \,|\, (\rho_0\rho_2)^2 = id \rangle$ covers all other polyhedra. Whenever the least common multiple of the co-degrees of the faces of a polyhedron $\mathcal{P}$ is p, and the least common multiple of the vertex degrees of $\mathcal{P}$ is q, $\mathcal{P}$ is covered by the tessellation $\{p, q\}$ whose automorphism group is isomorphic to the string Coxeter group

$$[p, q] := \langle \rho_0, \rho_1, \rho_2 \,|\, (\rho_0\rho_2)^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = id \rangle.$$

(Recall that $\{p, q\}$ can be viewed as a regular tessellation of the sphere, Euclidean plane or hyperbolic plane, depending on whether $\frac{1}{p} + \frac{1}{q}$ is bigger than, equal to, or less than $\frac{1}{2}$, respectively.)

Whenever $\mathcal{P} \searrow \mathcal{Q}$, $\mathcal{Q}$ is totally determined by $\mathcal{P}$ and the stabilizer N of a chosen base flag Φ of $\mathcal{Q}$ under the flag action of $\Gamma(\mathcal{P})$. Indeed, $\mathcal{Q} = \mathcal{P}/N$, the polytope whose faces are orbits under the action of N on $\mathcal{P}$. For further details, see [Har99a].

The monodromy groups of polyhedra are particularly nice, as seen in the following theorem, whose proof will appear in [MPW].

Theorem 2. *Let $\mathcal{Q}$ be an abstract polyhedron, and Mon($\mathcal{Q}$) its monodromy group. Then Mon($\mathcal{Q}$) is a string C-group.*

Since Theorem 2 says that $\mathrm{Mon}(\mathcal{Q})$ is a string C-group for any polyhedron $\mathcal{Q}$, then by the identification between string C-groups and regular abstract polytopes, any polyhedron is automatically equipped with a regular cover determined by its monodromy group. Moreover, it is straightforward to demonstrate that $\mathrm{Mon}(\mathcal{Q}) \cong \Gamma/\mathrm{Core}(\Gamma, N)$, if Γ is the automorphism group of a regular cover of $\mathcal{Q}$, N is the stabilizer in Γ of a flag in $\mathcal{Q}$ under the flag action of Γ and the *core*, $\mathrm{Core}(\Gamma, N)$, is the largest normal subgroup of Γ in N (see[MPW] for details). Note that

$$\mathrm{Core}(\Gamma, N) = \bigcap_{g \in \Gamma} g^{-1} N g.$$

Theorem 2.3 of [HW10] states the following. Let Γ be the automorphism group of a regular polytope $\mathcal{P}$ acting on a polytope $\mathcal{Q}$ via the flag action, and let $\mathcal{R}$ be any other regular cover of $\mathcal{Q}$. If $\Gamma/\mathrm{Core}(\Gamma, N)$ is a string C-group, then $\mathcal{R}$ also covers the polytope determined by $\Gamma/\mathrm{Core}(\Gamma, N)$. In other words, applying Theorem 2 we see that $\mathrm{Mon}(\mathcal{Q})$ is a minimal regular cover for $\mathcal{Q}$.

As a consequence of the previous discussion, the monodromy group of a regular polytope $\mathcal{P}$ is isomorphic to $\Gamma(\mathcal{P})/\mathrm{Core}(\Gamma, N)$, where N is the stabilizer of any flag in $\mathcal{P}$. Note that in this case $\mathrm{Core}(\Gamma, N)$ is trivial, since the stabilizer of all flags is the same as the stabilizer of any individual flag (due to regularity) and every conjugate of the stabilizer N of a flag of $\mathcal{P}$ is the stabilizer of another flag of $\mathcal{P}$. Hence $\Gamma(\mathcal{P}) \cong \mathrm{Mon}(\mathcal{P})$ (for further details of this fact see [MPW]).

This leads to a useful reinterpretation of the condition for a regular polytope $\mathcal{P}$ to be a cover of $\mathcal{Q}$. The fact that $\mathcal{Q}$ admits a flag action by $\Gamma(\mathcal{P})$ is equivalent to observing that there is an epimorphism from $\mathrm{Mon}(\mathcal{P})$ to $\mathrm{Mon}(\mathcal{Q})$. Thus, we find it more natural to understand the cover $\mathcal{P} \searrow \mathcal{Q}$ as an epimorphism of monodromy groups, instead of as a contravariant homomorphism from an automorphism group to a monodromy group. This perspective is motivated by the natural way in which i-adjacent flags of $\mathcal{P}$ are mapped into i-adjacent flags of $\mathcal{Q}$. Henceforth we shall proceed according to this notion and use the generators r_0, r_1, r_2 of $\mathrm{Mon}(\mathcal{P})$ instead of those of $\Gamma(\mathcal{P})$ to denote the action on the flags of $\mathcal{Q}$.

For a given abstract polyhedron $\mathcal{P}$, we define its flag graph $\mathcal{GF}(\mathcal{P})$ as the edge-labeled graph whose vertex set consists of all flags of $\mathcal{P}$, where two vertices (flags) are joined by an edge labeled i if and only if they are i-adjacent for some $i = 0, 1, 2$.

3 Prior Results on Covers of Archimedean Tilings

In [PW11], we provided explicit presentations for regular covers of the Archimedean tilings. In the current work, we seek to improve on those presentations by providing minimal covers. Before doing that however, some discussion of the results in [PW11] is in order. To construct an explicit representation of an abstract polytope $\mathcal{Q}$ using Hartley's theoretical framework it is necessary to find an adequate description for the stabilizer of a base flag under the flag action of the automorphism group of a regular cover of $\mathcal{Q}$. As a first step, we recall the following theorem from [PW11, Theorem 4]:

Theorem 3. *Let the polyhedron $\mathcal{Q}$ be a map on the sphere or the Euclidean plane, Φ the base flag of $\mathcal{Q}$, and Γ a string C-group with generators ρ_0, ρ_1, ρ_2 and a flag action on $\mathcal{Q}$. Then $Stab_\Gamma(\Phi)$ is generated by the set of elements of the form*

$$W_v = w_v^{-1}(\rho_1\rho_2)^{q_v} w_v \quad \text{and} \quad W_f = w_f^{-1}(\rho_0\rho_1)^{p_f} w_f,$$

where v is any vertex of $\mathcal{Q}$ of degree q_v, f is any face of $\mathcal{Q}$ with p_f edges, and w_v and w_f are words which map Φ to a flag containing v or f, respectively.

In light of the contravariant isomorphism between the automorphism group of the covering polytope and the monodromy group of $\mathcal{Q}$, in the language of this paper we would instead describe this relationship in terms of the subgroups of $\mathrm{Mon}(\mathcal{Q})$ generated by $\tilde{w}_v(r_2r_1)^{q_v}\tilde{w}_v^{-1}$ and $\tilde{w}_f(r_1r_0)^{p_f}\tilde{w}_f^{-1}$, where $\tilde{w}_f, \tilde{w}_v$ are the elements in $\mathrm{Mon}(\mathcal{Q})$ corresponding to the flag actions of w_f and w_v. Notice that the generating set for the stabilizer of a base flag for a tiling given by Theorem 3 is stated in the form of sufficiency, not necessity. In particular, the theorem says that the set containing each of the generators is determined by infinitely many words w_f and w_v sending Φ to a flag on f or v, for each of the infinite number of choices for f and v will be sufficient. However, there is likely to be an enormous amount of redundancy in such a description. In [PW11, Lemma 5] we tried to reduce that reduncancy via the application of the following lemma:

Lemma 4. *Let $\mathcal{Q}$ be a polyhedron, Φ a flag of $\mathcal{Q}$, and Γ a string C-group with generators ρ_0, ρ_1, ρ_2 and flag action on $\mathcal{Q}$. If $w(\rho_i\rho_{i+1})^q w^{-1} \in Stab_\Gamma(\Phi)$ then $w'(\rho_i\rho_{i+1})^q w'^{-1} \in Stab_\Gamma(\Phi)$ for any w' such that $\Phi w'$ and Φw coincide in their face if $i = 0$, and in their vertex if $i = 1$.*

On the basis of this lemma, we proposed that a generating set for the stabilizer of a specified base flag in each of the Archimedean tilings is a collection of elements of the form $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}$, where β and γ are words acting on the base flag as linearly independent translations, and α_k is a conjugate of $(\rho_0\rho_1)^q$ by a short word which will take the base flag into a flag containing a q-gonal face. Informally, each of these words corresponds to a walk in the flag graph involving first a number of steps to the left or the right (determined by the exponent i on the β), followed by a number of steps up or down (determined by the exponent j on the γ), plus a little loop at the end determined by α_k, where the entire element $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}$ corresponds to a walk in the flag graph shaped like a lollipop with a potentially very long, and somewhat bent, stem. In this formulation, each face of the tiling determines a single generator (see [PW11, Section 4]). Note that analogous words corresponding to edges or vertices of the Archimedean tilings are trivial in the covering group, so it is not necessary to included them in the generating set for the stabilizer subgroup.

In [PW11] we tried to argue using Lemma 4 that only one word w_f for each face and one word w_v for each vertex of $\mathcal{Q}$ in Theorem 3 is required to generate the stabilizer subgroup for the base flag (instead of infinitely many for each face and vertex). However, the presented argument instead demonstrated that when the last flags traversed by w and w' agree in the face or vertex, then the elements $w(\rho_i\rho_{i+1})^q w^{-1}$ and $w'(\rho_i\rho_{i+1})^q(w')^{-1}$ are conjugates in Γ. This is insufficient to show a single generator suffices for each face or vertex of the tiling. Although we conjecture that one generator per vertex or face of $\mathcal{Q}$ is sufficient to generate the stabilizer of a base flag in $\mathcal{Q}$ whenever the flag graph of $\mathcal{Q}$ is planar, in the remainder of this section we provide an alternative demonstration that the generating sets for the stabilizers of a base flag given below are sufficient for the Archimedean tilings.

Theorem 5. *The sets $\{\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i} \mid i,j \in \mathbb{Z}, 0 \leqslant k \leqslant n\}$ as given in Table 1 (following the descriptions given in [PW11]) are sufficient to generate the stabilizer subgroup of the specified base flags in the corresponding Archimedean tilings.*

For each tiling listed in Table 1 the location of the base flag is indicated in the tiling symbol by an arrow. For example, $(3.3.\overrightarrow{4.3}.4)$ indicates the base flag's position in the vertex figure, namely that it contains a square and shares an edge with a triangle. For more details on base flag selection see [PW11]. This is essentially a restatement of the results included in Section 4 of [PW11]. Note that in the notation of this paper, $r_0 = a, r_1 = b, r_2 = c$, and we are using the a, b, c notation here to be consistent with the remainder of the statements in [PW11] about these coverings. Likewise, for words w, y in Γ or $\mathrm{Mon}(Q)$, $w^y = y^{-1}wy$. The generator definitions given in Table 1 differ from those given in [PW11] as follows:

- In the tiling (3.3.4.3.4) we replaced $\alpha_3 = ((ab)^3)^{cbcbc}$ with $((ab)^3)^{bcbcb}$.
- In the tiling (3.3.3.3.6) we replaced $\alpha_1 = \alpha_0^{cbacbc}$ with α_0^{cbcabc} and $\alpha_7 = \alpha_0^{cbca}$ with α_0^{cbac}.

These changes are necessary only so that the portions of the walk determined by the word w with which we are conjugating α_k *does not* circuit a face of the tiling and so may be included in the spanning tree without forming closed loops. For example, the original walk α_3 for (3.3.4.3.4) included the string *cbcbc* which induces a walk that forms a closed loop with the walk induced by β^{-1} since the last five terms of β are *bcbcb* (recall $(bc)^5$ is trivial in the cover). These changes have no impact algebraically since in (3.3.4.3.4) $bcbcb = cbcbc$ and in (3.3.3.3.6) $ac = ca$ (as it does in all other intermediate covering groups).

To establish the validity of the smaller generating sets we will make use of a more fundamental theorem from [PW11, Theorem 1], which we restate here as Theorem 6.

Theorem 6. *Let T be a spanning tree in $\mathcal{GF}(\mathcal{Q})$ rooted at Φ, a specified (base) flag of $\mathcal{Q}$. For each edge $e = (\Psi, \Upsilon)$ of $\mathcal{GF}(\mathcal{Q})$, define the unique walk β_e as the unique path from Φ to Ψ in T, across e and followed by the unique path from Υ to Φ. Let w_{β_e} be the word in Γ inducing the walk β_e. Then $S = \{w_{\beta_e} : e \in \mathcal{GF}(\mathcal{Q}) \setminus T\}$ is a generating set for $Stab_\Gamma(\Phi)$.*

To prove Theorem 5, it is sufficient, by Theorem 6, to construct for each Archimedean tiling a spanning tree in the flag graph with the property that each of the w_{β_e} can be expressed as a product of words corresponding to lollilipops from the statement of Theorem 5. Such a spanning tree must have the following properties:

- The spanning tree is rooted at the base flag.
- The edges of the spanning tree contain all of the edges traversed by the walks from the base flag induced by $\beta^i\gamma^j w$, where $\alpha_k = w(r_0r_1)^q w^{-1}$ to each (vertex and) face of the polyhedron.
- The omitted edges of the spanning tree may be identified with cells of the flag graph so that each omitted edge is associated to a walk in the tree to a cell of the flag graph, a walk around that cell and a return walk along that same path in the tree.

Table 1: Generating sets for the stabilizer subgroup of a regular cover for each of the Archimedean tilings. Definitions modified from those give in [PW11] are indicated with *.

Tiling	Covering group	α_i	β	γ
$(3.\overrightarrow{6.3}.6)$	$[6,4]$	$\alpha_0 := ((ab)^3)^c$ $\alpha_1 := ((ab)^3)^{cb}$	$ababacbc$	$abcbabcb$
$(4.\overrightarrow{8.8})$	$[8,3]$	$\alpha_0 := ((ab)^4)^{cb}$	$ababcbab$	$cbababab$
$(3.3.\overrightarrow{4.3}.4)$	$[12,5]$	$\alpha_0 := (ab)^4$ $\alpha_1 := ((ab)^3)^c$ $\alpha_2 := \alpha_0^{cbc}$ $*\alpha_3 := ((ab)^3)^{bcbcb}$ $\alpha_4 := ((ab)^3))^{cb}$ $\alpha_5 := ((ab)^3)^{cbac}$	$abcbabcbcb$	$cabcbacbcbabcb$
$(3.3.\overleftarrow{3.4}.4)$	$[12,5]$	$\alpha_0 := (ab)^4$ $\alpha_1 := ((ab)^3)^c,$ $\alpha_2 := ((ab)^3)^{cbc}$	$abcb$	$cbab(cb)^2ab$
$(\overrightarrow{3.4}.6.4)$	$[12,4]$	$\alpha_0 := (ab)^3,$ $\alpha_1 := ((ab)^4)^{cba}$ $\alpha_2 := ((ab)^4)^{cb}$ $\alpha_3 := ((ab)^4)^c$ $\alpha_4 := ((ab)^6)^{cbc}$ $\alpha_5 := ((ab)^3)^{cbabc}$	$cbabcbcbabcbab$	$caba(bc)^2babcab$
$(3.3.3.\overrightarrow{3.6})$	$[6,5]$	$\alpha_0 := (ab)^3$ $*\alpha_1 := \alpha_0^{cbcabc}$ $\alpha_2 := \alpha_0^{cbc}$ $\alpha_3 := \alpha_0^{cbcb}$ $\alpha_4 := \alpha_0^{cb}$ $\alpha_5 := \alpha_0^{cba}$ $\alpha_6 := \alpha_0^{cbcba}$ $*\alpha_7 := \alpha_0^{cbac}$	$ab(cb)^3(abcb)^2cb$	$ca(ba)^2(bc)^2ab$
$(3.\overrightarrow{12.12})$	$[12,3]$	$\alpha_0 := ((ab)^3)^{cb}$ $\alpha_1 := ((ab)^3)^{cbabab}$	$(bcba)^2(ba)^2$	$(ba)^2(bcba)^2$
$(4.\overleftarrow{6.12})$	$[12,3]$	$\alpha_0 := ((ab)^4)^{cbabab}$ $\alpha_1 := ((ab)^6)^{cbab}$ $\alpha_2 := ((ab)^4)^{cb}$ $\alpha_3 := ((ab)^6)^c$ $\alpha_4 := ((ab)^4)^{cba}$	$(ab)^3(cbab)^2ab$	$(ab)^5cbabcb$

In practice, this final constraint must be done in a coherent fashion so that the cells may be ordered in such a way that some corresponding product of trivial elements in the universal cover Γ and elements of our generating set yields the desired enclosing walk determined by the omitted edge.

Figures 4 and 6 provide templates for the requisite spanning trees for all of the tilings. To explain how to interpret Figures 4 and 6, as well as to explain how we know these trees achieve the desired objectives, we will explore the case of the tiling (3.3.3.4.4) in some detail.

3.1 An Example Spanning Tree

We now consider the case of the tiling (3.3.3.4.4). For this tiling we choose a base flag Φ containing an edge shared by a triangle and a square, and also containing a square of the tiling (as indicated in Figure 1 by a large pale blue dot). Recall from Table 1, $\alpha_0 = (r_0r_1)^4$, $\alpha_1 = r_2(r_0r_1)^3r_2$, $\alpha_2 = r_2r_1r_2(r_0r_1)^3r_2r_1r_2$, $\beta = r_0r_1r_2r_1, \gamma = r_2r_1r_0r_1(r_2r_1)^2r_0r_1$; then $Stab_{\Gamma(\mathcal{P})}(\Phi) = \langle \beta^j\gamma^k\alpha_i\gamma^{-k}\beta^{-j}\rangle$ where $i = 0, 1, 2$ and $j, k \in \mathbb{Z}$.

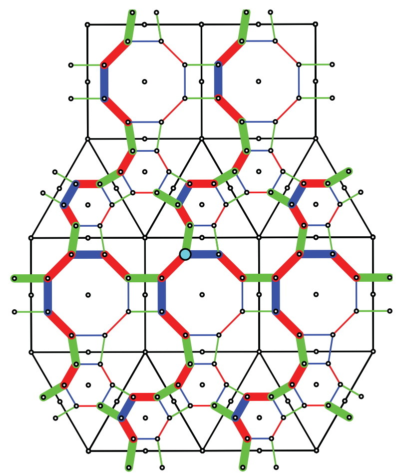


Figure 1: The main branches of our spanning tree in the flag graph rooted at the base flag (indicated by a cyan dot). Edges in the main branches of the tree are indicated by thick colored lines, edges not in the main branches of the tree are indicated with colored hairlines. Colors indicate adjacency relationships between flags. Edges of $\mathcal{GF}(\mathcal{Q})$ will be colored blue if they connect to 0-adjacent flags, red if they connect to 1-adjacent flags, and green if they connect to 2-adjacent flags.

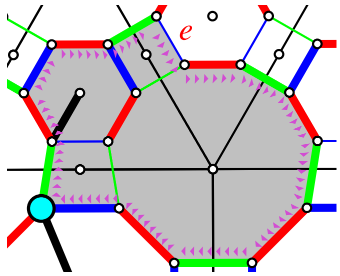


Figure 2: The walk β_e, which is equivalent to α_1. The base flag is indicated by a cyan dot, thick edges correspond to edges of the spanning tree, while the walk β_e is indicated by a sequence of purple arrows in the direction of the walk adjacent to the corresponding edges of $\mathcal{GF}(3.3.3.4.4)$.

By way of example, consider the walk β_e indicated in Figure 2. To establish equivalence with $\alpha_1 = r_2(r_0r_1)^3r_2$ we multiply it by three carefully chosen trivial elements in Mon((3.3.3.4.4)), namely $\beta(r_2r_1)^5\beta^{-1}$, $(r_0r_2)^2$, $((r_2r_0)^2)^{r_2r_1r_0r_1r_2}$, corresponding to walks in $\mathcal{GF}((3.3.3.4.4))$ of circuits of cells corresponding to two edges and a vertex of (3.3.3.4.4), respectively, as follows:

$$\begin{aligned}\beta(r_2r_1)^5\beta^{-1} \cdot (r_0r_2)^2 \cdot ((r_0r_1)^3)^{r_2} \cdot ((r_2r_0)^2)^{r_2r_1r_0r_1r_2}&\\ = (r_0r_1r_2r_1r_2r_1r_2r_1r_2r_1r_2r_1r_2r_1r_1r_2r_1r_0)(r_0r_2r_0r_2)\cdot&\\ (r_2r_0r_1r_0r_1r_0r_1r_2)(r_2r_1r_0r_1r_2r_2r_0r_2r_0r_2r_1r_0r_1r_2).&\end{aligned}$$

Trimming all words r_0r_0, r_1r_1 and r_2r_2, which are equivalent to the identity, we get the word $r_0r_1r_2r_1r_2r_1r_2r_1r_0r_2r_1r_0r_1r_2$, which is precisely the one corresponding to the walk β_e in the figure. Thus we have written a generator (in this case w_{β_e}) determined by an edge in the complement of the spanning tree as a product of elements of the form $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-1}$ (with trivial elements), i.e., $w_{\beta_e} = \beta^0\gamma^0\alpha_1\gamma^{-0}\beta^{-0}$.

To describe the means by which such a spanning tree is constructed, we present now the steps in constructing the spanning tree for (3.3.3.4.4). The main branches of our spanning tree are the walks induced by $w_{ij} := \beta^i\gamma^j$ and by $w_{ij}r_2r_1r_2$ (see Figure 3). Note that these walks naturally form a tree since β and γ correspond to independent translations of the base flag, and the additional steps induced by $r_2r_1r_2$ don't introduce any loops in the graph.

In Figure 4 we have indicated two copies of collections of cells of the flag graph that constitute regions whose borders are determined by walks of the form $\beta\gamma\beta^{-1}\gamma^{-1}$ and $\beta\gamma^{-1}\beta^{-1}\gamma$, in which one edge has been omitted from the path (adjacent to a red triangle in the figure). We next construct a spanning tree made of thick orange lines in the dual graph of each region. Edges crossed by the orange lines will be omitted edges of the spanning tree, and so we add to the main branches of our spanning tree those edges that weren't crossed by orange lines to complete our spanning trees in those regions (with the exception of those traversed by β^i as indicated by a dotted line in Figure 4). We also draw triangles connecting nodes of the dual tree to omitted edges of the spanning tree

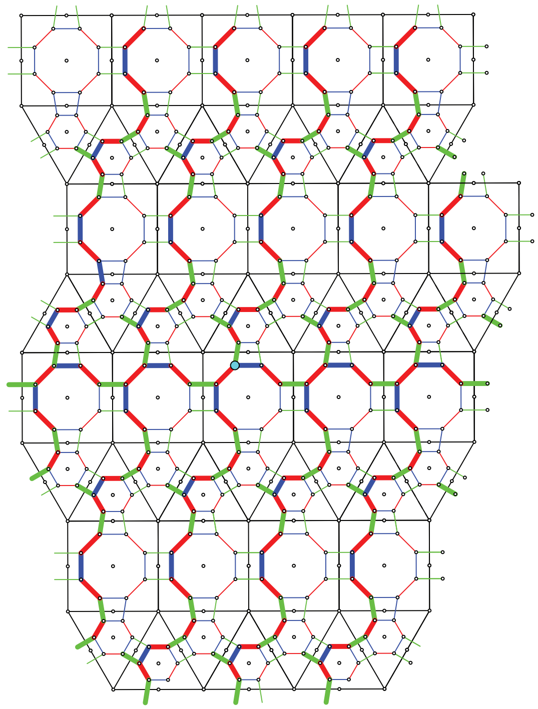


Figure 3: A view of a larger region of the tiling (3.3.3.4.4) with the flag graph and main branches of the spanning tree superimposed.

oriented towards the open end of each region (one triangle per omitted edge). We also draw black edges connecting nodes of the dual graph corresponding to faces of the tiling to the nodes of the flag graph determined by the end of the corresponding walk induced by $\beta^i\gamma^j w$. These triangles and black edges turn out to be helpful in ordering products equivalent to the w_{β_e}, as we will see below.

If we copy the introduced edges (and dual complementing forest) into the other regions enclosed by the walks starting on the base flag $\beta^i\gamma^j\beta^{-1}\gamma^{-1}\beta$ with $j \geqslant 0$ or $\beta^i\gamma^j\beta^{-1}\gamma\beta$ where $j \leqslant -1$, we obtain a spanning tree for the flag graph since the presence of the dual trees prevent these edges from forming closed loops, and every node is on a branch of the spanning tree since each node in the region enclosed by $\beta\gamma\beta^{-1}\gamma^{-1}$ is on a branch of the tree. The only thing exceptional in the construction process is that for those regions enclosed by $\beta^i\gamma^j\beta^{-1}\gamma\beta$ where $j \leqslant -1$ the orientation of the triangles for some of the nodes is flipped across the omitted edges from those where $j > -1$, as seen in the aqua colored region in Figure 4 (notice that some of these differ in orientation from those given in the yellow region).

To see that the $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}$ are adequate to generate the stabilizer subgroup of the base flag, it suffices to show that any of the w_{β_e} from Theorem 6 for this spanning tree may be obtained as a product of these generators. Since the regular universal cover is

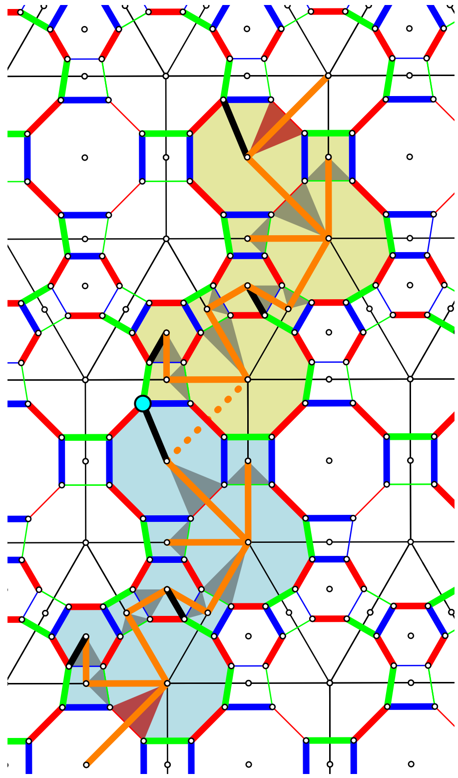
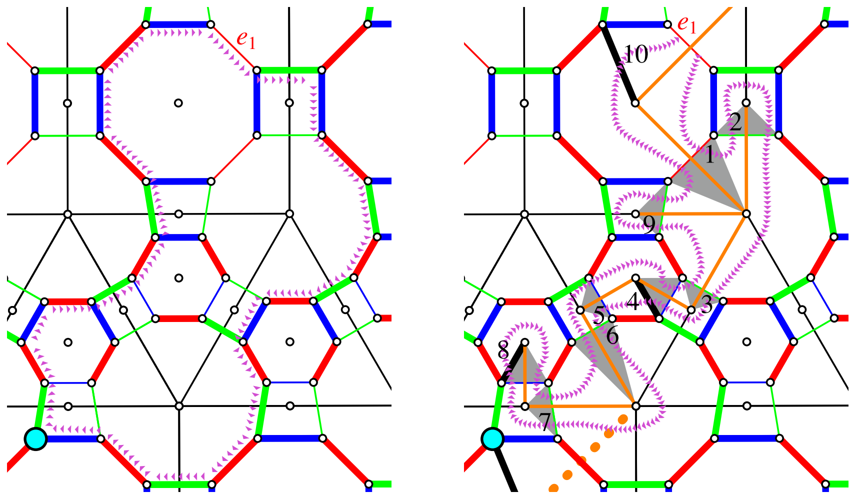


Figure 4: The dashed orange line indicates an edge of the tree in the dual flag graph that is omitted because of the presence of a walk induced by β^i. Gray triangles indicate correspondences between omitted edges of the flag graph and centers of the dual flag graph used to associate vertex, edge and face walks with those edges.

$\Gamma = [12, 5]$, all of the walks along the tree to a cell of the flag graph corresponding to an edge or vertex (and so not corresponding to a face of the tiling), around that cell, and back are trivial (the corresponding words are equivalent to the identity in $[12, 5]$). Therefore, any of their conjugates may be inserted as needed anywhere in a product of the given generators $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}$.

Let e be any omitted edge of the walk; then, as in Figure 5a, there is a unique walk (or *lasso*) β_e from the base flag, through the tree to one endpoint of e, across e and finally back along the tree from the other endpoint. The region enclosed by this walk is necessarily simply connected and contains a finite number of face cells. The product of the W_{f_i} corresponding to these enclosed face cells, suitably ordered, is equivalent to w_{β_e}. To help us understand this equivalence, consider the walk induced by β_{e_1} as indicated in Figure 5a. We construct an ordering on the words corresponding to the face, vertex, and edge lollipops for the enclosed cells of the flag graph by starting on the right hand side of the edge e_1 and to the right of the orange dual tree. Proceeding in a clockwise fashion around the dual tree (as seen in Figure 5b), we add terms to the product corresponding to nodes of the dual flag graph in the order we cross the corresponding edge of the flag graph (the correspondence being indicated by the triangles connecting omitted edges to the centers of vertices, edges and faces of the tiling), being sure not to add those corresponding to a cell center connected by a black edge to the spanning tree of the flag graph until we cross that black edge (thus ensuring we introduce elements to the product about which we have little flexibility while we are adding cells on the appropriate branch of the spanning tree). Such an ordering is given in Figure 5b for the walk induced by β_{e_1}, and guarantees the reducibility of the product to w_{β_e}.



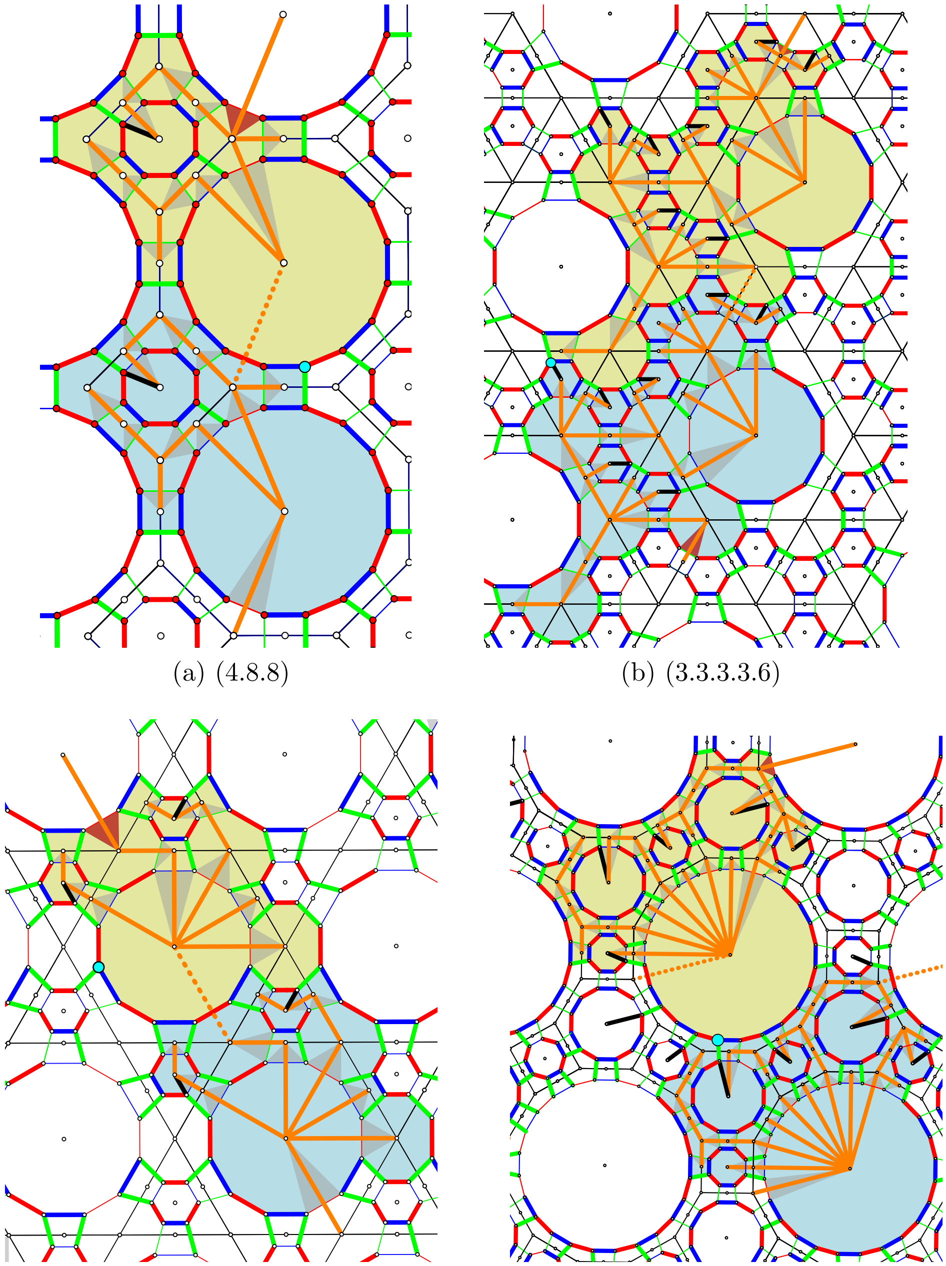
(a) The walk induced by β_{e_1}

(b) A walk around the dual tree induces a product ordering.

Figure 5: In each of these figures, the cyan dot indicates a base flag. In (a) we see the walk induced by β_{e_1}, indicated by a sequence of purple arrows in the direction of the walk, adjacent to the corresponding edges of the flag graph. In (b) traversals of grey triangles are used to induce an order on the products of vertex, edge and face walks determined by the centers of the cells of the dual graph. Cell centers are labeled in the order they need to be added to the product.

Since such an ordering is possible for either orientation of the triangles, and for any choice of omitted edge in the regions enclosed by $\beta\gamma\beta^{-1}\gamma^{-1}$ and $\beta\gamma^{-1}\beta^{-1}\gamma$, we can construct such a product for any omitted edge of the flag graph.

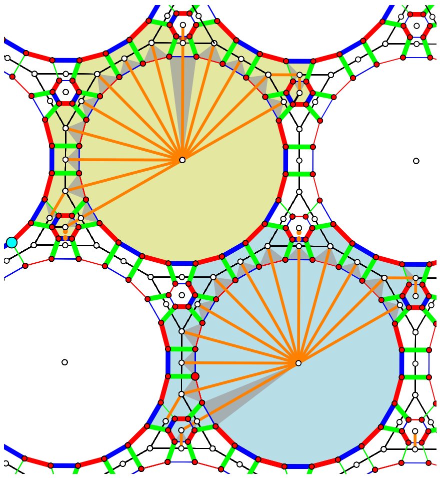
We conclude this section by providing specific examples of spanning trees sufficient for proving the remaining of Theorem 5, given in Figure 6. Note that the trees shown here reflect the changes to the generating sets mentioned in the statement of Theorem 5.



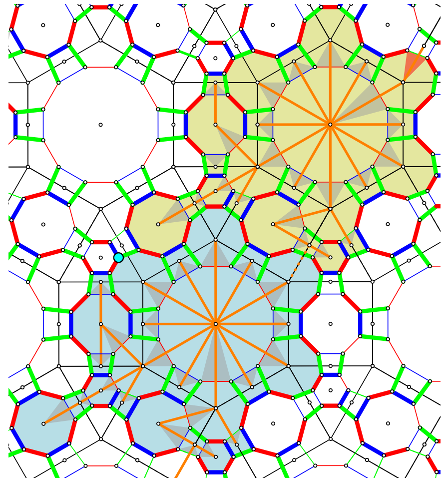
(a) (4.8.8) (b) (3.3.3.3.6)

(c) (3.6.3.6) (d) (4.6.12)

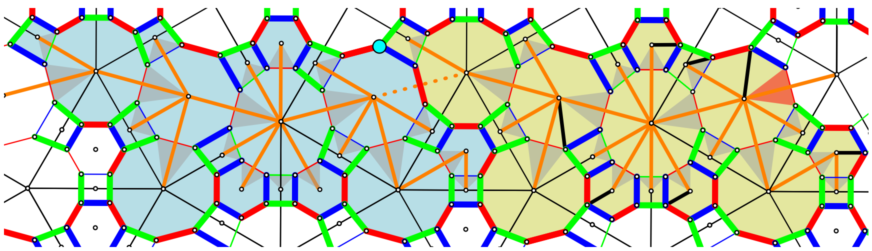
Figure 6: Sample regions for the spanning trees of the remaining Archimedean tilings necessary to establish the proof of Theorem 5. The vertex corresponding to the specified base flag in the flag graph is indicated with a large cyan dot. Figure continues on the next page.



(e) (3.12.12)



(f) (3.4.6.4)



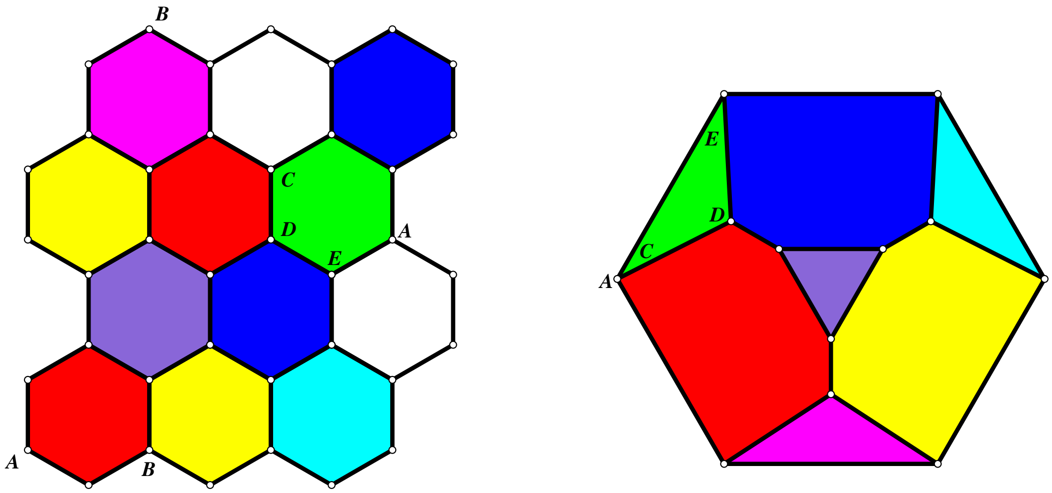
(g) (3.3.4.3.4)

Figure 6: The remaining three sample regions required for the proof of Theorem 5.

4 Covers as Identification Mappings

A natural question to ask in the context of these covers is how the numbers of flags in the covering polytope relate to the numbers of flags in the covered polytope. As an example, consider the minimal cover of the truncated tetrahedron $\mathcal{P}_{3.6.6}$ by the regular polytope $\{6,3\}_{(2,2)}$ (see Figure 7). The polytope $\{6,3\}_{(2,2)}$ has 12 hexagonal faces, of which four hexagons get wrapped twice around the triangles they get mapped to, and the remaining eight hexagons are mapped to the four hexagons of the truncated tetrahedron in pairs (for more details, see [HW10]). Thus, each flag of the truncated tetrahedron is covered by two flags of the polytope $\{6,3\}_{(2,2)}$. We define a cover $\mathcal{P} \searrow \mathcal{Q}$ to be a *k-fold cover* if the preimage of every flag in $\mathcal{Q}$ has k flags in $\mathcal{P}$. In this notation, $\{6,3\}_{(2,2)} \searrow \mathcal{P}_{3.6.6}$ is a 2-fold cover.

There is a useful relationship between the number of cosets of the core in the stabilizer of a flag Φ and the number of flags in the preimage of Φ, summarized by the following lemma. Let $|G : K|$ denote the index in G of the subgroup K.



$\{6,3\}_{(2,2)}$ (3.6.6)

Figure 7: The regular polytope $\{6,3\}_{(2,2)}$ and its relationship to the truncated tetrahedron (3.6.6). The regular polytope $\{6,3\}_{(2,2)}$ is formed by quotienting the regular tessellation $\{6,3\}$ by the translations that identify the vertices labeled A and B. The polyhedron (3.6.6) is represented here by its Schlegel diagram with the outside face colored white. The polytope $\{3,6\}_{(2,2)}$ is a minimal cover of (3.6.6) (see [HW10]). Face identifications formed by the quotient are indicated by tiles of matching colors. For example, the two red hexagons in $\{6,3\}_{(2,2)}$ cover the red hexagon in (3.6.6), and the two white hexagons are mapped to the outside face of the Schlegel diagram. Triangles in (3.6.6) are double covered by a single hexagon of $\{6,3\}_{(2,2)}$. For example, the single green hexagon in $\{6,3\}_{(2,2)}$ wraps twice around the green triangle, identifying vertices C and A.

Lemma 7. *Let $\mathcal{P} \searrow \mathcal{R} \searrow \mathcal{Q}$ with $\mathcal{P}$ and $\mathcal{R}$ regular, Γ the automorphism group of $\mathcal{P}$, $\mathcal{Q} = \mathcal{P}/N$ (i.e., N the stabilizer of a flag in $\mathcal{Q}$ under the flag action of Γ), and $\mathcal{R} = \mathcal{P}/Core(\Gamma, N)$. Then $\mathcal{R} \searrow \mathcal{Q}$ is a k-fold cover if $k = |N : Core(\Gamma, N)|$.*

Proof. Note that $Core(\Gamma, N)$ is the stabilizer of every flag of $\mathcal{R}$ in Γ (and therefore also in N) since it is the kernel of the first quotient map. By the Orbit-Stabilizer Theorem, there is a bijection between the orbit of an element and the set of left cosets of the stabilizer of that element. In this context, therefore, the elements of the flag orbits under the action of N on $\mathcal{R}$ are in one to one correspondence with the left cosets of $Core(\Gamma, N)$ in N. In other words, the flag orbits in $\mathcal{R}$ are all of size $|N : Core(\Gamma, N)|$. This also shows that the sizes of the orbits are all the same, so there is only one stabilizer subgroup in this case, namely $Core(\Gamma, N)$. Therefore $\mathcal{R} \searrow \mathcal{Q}$ is a k-fold cover. $\square$

It seems not unreasonable to suppose that in the context of covers of the uniform tilings we ought to be able to identify such a value of k, or at least determine that a finite value for k exists. As we shall see, this is not the case.

Table 2: Elements w and v such that the base flag Φ of the tiling is stabilized by w and acts as a translation on Φ^v via the flag action. Arrows are used to indicate the location of the base flag.

Vertex Type	w	v
$(\overrightarrow{3.6}.3.6)$	$[(r_0r_1)^3r_2r_1r_2]^2$	r_2
$(4.\overrightarrow{8.8})$	$[r_1r_2(r_1r_0)^3]^4$	r_1
$(3.3.4.\overleftarrow{3.4})$	$[(r_0r_1r_2)^2(r_0r_1)^2(r_2r_1)^2]^4$	r_1
$(3.3.\overleftarrow{3.4}.4)$	$[r_2(r_1r_0)^5r_1]^4$	r_1
$(\overleftarrow{3.4}.6.4)$	$[(r_1r_0)^2r_1r_2r_1r_0r_1r_2]^4$	r_1r_2
$(3.3.3.\overleftarrow{3.6})$	$[r_2r_1r_2r_0r_1r_2(r_1r_0)^3]^4$	r_1
$(3.\overrightarrow{12.12})$	$[r_1r_2(r_1r_0)^5]^4$	r_1
$(4.\overleftarrow{6.12})$	$[r_1r_2(r_1r_0)^3]^6$	r_1

Proposition 8. *Let $\mathcal{Q}$ be a uniform tessellation of the plane, Γ the string C-group of any regular cover of $\mathcal{Q}$, N the stabilizer in Γ of a flag in $\mathcal{Q}$, and C the core of N in Γ. Then C is a subgroup of infinite index in N.*

Proof. In Table 2 we list each of the uniform tilings of the plane. The symbols have been modified with an overset arrow to indicate the symmetry type of a base flag Φ of the tiling. For example, if the arrow is over the substring $a.b$ in a symbol, with the tail over a and arrowhead over b, then the base flag contains a face with a sides and shares its edge with a face that has b sides, and conversely if the arrow is oriented in the opposite direction. Note that for (3.3.3.3.6) the given symbol is ambiguous (there are two possible choices for Φ). Our presentation assumes that $\Phi r_2r_1r_2r_0r_1r_2$ is stabilized by $(r_0r_1)^3$.

Let Φ be a base flag of $\mathcal{Q}$ of the type specified in Table 2, w the word associated to the tiling, and Ψ the flag obtained from Φ by the flag action of the given element $v \in \Gamma$. Then $\Phi^w = \Phi$ and $\Psi^w = \Psi t$ for some translation t in the symmetry group of $\mathcal{Q}$. Clearly, $\Phi^{w^k} = \Phi$ and $\Psi^{w^k} = \Psi t^k$ for any $k \in \mathbb{Z}$. In particular, it follows that $w^k \in N$ for all k.

We know that the coset w^kC coincides with the coset w^mC (as cosets in N) if and only if $w^{k-m} \in C$; that is, $\Upsilon^{w^{k-m}} = \Upsilon$ for every flag Υ of $\mathcal{Q}$. In particular, $\Psi^{w^{k-m}} = \Psi$. However, this occurs if and only if $k = m$. Therefore $\{w^kC \mid k \in \mathbb{Z}\}$ is an infinite family of left cosets of C in N implying that C has index ∞ in N. $\square$

5 Minimal Covers for Some of the Uniform Tilings of $\mathbb{E}^2$

In what follows, we present the minimal regular covers of the tessellations (3.6.3.6), (4.8.8) and (3.12.12). The remaining, more complicated cases will appear in a subsequent work.

The tiling (3.6.3.6) has two orbits of flags; type A containing hexagonal faces and type B containing triangular faces. The tilings (4.8.8) and (3.12.12) each have three orbits of flags. For the tiling (4.8.8) type A consists of those flags containing a square, those of type B contain an octagon and share an edge with another octagon while those of type C are the remaining flags on the octagons. For the tiling (3.12.12) type A consists of those flags containing a 12-gon and sharing an edge with another 12-gon, type B the remaining flags containing a 12-gon, and type C being those flags containing a triangle.

5.1 General procedure

Let $\mathcal{T}$ be a uniform tiling, $\Gamma := [p,q]$ its universal covering group, N_A the stabilizer of a specified flag Φ of type A (say), $\mathcal{C}$ the core in $[p,q]$ of N_A and therefore $\Gamma/\mathcal{C}$ is the automorphism group of the minimal regular cover of $\mathcal{T}$. As we mentioned earlier, there is a contravariant isomorphism between Γ and $\mathrm{Mon}([p,q])$; we shall abuse notation and interpret Γ as equal to $\mathrm{Mon}([p,q])$ with generators r_i. To describe the minimal regular covers of $\mathcal{T}$ we provide defining relations in terms of the generators of $[p,q]$ for its automorphism group $\Gamma/\mathcal{C}$. The outline in each of the cases is as follows.

5.1.1 Step 1: Reduce to a finite generating set for N_A

The group N_A is generated by the infinite set $\{\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^i\}$ as in Theorem 5 (see also [PW11]). Each element $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^i$ acts on flags of type B (and C, when applicable) in a particular geometric way. Some compositions of these actions act like the identity on all flags: for example, if two generators g_1, g_2 act like linearly independent translations on flags of type B (and C), then $g_1g_2g_1^{-1}g_2^{-1}$ must act like the identity on all flags. This is equivalent to saying that $g_1g_2g_1^{-1}g_2^{-1} \in \mathcal{C}$. We construct a finite number of elements η_l, $l = 1,\ldots,m_1$ found this way.

Additionally, in the tilings considered in this paper, the commutator of some powers of β and of γ also belong to $\mathcal{C}$. We denote these elements by η_l, $l = m_1+1,\ldots,m_2$.

The first step to finding $\mathcal{C}$ is to consider the normal closure $Cl_1 := Cl_\Gamma(\{\eta_1,\ldots,\eta_{m_2}\})$ of $\eta_1,\ldots,\eta_{m_2}$ in Γ. Clearly Cl_1 is a subgroup of $\mathcal{C} \subseteq N_A$ and it is normal in Γ (and hence in N_A). Furthermore, $Cl_1 = \mathcal{C}$ if and only if for every element in N_A/Cl_1 there exists a flag it acts on in a nontrivial way. Unfortunately, this is typically not the case, but we are able to show at least that N_A/Cl_1 is generated by a finite list $w_1,\ldots,w_s$ of elements $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}\cdot Cl_1$. This makes it easier to analyze the group and its action on the flags of the tiling.

5.1.2 Step 2 : Intersect with N_B

Let N_B be the stabilizer of flags of type B. In Step 2 we a generating set for $N_A \cap N_B$. To do this, we first determine the action of each of the generators w_i of N_A/Cl_1 on a given flag Ψ of type B. When $\mathcal{T}$ is either (3.6.3.6) or (4.8.8), the generators w_i preserve all flag orbits. Consequently, $N_A \cap N_B$ is a normal subgroup of N_A. (Note that this is not true for tilings where the elements of N_A do not preserve the flag orbits.) Moreover,

$N_A/(N_A \cap N_B)$ is the largest quotient of N_A with the property that no non-trivial element acts trivially on flags of type B of $\mathcal{T}$.

For the tilings (3.6.3.6) and (4.8.8), the generators $w_1, \ldots, w_s$ of N_A/Cl_1 act on a given flag Ψ of type B like half-turns $h_1, \ldots, h_s$. It suffices to consider the action on only one flag since the monodromy group commutes with the automorphism group. Now we use the contravariant group homomorphism from N_A/Cl_1 to the isometry group $\langle h_1, \ldots, h_s \rangle$ described in Proposition 1 (recall that h_i is the unique isometry mapping Ψ to Ψw_i), whose kernel is $(N_A \cap N_B)/Cl_1$. Consequently, $N_A/(N_A \cap N_B) \cong \langle h_1, \ldots, h_s \rangle$. By inspecting the actions of the elements h_i on Ψ we can find a set of defining relations $\mathcal{R} := \{R_1 = id, \ldots, R_t = id\}$ for $\langle h_1, \ldots, h_s \rangle$. We shall abuse notation and consider R_i as a word on the generators of N_A/Cl_1 corresponding to R_i's inverse image. Note that R_i is then an element in $(N_A \cap N_B)/Cl_1$. Furthermore, $(N_A \cap N_B)/Cl_1$ is the normal closure in N_A/Cl_1 of $\{R_1, \ldots, R_t\}$. In fact, any element in the normal closure in N_A/Cl_1 of $\{R_1, ..., R_t\}$ must stabilize flags of both type A and type B, and so is an element of $(N_A \cap N_B)/Cl_1$. On the other hand, any element of $(N_A \cap N_B)/Cl_1$ stabilizes flags of type B, and must also be an element in N_A/Cl_1, so it must be in the normal closure in N_A/Cl_1 of $\{R_1, \ldots, R_t\}$, since the R_i generate all actions of the elements of N_A/Cl_1 that fix flags of type B.

When the tiling $\mathcal{T}$ being considered is (3.12.12), each of the generators $w_1, \ldots, w_s$ in our presentation of N_A/Cl_1 interchanges flag orbits B and C. Hence, every element in $\mathcal{C}/Cl_1$ must belong to the even subgroup $(N_A/Cl_1)^+$ consisting of all elements corresponding to words with even length on the generators $w_1, \ldots, w_s$. All products of any two generators w_i (not necessarily distinct) form a finite generating set of $(N_A/Cl_1)^+$. Each of these generators acts like a half-turn or a translation on flags of type B. Therefore, to find $(N_A \cap N_B)/Cl_1$ we can proceed as for the tilings (3.6.3.6) and (4.8.8) with the generators w_i replaced by the generators of $(N_A/Cl_1)^+$.

We frequently make use of the following lemma from [CBGS08] to determine some of the R_i's.

Lemma 9. *Let G be a group generated by three involutions g, h, k such that the product ghk is also an involution. If the elements gh and hk have infinite order then G is isomorphic to the group* 2222 *(in the notation of [CBGS08]) of symmetries of the Euclidean plane generated by three half-turns with respect to three non-co-linear points. Furthermore, $G = \langle g, h, k \mid g^2 = h^2 = k^2 = (ghk)^2 = id \rangle$ is a group presentation for G.*

5.1.3 Step 3: Intersect with N_C

For the tiling (3.6.3.6) which has only two orbits of flags, $\mathcal{C} = N_A \cap N_B$. For the remaining two tilings, $\mathcal{C} = N_A \cap N_B \cap N_C$, where N_C is the stabilizer of flags of type C. Our first objective is to reduce to a finite set of generators as we did in Step 1. Note that the group $(N_A \cap N_B)/Cl_1 = Cl_{N_A/Cl_1}(\{R_1, \ldots, R_t\})$ is generated by the set of all conjugates in N_A of $R_1, \ldots, R_t$, which in principle is an infinite set. However, these elements correspond to only finitely many distinct actions on flags of type C. Using the actions of the generators of N_A/Cl_1 and of $R_1, \ldots, R_t$, and the isometries determined by their actions on a given flag, we find elements stabilizing all flags, that is, elements in $\mathcal{C}$. Moreover, the set of generators of $(N_A \cap N_B)/Cl_1$ is reduced considerably after taking a quotient by the normal closure of these elements. We carefully choose a finite number of these elements in $\mathcal{C}$, called $\eta_{m_2+1}, \dots, \eta_{m_3}$, and we consider the normal closure $Cl_2 := Cl_\Gamma(\{\eta_1, \dots, \eta_{m_3}\})$. We prove in §5.2.2, 5.3.2 and 5.4.2 that for some choice of elements η_i, $(N_A \cap N_B)/Cl_2$ is finitely generated.

To find $\mathcal{C} = N_A \cap N_B \cap N_C$ we consider the action on flags of type C of the generators of $(N_A \cap N_B)/Cl_2$ and apply an analogous procedure to that used to determine $(N_A \cap N_B)/Cl_1$ in Step 2. Instead of R_i we denote the defining relations of $N_A/(N_A \cap N_B \cap N_C)$ by T_i.

The final result is that $\mathcal{C}$ is the normal closure in Γ of the elements $\eta_1, \dots, \eta_{m+3}$ together with (representatives of) the elements T_i whose normal closure in $(N_A \cap N_B)/Cl_2$ is $\mathcal{C}/Cl_2$. To see this we note first that each of the η_i and T_i were chosen so as to stabilize all three classes of flags, so all of the elements in the subgroup generated by elements of Cl_2 and $Cl_\Gamma(\{T_i\})$ must stabilize all three classes of flags. Thus once one demonstrates that any element of $(N_A \cap N_B \cap N_C)/\langle Cl_2 \cup Cl_\Gamma\{T_i\}\rangle$ is trivial, we know that $\mathcal{C} = \langle Cl_2 \cup Cl_\Gamma\{T_i\}\rangle$. In other words, the elements η_i and T_i induce a set of defining relations for the automorphism group of the regular cover of $\mathcal{T}$.

5.2 The tiling (3.6.3.6)

Throughout, let

$$\Gamma = [6,4] := \langle r_0, r_1, r_2 \,|\, r_0^2 = r_1^2 = r_2^2 = (r_0r_2)^2 = (r_0r_1)^6 = (r_1r_2)^4 = id\rangle,$$

let $\mathcal{T}$ be the uniform tiling (3.6.3.6), designate flags containing a hexagon as being of type A, N_A the stabilizer of a flag Φ of type A under the flag action of $[6,4]$, N_B the stabilizer of a flag of type B containing a triangle under the same action, $\mathcal{C} = N_A \cap N_B$ the core of N_A in $[6,4]$, and $[6,4]/\mathcal{C}$ the automorphism group of the minimal cover of $\mathcal{T}$.

Let $\alpha_0 = r_2(r_0r_1)^3r_2$, $\alpha_1 = r_1r_2(r_0r_1)^3r_2r_1$, $\beta = r_0r_1r_0r_1r_0r_2r_1r_2$, $\gamma = (r_0r_1r_2r_1)^2$ and $\delta = r_1r_0\beta r_0r_1 = r_0r_1r_0r_2r_1r_2r_0r_1$. By Theorem 5 we know that

$$N_A = \langle \beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i}\rangle \tag{2}$$

for $i, j \in \mathbb{N}$, $k \in \{0,1\}$.

Before doing step (1) we show that N_A is also generated by all elements of the form $\beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i}$ with $i, j \in \mathbb{Z}$ and $k \in \{0,1\}$. To see this, note that

$$\gamma = (r_0r_1r_2r_1)^2 = r_0r_1r_2(r_0r_1r_0)^2r_1r_0r_1r_2r_1 = r_0r_1r_2r_0r_1r_0(r_0r_1)^3r_2r_1 = \delta\alpha_1.$$

It follows that

$$\gamma^j\alpha_k\gamma^{-j} = (\delta\alpha_1)^j\alpha_k(\delta\alpha_1)^{-j},$$

which equals

$$\delta\alpha_1\delta^{-1}\delta^2\alpha_1\delta^{-2}\dots\delta^{j-1}\alpha_1\delta^{-(j-1)}\delta^j\alpha_1\delta^{-j}\delta^j\alpha_k\delta^{-j}\delta^j\alpha_1\delta^{-j}\delta^{j-1}\alpha_1\delta_{-(j-1)}\dots\delta\alpha_1\delta^{-1}$$

for $i, j \in \mathbb{N}$, $k \in \{0,1\}$. It follows that

$$\langle\gamma^j\alpha_k\gamma^{-j}\rangle = \langle\delta^j\alpha_k\delta^{-j}\rangle,$$

and therefore

$$N_A = \langle \beta^i \delta^j \alpha_k \delta^{-j} \beta^{-i} \rangle. \tag{3}$$

5.2.1 Reducing to a finite generating set for N_A

Note that β^2 and δ^2 act like the identity on flags containing a triangle of $\mathcal{T}$. Since α_0 and α_1 are involutions, the elements $\eta_1 := \delta^2\alpha_0\delta^{-2}\alpha_0$, $\eta_2 := \delta^2\alpha_1\delta^{-2}\alpha_1$, $\eta_3 := \beta^2\alpha_0\beta^{-2}\alpha_0$, $\eta_4 := \beta^2\alpha_1\beta^{-2}\alpha_1$ and $\eta_5 := \delta^{-1}\beta^{-2}\delta\beta^2$ are all elements of $\mathcal{C}$.

Let $Cl_1 := Cl_\Gamma(\{\eta_1, \ldots, \eta_5\})$. The generating set of N_A given in (3) induces the generating set

$$\left\{\beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i} \cdot Cl_1 \mid \; k \in \{0,1\}\right\}$$

of N_A/Cl_1. Because α_k is an involution, we have that $\beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i} \cdot Cl_1$ equals

$$\begin{aligned}
& \beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i} \cdot \beta^i\delta^{j-2}(\eta_{k+1})\delta^{-j+2}\beta^{-i} \cdot Cl_1 \\
= \; & \beta^i\delta^j\alpha_k\delta^{-2}(\delta^2\alpha_k\delta^{-2}\alpha_k)\delta^{-j+2}\beta^{-i} \cdot Cl_1 \\
= \; & \beta^i\delta^{j-2}\alpha_k\delta^{-j+2}\beta^{-i} \cdot Cl_1.
\end{aligned}$$

Let $\hat{j} = 0$ if j is even and $\hat{j} = 1$ if j is odd. Then

$$\beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i} \cdot Cl_1 = \beta^i\delta^{\hat{j}}\alpha_k\delta^{-\hat{j}}\beta^{-i} \cdot Cl_1. \tag{4}$$

Now,

$$\begin{aligned}
\beta^i\alpha_k\beta^{-i} \cdot Cl_1 &= \beta^i\alpha_k\beta^{-i} \cdot \beta^{i-2}\eta_{k+3}\beta^{-i+2} \cdot Cl_1 \\
&= \beta^i\alpha_k\beta^{-i} \cdot \beta^{i-2}\beta^2\alpha_k\beta^{-2}\alpha_k\beta^{-i+2} \cdot Cl_1 \\
&= \beta^{i-2}\alpha_k\beta^{-i+2} \cdot Cl_1.
\end{aligned}$$

Thus if we let $\hat{i} = 0$ if i is even and $\hat{i} = 1$ if i is odd, then

$$\beta^i\alpha_k\beta^{-i} \cdot Cl_1 = \beta^{\hat{i}}\alpha_k\beta^{-\hat{i}} \cdot Cl_1. \tag{5}$$

Finally,

$$\begin{aligned}
\beta^i\delta\alpha_k\delta^{-1}\beta^{-i} \cdot Cl_1 &= \beta^i\delta\alpha_k\delta^{-1}\beta^{-i} \cdot \beta^i\delta\alpha_k\eta_5\alpha_k\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^i\delta\alpha_k\delta^{-1}\beta^{-i} \cdot \beta^i\delta\alpha_k(\delta^{-1}\beta^{-2}\delta\beta^2)\alpha_k\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\beta^2\alpha_k\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\beta^2\alpha_k\delta^{-1}\beta^{-i} \cdot \beta^i\delta(\beta^{-2}\eta_{k+3}\beta^2)\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\beta^2\alpha_k\delta^{-1}\beta^{-i} \cdot \beta^i\delta(\alpha_k\beta^{-2}\alpha_k\beta^2)\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\alpha_k\beta^2\delta^{-1}\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\alpha_k\beta^2\delta^{-1}\beta^{-i} \cdot \beta^i(\delta\eta_5^{-1}\delta^{-1})\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\alpha_k\beta^2\delta^{-1}\beta^{-i}\beta^i(\delta\beta^{-2}\delta^{-1}\beta^2)\beta^{-i} \cdot Cl_1 \\
&= \beta^{i-2}\delta\alpha_k\delta^{-1}\beta^{-i+2} \cdot Cl_1.
\end{aligned}$$

Consequently,

$$\beta^i\delta\alpha_k\delta^{-1}\beta^{-i}\cdot Cl_1 = \beta^{\hat{i}}\delta\alpha_k\delta^{-1}\beta^{-\hat{i}}\cdot Cl_1, \tag{6}$$

where $\hat{i}=0$ if i is even and $\hat{i}=1$ if i is odd.

From (4), (5) and (6) we conclude that every generator $\beta^i\delta^j\alpha_k\delta^{-j}\beta^{-i}$ of N_A is equivalent modulo Cl_1 to $\beta^{\hat{i}}\delta^{\hat{j}}\alpha_k\delta^{-\hat{j}}\beta^{-\hat{i}}$ for some $\hat{i},\hat{j}\in\{0,1\}$. Therefore N_A/Cl_1 is generated by the eight elements $\beta^{\hat{i}}\delta^{\hat{j}}\alpha_k\delta^{-\hat{j}}\beta^{-\hat{i}}$ with $\hat{i},\hat{j},k\in\{0,1\}$.

5.2.2 Intersect with N_B

We note that the generators

$$\alpha_0\cdot Cl_1, \quad \alpha_1\cdot Cl_1, \quad \delta\alpha_0\delta^{-1}\cdot Cl_1 \tag{7}$$

of N_A/Cl_1 act on flags of type B like three half-turns with respect to three non-colinear points. The action of the remaining 5 generators can be described as a product of the actions of the previous three in the following way:

$$\begin{aligned}
&\beta\alpha_0\beta^{-1}\cdot Cl_1 = \alpha_1\cdot Cl_1,\\
&\beta\alpha_1\beta^{-1}\cdot Cl_1 = \alpha_0\cdot Cl_1,\\
&\delta\alpha_1\delta^{-1}\cdot Cl_1 = \alpha_0\alpha_1\delta\alpha_0\delta^{-1}\cdot Cl_1,\\
&\beta\delta\alpha_0\delta^{-1}\beta^{-1}\cdot Cl_1 = \alpha_0\delta\alpha_0\delta^{-1}\alpha_1\cdot Cl_1,\\
&\beta\delta\alpha_1\delta^{-1}\beta^{-1}\cdot Cl_1 = \alpha_1\delta\alpha_0\delta^{-1}\alpha_1\cdot Cl_1.
\end{aligned}$$

Since all generators of N_A/Cl_1 are involutions it follows that the five elements

$$\begin{aligned}
\zeta_1 &:= \beta\alpha_0\beta^{-1}\alpha_1,\\
\zeta_2 &:= \beta\alpha_1\beta^{-1}\alpha_0,\\
\zeta_3 &:= \delta\alpha_1\delta^{-1}\alpha_0\alpha_1\delta\alpha_0\delta^{-1},\\
\zeta_4 &:= \beta\delta\alpha_0\delta^{-1}\beta^{-1}\alpha_0\delta\alpha_0\delta^{-1}\alpha_1,\\
\zeta_5 &:= \beta\delta\alpha_1\delta^{-1}\beta^{-1}\alpha_1\delta\alpha_0\delta^{-1}\alpha_1
\end{aligned}$$

belong to $\mathcal{C}$. We define $\eta_6=\zeta_1$; note that ζ_2 is the conjugate of η_6 by r_1, since $\alpha_1=r_1\alpha_0r_1$ and $r_1\beta r_1=\beta$ (here we use the fact that $(r_0r_1)^6=(r_1r_2)^4=id$). For convenience we let η_7 be the conjugation of ζ_3 by $\delta\alpha_1\delta^{-1}$ to obtain the simpler generator $\eta_7 := \alpha_0\alpha_1\delta\alpha_0\alpha_1\delta^{-1}$. Finally, let $\eta_8 := \zeta_4$ and $\eta_9 := \zeta_5$.

Let $Cl_2 := Cl_\Gamma(\{\eta_1,\ldots,\eta_9\})$. Then by construction $Cl_2\subseteq\mathcal{C}$.

5.2.3 Show that $Cl_2\cong\mathcal{C}$.

Note that the three generators in (7) are involutions, the product of all three of them (in any order) is another involution, and the product of any two of them acts as a translation on any flag containing a triangle. In particular, the product of any two of them has infinite order. Now we make use of Lemma 9 to conclude that a sufficient set of defining relations for N_A/Cl_1 is

I $(\alpha_0 \cdot Cl_1)^2 = Cl_1$,

II $(\alpha_1 \cdot Cl_1)^2 = Cl_1$,

III $(\delta\alpha_0\delta^{-1} \cdot Cl_1)^2 = Cl_1$, and

IV $[(\alpha_0 \cdot Cl_1)(\alpha_1 \cdot Cl_1)(\delta\alpha_0\delta^{-1} \cdot Cl_1)]^2 = Cl_1$.

These relations correspond to T_1, T_2, T_3 and T_4 in the notation of Section 5.1.3.We observe that the relations (I), (II) and (III) are trivial in Γ/Cl_1 since α_0 and α_1 are involutions in Γ. To see that the last one is equivalent to η_8, rewrite

$$\beta\delta\alpha_0\delta^{-1}\beta^{-1} = \alpha_0\delta\alpha_0\delta^{-1}\alpha_1$$

and note that the left hand side is an involution. Hence the right hand side is also an involution and it is a conjugate of the desired relation. We recall that there is a natural morphism $\phi : N_A/Cl_2 \to N_A/\mathcal{C}$ mapping $w \cdot Cl_2$ to $w \cdot \mathcal{C}$, and that an element $w \cdot Cl_2$ is in the kernel of ϕ if and only if $w \in \mathcal{C}$, that is, w fixes all flags in $\mathcal{T}$.

It is easy to see that the elements $\alpha_0\alpha_1$ and $\alpha_0\delta\alpha_0\delta^{-1}$ act like translations with respect to linearly independent vectors on any given flag containing a triangle. For a given group G with generators g_i we denote by G^+ the even subgroup of G, consisting of the elements that can be expressed as a product of an even number of these generators. Moreover, $\alpha_0\alpha_1 \cdot Cl_2$ and $\alpha_0\delta\alpha_0\delta^{-1} \cdot Cl_2$ generate the even subgroup $(N_A/Cl_2)^+$ of N_A/Cl_2. Since no non-trivial element in $(N_A/Cl_2)^+$ acts trivially on a flag containing a triangle, we conclude that $(N_A/Cl_2)^+ \cap \ker(\phi)$ is trivial. On the other hand, any word w such that $w \cdot Cl_2 \notin (N_A/Cl_2)^+$ maps any flag Ψ containing a triangle into a flag obtained from Ψ by a rotation by an angle of π. Therefore, none of these words can belong to $\ker(\phi)$. Hence $\ker(\phi)$ is trivial, so $\mathcal{C} = Cl_2$.

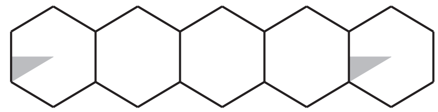
It can be proved that the set of relations obtained by setting η_i equal to the identity id can be reduced to relations $[(r_1r_0)^2r_1r_2]^4 = id$ and $[(r_1r_0)^2r_2]^6 = id$. This implies that the minimal regular cover of (3.6.3.6) can be obtained from the regular tessellation of the hyperbolic plane with hexagons meeting six around each vertex, by making the identifications indicated in Figure 8. Details are available on the second author's website ([Wil11]).

5.3 The tiling (4.8.8)

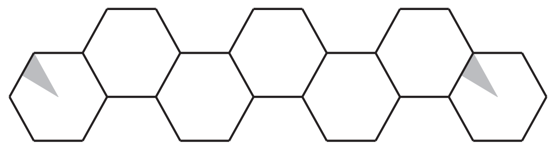
Throughout what follows let

$$\Gamma := [8,3] = \langle r_0, r_1, r_2 |\, r_0^2 = r_1^2 = r_2^2 = (r_0r_2)^2 = (r_0r_1)^8 = (r_1r_2)^3 = id\rangle,$$

$\mathcal{T}$ be the uniform tiling (4.8.8), designate flags to be of type A if they contain a square, let N_A the stabilizer of a flag Φ of type A under the action of $[8,3]$, N_B the stabilizer of



$[(r_1r_0)^2r_1r_2]^4$



$[(r_1r_0)^2r_2]^6$

Figure 8: Identifications of the hyperbolic tessellation yielding the minimal regular cover of 3.6.3.6

a flag in an octagon sharing an edge with another octagon, and N_C the stabilizer of a flag in an octagon sharing an edge with a square under the same action. Since there are only three transitivity classes of flags, $\mathcal{C} = N_A \cap N_B \cap N_C$ is the core of N_A in $[8,3]$, and $\Gamma = [8,3]/\mathcal{C}$ is the automorphism group of the minimal cover of $\mathcal{T}$.

Let $\beta := (r_0r_1)^2r_2r_1r_0r_1$, $\gamma := r_2(r_1r_0)^3r_1$, $\alpha_0 = ((r_0r_1)^4)^{r_2r_1} = ((r_0r_1)^{-4})^{r_2r_1}$, $\alpha = (r_0r_1)^4 = (r_0r_1)^{-4}$, $\delta = r_2r_1r_0r_1r_2r_0r_1r_0$ and $\epsilon = r_2r_0r_1r_0r_1r_2r_0r_1$. By Theorem 5 we know that $N_B = \langle \beta^i\gamma^j\alpha_0\gamma^{-j}\beta^{-i} \mid i,j \in \mathbb{Z}\rangle$. For the purposes of this argument, we found the analysis easier to do looking at $N_A = \langle \delta^i\epsilon^j\alpha\delta^{-j}\epsilon^{-j}\rangle$. As Hartley showed in [Har99a], different choices of the base flag Φ in the construction of Γ/N_Φ correspond to conjugate stabilizers, and in our case $N_A = r_2r_1N_Br_1r_2$. It suffices therefore to show that $\{\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i}\} = r_2r_1\{\beta^i\gamma^j\alpha_0\gamma^{-j}\beta^{-i} \mid i,j \in \mathbb{Z}\}r_1r_2$. To show this claim, it suffices to observe that

$$r_2r_1\alpha_0r_1r_2 = \alpha, r_2r_1\gamma r_1r_2 = \epsilon^{-1}, \qquad \text{and} \qquad r_2r_1\beta r_1r_2 = \delta.$$

As an example calculation, note that

$$\begin{aligned} r_2r_1\gamma r_1r_2 &= r_2r_1r_2r_1r_0r_1r_0r_1r_0r_1r_1r_2 = r_1r_2r_1r_1r_0r_1r_0r_1r_0r_2 \\ &= r_1r_2r_0r_1r_0r_1r_0r_2 = r_1r_0r_2r_1r_0r_1r_0r_2 = \epsilon^{-1}. \end{aligned}$$

5.3.1 Reducing to a finite generating set for N_A

We begin with a description of some elements $\eta_i \in \mathcal{C}$. Let $\eta_{j+1} = \epsilon^{-j}\delta^{-4}\epsilon^j\delta^4$ for $j \in \{0,1,2,3\}$, and let $\eta_5 = \alpha\delta^{-4}\alpha\delta^4, \eta_6 = \epsilon^{-4}\alpha\epsilon^4\alpha$. Note that δ^4 and ϵ^4 act like the identity on flags of type B and C, and δ and ϵ act like translations on flags of type A. Since α^2 is equal to the identity in Γ, and α preserves flag-orbits, each of the $\eta_i \in \mathcal{C}$. Let $E_1 = \{\eta_1, \cdots, \eta_6\}$, and define $Cl_1 := Cl(E_1)$.

Consider N_A/Cl_1. The generating set for N_A induces the generating set $\{\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1\}$ for N_A/Cl_1.

The group N_A/Cl_1 is generated by the 16 elements of the form $\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i}$ for $i, j \in \{0, 1, 2, 3\}$. To see this, consider a generator $\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1$ of N_A/Cl_1. Since $\delta^i\epsilon^j\eta_6^{-1}\epsilon^{-j}\delta^{-i} \in Cl_1$,

$$\begin{aligned}
\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 &= \delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot \delta^i\epsilon^j\eta_6^{-1}\epsilon^{-j}\delta^{-i} \cdot Cl_\Gamma(E) \\
&= \delta^i\epsilon^j\epsilon^{-4}\alpha\epsilon^4\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^i\epsilon^{j-4}\alpha\epsilon^{4-j}\delta^{-i} \cdot Cl_1.
\end{aligned}$$

Thus $\{\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \mid i \in \mathbb{Z}, 0 \leqslant j < 4\}$ form a generating set for N_A/Cl_1. To reduce this infinite generating set to the claimed 16 elements, we use η_{j+1} for $j = 0, 1, 2, 3$ as follows:

$$\begin{aligned}
\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 &= \delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot \delta^i\epsilon^j\alpha\eta_{j+1}\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^i\epsilon^j\eta_{j+1}\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^i\epsilon^j \cdot \epsilon^{-j}\delta^{-4}\epsilon^j\delta^4 \cdot \alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\delta^4\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\delta^4\alpha\epsilon^{-j}\delta^{-i} \cdot \delta^i\epsilon^j\eta_5\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\delta^4\alpha \cdot \alpha\delta^{-4}\alpha\delta^4 \cdot \epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\delta^4\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\delta^4\epsilon^{-j}\delta^{-i} \cdot \delta^i\epsilon^j\eta_{j+1}^{-1}\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\delta^4\eta_{j+1}^{-1}\epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\delta^4 \cdot \delta^{-4}\epsilon^{-j}\delta^4\epsilon^j \cdot \epsilon^{-j}\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\epsilon^{-j}\delta^4\delta^{-i} \cdot Cl_1 \\
&= \delta^{i-4}\epsilon^j\alpha\epsilon^{-j}\delta^{4-i} \cdot Cl_1
\end{aligned}$$

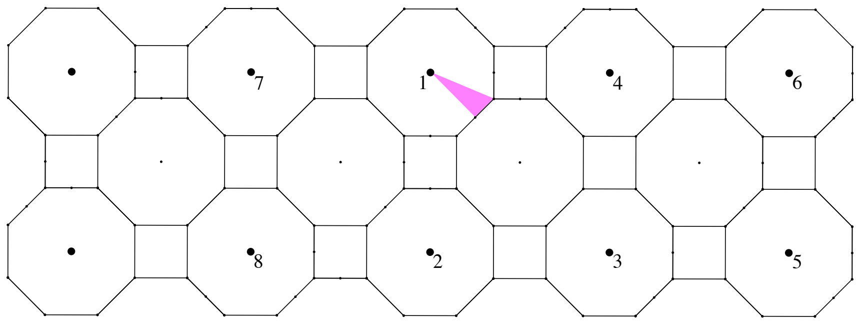
As such, by induction, a finite generating set for N_A/Cl_1 is $\{\delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \cdot Cl_1 \mid 0 \leqslant i, j < 4\}$.

5.3.2 Intersect with N_B

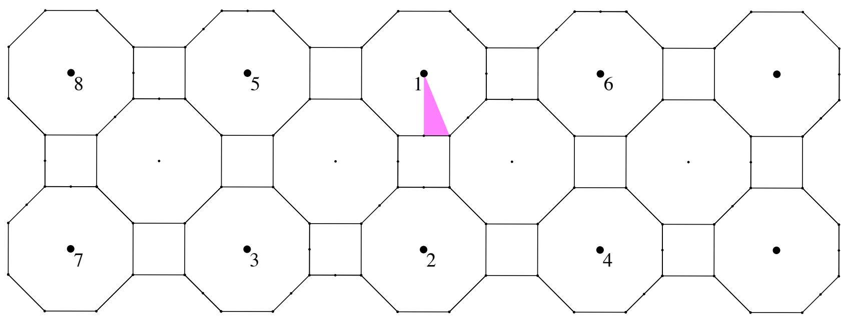
Let $w_{ij} := \delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i}$, $0 \leqslant i, j < 4$. We now investigate the action of the $\{w_{ij}\}$ on the remaining flags of (4.8.8). In particular, we wish to reduce the number of generators to the minimal set necessary, and to determine what (if any) further identifications are made by taking the quotient by $\mathcal{C}$.

We observe that the action of each of the w_{ij} on flag types B and C is a rotation by π radians about the center of an octagon.

In Table 3 we indicate which labeled center relative to the indicated flag in Figure 9 corresponds to the action of each of the generating w_{ij} on the indicated flag. For example, the $(1, 2)$ entry in Table 4B corresponds to the rotation around the center marked “5” in



(B) Half-turn centers for a flag whose edge is contained in two octagons (indicated by the shaded triangle).



(C) Half-turn centers for a flag on an octagon whose edge is contained in a square(indicated by the shaded triangle).

Figure 9: Rotation centers for the flags containing octagons in (4.8.8) as determined by the generators of N_A/Cl_1.

Table 3: In Table (B) we see the action table for flags of type B whose edges are contained in two octagons. In Table (C) we see the action table for flags of type C whose maximal proper face is an octagon, and whose edge is also contained in a square.

δ \ ϵ	0	1	2	3
0	1	2	3	4
1	4	3	5	6
2	3	4	1	2
3	2	1	7	8

(B) Type B

δ \ ϵ	0	1	2	3
0	1	2	4	6
1	2	1	5	3
2	3	5	8	7
3	5	3	2	1

(C) Type C

Figure 9B, and it corresponds to the action of the word $w_{12} = \delta\epsilon^2\alpha\epsilon^{-2}\delta^{-1}$ on the indicated flag of type B.

Since $\alpha^2 = id$, each of the w_{ij} is an involution. Because many of the w_{ij} share the same actions on flags of type B, we may define some relations in N_A to help us reduce the number of relations needed to describe the intersection of N_A with N_B. For example, we observe that the actions of w_{03} and w_{10} on flags of type B are the same (since the $(0, 3)$ and $(1, 0)$ entry in Table 4B both equal 4). The list of equivalent actions from generators on flags of type B in terms of w_{10}, w_{30}, w_{11} obtained in this form is

$$\begin{aligned}
&w_{30} = w_{01} = w_{23}\\
&w_{11} = w_{20} = w_{02}\\
&w_{10} = w_{21} = w_{03}\\
&w_{00} = w_{31} = w_{22} = w_{30}w_{11}w_{10}\\
&w_{12} = w_{11}w_{30}w_{11}\\
&w_{13} = w_{10}w_{30}w_{11}\\
&w_{32} = w_{30}w_{11}w_{30}w_{11}w_{10}\\
&w_{33} = w_{30}w_{11}w_{30}.
\end{aligned}$$

Since the rotation centers of the actions of w_{10}, w_{11} and w_{30} on flags of class B and C don't all lie on a line, the action of $(w_{10}w_{11}w_{30})^2$ on all flags is trivial. On the other hand, the product of any two of w_{10}, w_{11} and w_{30} acts like a translation on flags of type B. Then, by Lemma 9, $\langle w_{10}, w_{11}, w_{30}\rangle \cong G_{2222}$, where G_{2222} is the group 2222 given in [CBGS08]. It follows that

$$\begin{aligned}
&(N_A \cap N_B)/Cl_1 =\\
&\qquad Cl_{N_A/Cl_1}(R_1 := w_{00}w_{10}w_{11}w_{30}, R_2 := w_{00}w_{31}, R_3 := w_{00}w_{22},\\
&\qquad\qquad R_4 := w_{30}w_{01}, R_5 := w_{30}w_{23}, R_6 := w_{11}w_{20}, R_7 := w_{11}w_{02},\\
&\qquad\qquad R_8 := w_{10}w_{21}, R_9 := w_{10}w_{03}, R_{10} := w_{12}w_{11}w_{30}w_{11},\\
&\qquad\qquad R_{11} := w_{32}w_{10}w_{11}w_{30}w_{11}w_{30}, R_{12} := w_{13}w_{11}w_{30}w_{10},\\
&\qquad\qquad R_{13} := w_{33}w_{30}w_{11}w_{30}, R_{14} := (w_{10}w_{11}w_{30})^2),
\end{aligned}$$

since under this set of relations every generator w_{ij} of N_A/Cl_1 is equivalent to an expression in terms of w_{10}, w_{11} and w_{30}.

We now have an expression for $(N_A \cap N_B)/Cl_1$ as a normal closure of a finite list of elements in N_A. This description is somewhat unwieldy for determining the actions of $(N_A \cap N_B)/Cl_1$ on flags of type C, since it has, in principle, an infinite generating set. We can improve this situation dramatically by observing that there is a contravariant group isomorphism from $(N_A \cap N_B)/Cl_1$ to the group generated by the isometries whose actions coincide with those of $R_1, \ldots, R_{14}$ on a fixed flag of type C. Since each of the w_{ij} acts like a half-turn on flags of type C, and each of the R_k acts like a translation on flags of type C then

$$w_{ij}R_kw_{ij} = R_k^{-1}. \tag{8}$$

In other words, $w_{ij}R_kw_{ij}R_k$ is in $\mathcal{C}$ for each choice of $0 \leqslant i,j \leqslant 3,\ 1 \leqslant k \leqslant 14$. Let

$$E_1' = E_1 \cup \{w_{ij}R_kw_{ij}R_k \ \mid \ 0 \leqslant i,j \leqslant 3, 1 \leqslant k \leqslant 14\}$$

and $Cl_1' = Cl_\Gamma(E_1')$. Consider $(N_A \cap N_B)/Cl_1'$. While the set of conjugates in N_A of $\{R_1, ..., R_{14}\}$ still form a generating set, we need only consider the elements obtained as products of the R_i, since any product of conjugates of the R_i may be reduced using the relations in Cl_1' to a product of the R_i and their inverses. Thus $(N_A \cap N_B)/Cl_1' = \langle R_i : 1 \leqslant i \leqslant 14\rangle \leqslant \Gamma/Cl_1'$.

5.3.3 Intersect with N_C

We now consider the actions of each of the generators $R_i \cdot Cl_1'$ for $(N_A \cap N_B)/Cl_1'$ on flags of type C. First note that $(w_{10}w_{11}w_{30})^2$ is in $\mathcal{C}$, and observe that each of the remaining generators $R_i \cdot Cl_1'$ acts like a translation on any given flag of type C. Moreover, many of them have equivalent actions on flags of type C. For example, both $R_2 = w_{00}w_{31}$ and $R_6 = w_{11}w_{20}$ translate a flag of type C by four adjacent octagons in the same direction, and so $R_2(R_6)^{-1}$ must be in $\mathcal{C}$. Thus we observe that in terms of their actions on flags of type C

$$\begin{aligned}
&R_6 = R_2 = R_1 = R_5 = (R_9)^{-1}\\
&R_3 = R_{11} = R_{13} = (R_{10})^{-1} = (R_{12})^{-1}\\
&R_4 = R_7 = (R_8)^{-1}
\end{aligned}$$

Note that the three sets of generators listed above correspond to translations in three different directions, so we may determine by inspection that

$$(R_4)^{-1}(R_3)^{-1}R_6 = w_{01}w_{30}w_{22}w_{00}w_{11}w_{20}$$

is a product of the three types of generators that has a trivial action. Consequently, we let

$$\begin{aligned}
E_2 = E_1' \cup \{&R_6(R_2)^{-1}, R_6(R_1)^{-1}, R_6(R_5)^{-1}, R_6R_9, R_3(R_{11})^{-1}, R_3(R_{13})^{-1},\\
&R_3R_{10}, R_3R_{12}, R_4(R_7)^{-1}, R_4(R_8)^{-1}, (R_4)^{-1}(R_3)^{-1}R_6\}.
\end{aligned}$$

We now consider $Cl_2 := Cl_\Gamma(E_2)$. We observe that the inclusion of the elements in E_2 allows us to significantly reduce the number of generators for $(N_A \cap N_B)/Cl_2$. For example, note that

$$R_2 \cdot Cl_2 = w_{00}w_{31} \cdot Cl_2 = w_{00}w_{31} \cdot w_{31}w_{00}w_{20}w_{11} \cdot Cl_2 = w_{11}w_{20} \cdot Cl_2.$$

Similar computations allow us to conclude that $(N_A \cap N_B)/Cl_2 = \langle w_{11}w_{20} \cdot Cl_2, w_{00}w_{22} \cdot Cl_2\rangle$. Each of these acts non-trivially as a translation on flags of type C in linearly independent directions, and so we observe that $T_1 := w_{11}w_{20}w_{00}w_{22}w_{20}w_{11}w_{22}w_{00} \in \mathcal{C}$. Let $C_* = Cl_\Gamma(E_2 \cup \{T_1\})$. Since $\mathcal{C} = N_A \cap N_B \cap N_C$, and since $w_{11}w_{20} \cdot Cl_*, w_{00}w_{22} \cdot Cl_*$

generate $(N_A \cap N_B)/Cl_*$, and we have accounted for all possible actions of an element of $(N_A \cap N_B)/Cl_*$ on a flag of type C, any element of $N_A \setminus Cl_*$ must have non-trivial action on a flag of either type B or of type C. Therefore, $\mathcal{C} = Cl_\Gamma(E_2 \cup \{T_1\})$. The set of defining relations we have found for this cover has 242 relations (listed in Table 4), this makes any description of the local geometry of the minimal cover complicated to obtain and so is material for another paper.

Table 4: The 242 elements needed to form the kernel of the cover of (4.8.8) from $[8,3]$, with the necessary labels for compactness of presentation.

$$
\begin{array}{r|l}
\text{Labels} & \alpha := (r_0r_1)^4,\ \delta := r_2r_1r_0r_1r_2r_0r_1r_0\ ,\ \epsilon := r_2r_0r_1r_0r_1r_2r_0r_1 \\
 & w_{ij} := \delta^i\epsilon^j\alpha\epsilon^{-j}\delta^{-i} \\
 & R_1 := w_{00}w_{10}w_{11}w_{30},\ R_2 := w_{00}w_{31}, R_3 := w_{00}w_{22}, \\
 & R_4 := w_{30}w_{01}, R_5 := w_{30}w_{23},\ R_6 := w_{11}w_{20}, R_7 := w_{11}w_{02}, \\
 & R_8 := w_{10}w_{21}, R_9 := w_{10}w_{03},\ R_{10} := w_{12}w_{11}w_{30}w_{11}, \\
 & R_{11} := w_{32}w_{10}w_{11}w_{30}w_{11}w_{30},\ R_{12} := w_{13}w_{11}w_{30}w_{10}, \\
 & R_{13} := w_{33}w_{30}w_{11}w_{30},\ R_{14} := (w_{10}w_{11}w_{30})^2.
\end{array}
$$

$$
\begin{array}{r|l}
\text{Relations} & \epsilon^{-1}\delta^{-4}\epsilon\delta^4, \epsilon^{-2}\delta^{-4}\epsilon^2\delta^4, \epsilon^{-3}\delta^{-4}\epsilon^3\delta^4, \epsilon^{-4}\delta^{-4}\epsilon^4\delta^4, \\
 & \alpha\delta^{-4}\alpha\delta^4, \epsilon^{-4}\alpha\epsilon^4\alpha, \\
 & w_{ij}R_kw_{ij}R_k \ \text{ for }\ 0 \leqslant i,j \leqslant 3, 1 \leqslant k \leqslant 14, \\
 & R_6(R_2)^{-1},\ R_6(R_1)^{-1},\ R_6(R_5)^{-1},\ R_6R_9,\ R_3(R_{11})^{-1}, \\
R_3(R_{13})^{-1}, & R_3R_{10},\ R_3R_{12},\ R_4(R_7)^{-1},\ R_4(R_8)^{-1},\ (R_4)^{-1}(R_3)^{-1}R_6 \\
 & w_{11}w_{20}w_{00}w_{22}w_{20}w_{11}w_{22}w_{00}.
\end{array}
$$

5.4 The tiling (3.12.12)

The first two steps for the determination of the core of the stabilizer of a flag in the tessellation (3.12.12) are similar to those of (3.6.3.6) and of (4.8.8). As we shall see, the third step is more involved than in the previous two cases. Throughout let

$$[12,3] := \langle r_0, r_1, r_2 |\, r_0^2 = r_1^2 = r_2^2 = (r_0r_2)^2 = (r_0r_1)^{12} = (r_1r_2)^3 = id\rangle.$$

Let $\mathcal{T}$ be the uniform tiling (3.12.12), N_A the stabilizer of a type A flag Φ containing an edge shared by two 12-gons under the action of $[12,3]$, N_B the stabilizer of a type B flag in an 12-gon sharing an edge with a triangle, and N_C the stabilizer of a type C flag in a triangle under the same action. Then $\mathcal{C} = N_A \cap N_B \cap N_C$ is the core of N_A in $[12,3]$, and $\Gamma = [12,3]/\mathcal{C}$ is the automorphism group of the minimal cover of $\mathcal{T}$.

Let $\alpha_0 := r_1r_2(r_0r_1)^3r_2r_1$, $\alpha_1 := (r_1r_0)^2r_1r_2(r_0r_1)^3r_2r_1(r_0r_1)^2$, $\beta := (r_1r_2r_1r_0)^2(r_1r_0)^2$ and $\gamma := (r_1r_0)^2(r_1r_2r_1r_0)^2$. By Theorem 5 we know that the set

$$\{\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i} \,|\, i,j \in \mathbb{Z}, k \in \{0,1\}\}$$

is a generating set for N_A.

5.4.1 Reducing to a finite generating set for N_A

First note that β^2 and γ^2 fix flags of types B and C. Therefore the elements $\eta_1 := \beta^2\alpha_0\beta^{-2}\alpha_0^{-1}$, $\eta_2 := \beta^2\alpha_1\beta^{-2}\alpha_1^{-1}$, $\eta_3 := \gamma^2\alpha_0\gamma^{-2}\alpha_0^{-1}$ and $\eta_3' := \gamma^2\alpha_1\gamma^{-2}\alpha_1^{-1}$ belong to $\mathcal{C}$. Moreover, since $\alpha_1 = (r_1r_0)^2\alpha_0(r_1r_0)^{-2}$ and $\gamma = (r_1r_0)^2\beta(r_1r_0)^{-2}$, it follows that η_3' is the conjugate of η_1 by $(r_1r_0)^2$ and so we need not include it in a generating set for $\mathcal{C}$ as a normal closure.

It can be verified easily that $\eta_4 := \beta^2\gamma\beta^{-2}\gamma^{-1}$ is also an element of $\mathcal{C}$. Let $Cl_1 := Cl_\Gamma(\{\eta_1, \eta_2, \eta_3, \eta_4\})$.

Following an argument analogous to those presented in §5.2.1 we conclude that N_A/Cl_1 is generated by the eight elements

$$\{\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i} \cdot Cl_1\}$$

with $i, j, k \in \{0, 1\}$.

We note that β and γ preserve all flag orbits, whereas α_0 and α_1 interchange flags of type B with flags of type C. This implies that any element in N_A/Cl_1 that fixes all flags must be expressed as a word with an even number of generators of the form $\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i} \cdot Cl_1$.

Before determining the even subgroup of N_A it is convenient to reduce the number of generators. To do this we determine a normal subgroup $Cl_1^{(a)}$ of Γ containing Cl_1, under which some of the generators of Cl_1 can be expressed it terms of the others. To find elements in $Cl_1^{(a)}$ we use the action of the eight generators of Cl_1 on flags of types B and C looking for products acting trivially on all flags.

Let $\eta_5 := \beta\gamma\alpha_0\gamma^{-1}\beta^{-1}\alpha_1\beta\gamma\alpha_1\gamma^{-1}\beta^{-1}\alpha_1^2\beta\alpha_1\beta^{-1}$, $\eta_6 := \beta\gamma\alpha_1\gamma^{-1}\beta^{-1}\alpha_1^2\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_0$, $\eta_7 := \gamma\alpha_1\gamma^{-1}(\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1})^{-1}$, $\eta_8 := \beta\alpha_1\beta^{-1}\alpha_0\alpha_1\beta\alpha_0\beta^{-1}$, $Cl_1^{(a)} := Cl_\Gamma(\{\eta_1, \dots, \eta_8\})$.

It can be verified that η_5, η_6, η_7 and η_8 belong to $\mathcal{C}$. Then $N_A/Cl_1^{(a)}$ is generated by the elements

$$\beta^i\gamma^j\alpha_k\gamma^{-j}\beta^{-i} \cdot Cl_1^{(a)} \tag{9}$$

with $i, j, k \in \{0, 1\}$. However, $\beta\gamma\alpha_0\gamma^{-1}\beta^{-1} \cdot Cl_1^{(a)}$ can be expressed in terms of the remaining generators (using relation η_5) and similarly $\beta\gamma\alpha_1\gamma^{-1}\beta^{-1} \cdot Cl_1^{(a)}$, $\gamma\alpha_1\gamma^{-1} \cdot Cl_1^{(a)}$ and $\beta\alpha_1\beta^{-1} \cdot Cl_1^{(a)}$ can be expressed in terms of the remaining four elements $\alpha_0 \cdot Cl_1^{(a)}$, $\alpha_1 \cdot Cl_1^{(a)}$, $\beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}$ and $\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}$ (using relations η_6, η_7 and η_8). Hence $N_A/Cl_1^{(a)}$ is generated by the four elements $\alpha_0 \cdot Cl_1^{(a)}, \alpha_1 \cdot Cl_1^{(a)}, \beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}$ and $\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}$.

By definition, $[N_A/Cl_1^{(a)}]^+$ is generated by the products of pairs of generators of $N_A/Cl_1^{(a)}$. In addition, if x and y are generators of $N_A/Cl_1^{(a)}$, then yx can be expressed as $y^2 \cdot (xy)^{-1} \cdot x^2$, so, to generate $[N_A/Cl_1^{(a)}]^+$, we only need one of the two products xy and yx for each pair of distinct generators x and y of $[N_A/Cl_1^{(a)}]$. Define X to be the following list of 10 generators of $[N_A/Cl_1^{(a)}]^+$:

$$\begin{aligned} X \;:=\; & \{\alpha_0^2 \cdot Cl_1^{(a)}, \alpha_0\alpha_1 \cdot Cl_1^{(a)}, \alpha_0\beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}, \alpha_0\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}, \\ & \alpha_1^2 \cdot Cl_1^{(a)}, \alpha_1\beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}, \alpha_1\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}, \beta\alpha_0^2\beta^{-1} \cdot Cl_1^{(a)}, \\ & \beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}, \gamma\alpha_0^2\gamma^{-1} \cdot Cl_1^{(a)}\}. \end{aligned} \tag{10}$$

Reduce the number of generators of the even subgroup

Similar to the analysis for (4.8.8), we shall represent $[N_A/Cl_1^{(a)}]^+ \cap N_B/Cl_1^{(a)}$ as the normal closure in $[N_A/Cl_1^{(a)}]^+$ of some set of elements. The set of elements is generated (in principle) by an infinite set, which we intersect with $N_C/Cl_1^{(a)}$. To do this we shall need analogs to the equations in (8). To reduce the number of these equations it is convenient to reduce the number of generators of $[N_A/Cl_1^{(a)}]^+$ itself. We do this by constructing a larger normal subgroup $Cl_1^{(b)}$ of Γ so that in the quotient some of the generators of $[N_A/Cl_1^{(a)}]^+$ may be expressed as products of the others.

To do this, first note that the element $\alpha_1\beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}$ acts like a translation on any given flag Φ of type B whereas the remaining nine elements in 10 act on Φ like half-turns. On the other hand, the elements $\alpha_0\alpha_1 \cdot Cl_1^{(a)}$, $\alpha_1\beta\alpha_0\beta^{-1} \cdot Cl_1^{(a)}$ and $\alpha_1\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(a)}$ act like translations on any given flag Ψ of type C (the last two actually act like the same translation), whereas the remaining seven elements in (10) act on Ψ like half-turns.

We now use the contravariant group homomorphism from $[N_A/Cl_1^{(a)}]^+$ to the corresponding isometry group described in Proposition 1. The kernel of this homomorphism is $[N_A/Cl_1^{(a)}]^+ \cap N_B/Cl_1^{(a)}$. There exists another group homomorphism from $[N_A/Cl_1^{(a)}]^+$ to the permutation group determined by the actions of its elements on flags of type C, and the intersection of the kernels of these two morphisms is $\mathcal{C}/Cl_1^{(a)}$. These two homomorphisms can be used to determine that the following elements belong to $\mathcal{C}$:

- $\eta_9 := \beta\alpha_0^2\beta^{-1}\alpha_0\beta\alpha_0\beta^{-1}\alpha_1^2(\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1})^{-1}\alpha_0\beta\alpha_0\beta^{-1}(\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1})^{-1}$,
- $\eta_{10} := \beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1}\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2$,
- $\eta_{11} := \alpha_1\gamma\alpha_0\gamma^{-1}(\alpha_1\beta\alpha_0\beta^{-1})^{-1}\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1}$,
- $\eta_{12} := \gamma\alpha_0^2\gamma^{-1}\alpha_1^2\alpha_1\beta\alpha_0\beta^{-1}$,
- $\eta_{13} := (\alpha_1\beta\alpha_0\beta^{-1})^{-1}\alpha_1^2\alpha_0\beta\alpha_0\beta^{-1}\alpha_0\alpha_1 = \beta\alpha_0^{-1}\beta^{-1}\alpha_1\alpha_0\beta\alpha_0\beta^{-1}\alpha_0\alpha_1$.

Define $Cl_1^{(b)} := Cl_\Gamma(\{\eta_1, \ldots, \eta_{13}\})$. Note that $[N_A/Cl_1^{(b)}]^+$ is generated by the left cosets of $Cl_1^{(b)}$ determined by (a representative of each of) the elements in (10). However, the elements $\beta\alpha_0^2\beta^{-1}\cdot Cl_1^{(b)}$, $\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1}\cdot Cl_1^{(b)}$, $\alpha_1\gamma\alpha_0\gamma^{-1}\cdot Cl_1^{(b)}$, $\gamma\alpha_0^2\gamma^{-1}\cdot Cl_1^{(b)}$ and $\alpha_1\beta\alpha_0\beta^{-1}\cdot Cl_1^{(b)}$ can be expressed in terms of the remaining five generators $\alpha_0^2 \cdot Cl_1^{(b)}$, $\alpha_0\alpha_1 \cdot Cl_1^{(b)}$, $\alpha_0\beta\alpha_0\beta^{-1} \cdot Cl_1^{(b)}$, $\alpha_0\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(b)}$ and $\alpha_1^2 \cdot Cl_1^{(b)}$ (by applying relations η_9, η_{10}, η_{11}, η_{12} and η_{13} respectively). Therefore $[N_A/Cl_1^{(b)}]^+$ is generated by the elements

- $\alpha_0^2 \cdot Cl_1^{(b)}$,
- $\alpha_0\alpha_1 \cdot Cl_1^{(b)}$,
- $\alpha_0\beta\alpha_0\beta^{-1} \cdot Cl_1^{(b)}$,
- $\alpha_0\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(b)}$, and
- $\alpha_1^2 \cdot Cl_1^{(b)}$.

5.4.2 Determination of $N_A \cap N_B$

We now consider the elements α_0^2, α_1^2 and $\alpha_0\beta\alpha_0\beta^{-1}$. Note that each of them acts like a half-turn on each flag with types B or C. Furthermore, the centers of these half-turns do not lie in a line. Therefore the elements $\eta_{14} := (\alpha_0\beta\alpha_0\beta^{-1})^2$ and $\eta_{15} := (\alpha_0^2\alpha_1^2\alpha_0\beta\alpha_0\beta^{-1})^2$ belong to $\mathcal{C}$. Let $Cl_1^{(c)} := Cl_\Gamma(\{\eta_1, \dots, \eta_{15}\})$.

On the other hand, $\alpha_0\alpha_1$ and $\alpha_0\gamma\alpha_0\gamma^{-1}$ act respectively like the identity and α_1^2 on flags of type B. Hence, by Lemma 9, $[N_A/Cl_1^{(c)}]^+ \cap N_B/Cl_1^{(c)}$ is generated by all conjugates in $[N_A/Cl_1^{(c)}]^+$ of $R_1 := \eta_{14} \cdot Cl_1^{(c)}$, $R_2 := \eta_{15} \cdot Cl_1^{(c)}$, $R_3 := \alpha_0\alpha_1 \cdot Cl_1^{(c)}$ and $R_4 := \alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_1^{(c)}$. Therefore $[N_A/Cl_1^{(c)}]^+ \cap N_B/Cl_1^{(c)}$ is generated by all conjugates in $[N_A/Cl_1^{(c)}]^+$ of $\alpha_0\alpha_1 \cdot Cl_1^{(c)}$ and $\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_1^{(c)}$.

5.4.3 Intersect with N_C.

In order to determine the elements in $[N_A/Cl_1^{(c)}]^+ \cap N_B/Cl_1^{(c)}$ acting trivially on flags type C, we must first find a normal subgroup Cl_2 such that $[N_A/Cl_2]^+ \cap N_B/Cl_2$ is finitely generated.

Above we showed that $[N_A/Cl_1^{(c)}]^+$ is generated by $\alpha_0^2 \cdot Cl_1^{(c)}$, $\alpha_0\beta\alpha_0\beta^{-1} \cdot Cl_1^{(c)}$, $\alpha_1^2 \cdot Cl_1^{(c)}$, $\alpha_0\alpha_1 \cdot Cl_1^{(c)}$ and $\alpha_0\gamma\alpha_0\gamma^{-1} \cdot Cl_1^{(c)}$. On any flag of type C, the first three act like half-turns and the last two act like translations. In addition, $\alpha_0\alpha_1$ and $\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2$ also act like translations on any flag of type C. We use the contravariant group homomorphism from $(N_A/Cl_1^{(c)})^+ \cap N_B/Cl_1^{(c)}$ to the corresponding isometry group described in Proposition 1, to verify that the elements

- $\eta_{16} := (\alpha_0^2)(\alpha_0\alpha_1)(\alpha_0^2)(\alpha_0\alpha_1)$,
- $\eta_{17} := (\alpha_0^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$,
- $\eta_{18} := (\alpha_0\beta\alpha_0\beta^{-1})(\alpha_0\alpha_1)(\alpha_0\beta\alpha_0\beta^{-1})^{-1}(\alpha_0\alpha_1)$,
- $\eta_{19} := (\alpha_0\beta\alpha_0\beta^{-1})(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\beta\alpha_0\beta^{-1})^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$,
- $\eta_{20} := (\alpha_1^2)(\alpha_0\alpha_1)(\alpha_1^2)(\alpha_0\alpha_1)$,
- $\eta_{21} := (\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$,
- $\eta_{22} := (\alpha_0\alpha_1)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\alpha_1)^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)^{-1}$,
- $\eta_{23} := (\alpha_0\gamma\alpha_0\gamma^{-1})(\alpha_0\alpha_1)(\alpha_0\gamma\alpha_0\gamma^{-1})^{-1}(\alpha_0\alpha_1)$,
- $\eta_{24} := (\alpha_0\gamma\alpha_0\gamma^{-1})(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1})^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$

belong to $\mathcal{C}$. Let $Cl_2 := Cl_\Gamma(\{\eta_1, \dots, \eta_{24}\})$. Then $[N_A/Cl_2]^+ \cap N_B/Cl_2$ is the group whose elements are products of conjugates in $[N_A/Cl_2]^+$ of $\alpha_0\alpha_1 \cdot Cl_2$ and $\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2$. Note that relations $\eta_{16}, \dots, \eta_{24}$ imply that all such conjugates belong to the set

$$\{\alpha_0\alpha_1 \cdot Cl_2, (\alpha_0\alpha_1)^{-1} \ \cdot \ Cl_2, \alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2, (\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)^{-1} \cdot Cl_2.\}$$

In other words, $[N_A/Cl_2]^+ \cap N_B/Cl_2$ is generated (as a group) by $\alpha_0\alpha_1 \cdot Cl_2$ and $\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2$. For example, relation η_{21} implies that

$$\alpha_1^2(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)\alpha_1^2 \cdot Cl_2 = (\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)^{-1} \cdot Cl_2.$$

Hence, $[N_A/Cl_2]^+ \cap N_B/Cl_2$ is generated by $\alpha_0\alpha_1 \cdot Cl_2$ and $\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2$. On the other hand, $[N_A/Cl_2]^+ \cap N_B/Cl_2 = (N_A^+ \cap N_B)/Cl_2$ because of the fourth isomorphism theorem and $N_A^+ \cap N_B = N_A \cap N_B$ since the elements in $N_A \setminus N_A^+$ map flags of type B onto flags of type C and hence $(N_A \setminus N_A^+) \cap N_B = \emptyset$. Therefore $(N_A \cap N_B)/Cl_2 = [N_A/Cl_2]^+ \cap N_B/Cl_2$. It only remains to determine the elements in $(N_A \cap N_B)/Cl_2$ which fix flags of type C.

Note that $\alpha_0\alpha_1 \cdot Cl_2$ and $\alpha_1\beta\alpha_0\beta^{-1}\alpha_1^2\alpha_0\beta\alpha_0\beta^{-1} \cdot Cl_2$ act on any flag type C like translations with respect to linearly independent vectors. Then a set of defining relations for

$$\langle \alpha_0\alpha_1 \cdot Cl_2, \alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2 \rangle$$

is determined just by their commutativity. It follows that

$$\eta_{25} := \alpha_0\alpha_1(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\alpha_1)^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)^{-1}$$

belongs to $\mathcal{C}$. (The element η_{25} corresponds to the element T_1 described in the outline.) Note, however that $\eta_{25} = \eta_{22}$. Then the set of elements in $N_A \cap N_B/Cl_2$ fixing all flags of type C is the normal closure in $(N_A/ \cap N_B)/Cl_2$ of $\eta_{25} \cdot Cl_2$, that is, it is trivial.

Since the core of N_A in Γ is $N_A \cap N_B \cap N_C$, and since the elements $\alpha_0\alpha_1 \cdot Cl_2, \alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2 \cdot Cl_2$ generate $(N_A \cap N_B)/Cl_2$, we were able to account for all of the actions on flags of type C, and so any element in $N_A \setminus Cl_2$ must have a nontrivial action on a flag of type B or on a flag of type C. Hence $\mathcal{C} = Cl_2$.

The set of relations $\{\eta_i = id \,|\, i = 1, \ldots, 24\}$ is not minimal (collected in Table 5). For example, $\eta_{16} = id$ can be rewritten as $(\alpha_0^{-1}\alpha_1)^2 = id$, which is the conjugate by α_1^{-1} of relation $\eta_{20} = id$. However, determining a minimal set of elements η_i required to generate $\mathcal{C}$ is beyond the scope of this paper. As in the case of (4.8.8), with this many relations it is difficult to provide a coherent picture of the local or global geometry of the cover.

6 Epilogue

6.1 Minimal Covers for the Remaining Archimedean Tilings

In the current work we have been able to provide descriptions of minimal regular covers of the Archimedean tilings (3.6.3.6), (4.8.8) and (3.12.12). Each of the remaining tilings is covered by a regular hyperbolic tiling of Schläfli type $\{p, q\}$, where p is the least common multiple of the number of sides of a polygon in the tiling, and q is the degree of any vertex in the tiling (since these tilings are uniform). In fact, these hyperbolic tilings are universal covers for any polyhedron where the least common multiples of the number of sides of a polygon and of the degrees of the vertices in the polyhedron are p and q respectively. A natural question to ask in the context of the current work is whether or not there are

Table 5: The list of elements set equal to id to form the minimal regular cover of (3.12.12) from $[12, 3]$ with requisite labels for compactness of presentation.

Labels	$\alpha_0 := r_1r_2(r_0r_1)^3r_2r_1$, $\alpha_1 := (r_1r_0)^2r_1r_2(r_0r_1)^3r_2r_1(r_0r_1)^2$, $\beta := (r_1r_2r_1r_0)^2(r_1r_0)^2$, $\gamma := (r_1r_0)^2(r_1r_2r_1r_0)^2$.
Relations	$\beta^2\alpha_0\beta^{-2}\alpha_0^{-1}$, $\beta^2\alpha_1\beta^{-2}\alpha_1^{-1}$, $\gamma^2\alpha_0\gamma^{-2}\alpha_0^{-1}$, $\beta^2\gamma\beta^{-2}\gamma^{-1}$,$\beta\gamma\alpha_0\gamma^{-1}\beta^{-1}\alpha_1\beta\gamma\alpha_1\gamma^{-1}\beta^{-1}\alpha_1^2\beta\alpha_1\beta^{-1}$, $\beta\gamma\alpha_1\gamma^{-1}\beta^{-1}\alpha_1^2\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_0$, $\gamma\alpha_1\gamma^{-1}(\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1})^{-1}$, $\beta\alpha_1\beta^{-1}\alpha_0\alpha_1\beta\alpha_0\beta^{-1}$, $\beta\alpha_0^2\beta^{-1}\alpha_0\beta\alpha_0\beta^{-1}\alpha_1^2(\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1})^{-1}\alpha_0\beta\alpha_0\beta^{-1}(\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1})^{-1}$, $\beta\alpha_0\beta^{-1}\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1}\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2$, $\alpha_1\gamma\alpha_0\gamma^{-1}(\alpha_1\beta\alpha_0\beta^{-1})^{-1}\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_0\beta\alpha_0\beta^{-1}$,$\gamma\alpha_0^2\gamma^{-1}\alpha_1^2\alpha_1\beta\alpha_0\beta^{-1}$, $(\alpha_1\beta\alpha_0\beta^{-1})^{-1}\alpha_1^2\alpha_0\beta\alpha_0\beta^{-1}\alpha_0\alpha_1$, $(\alpha_0\beta\alpha_0\beta^{-1})^2$, $(\alpha_0^2\alpha_1^2\alpha_0\beta\alpha_0\beta^{-1})^2$, $(\alpha_0^2)(\alpha_0\alpha_1)(\alpha_0^2)(\alpha_0\alpha_1)$, $(\alpha_0^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$, $(\alpha_0\beta\alpha_0\beta^{-1})(\alpha_0\alpha_1)(\alpha_0\beta\alpha_0\beta^{-1})^{-1}(\alpha_0\alpha_1)$, $(\alpha_0\beta\alpha_0\beta^{-1})(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\beta\alpha_0\beta^{-1})^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$, $(\alpha_1^2)(\alpha_0\alpha_1)(\alpha_1^2)(\alpha_0\alpha_1)$, $(\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$, $(\alpha_0\alpha_1)(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\alpha_1)^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)^{-1}$, $(\alpha_0\gamma\alpha_0\gamma^{-1})(\alpha_0\alpha_1)(\alpha_0\gamma\alpha_0\gamma^{-1})^{-1}(\alpha_0\alpha_1)$, $(\alpha_0\gamma\alpha_0\gamma^{-1})(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)(\alpha_0\gamma\alpha_0\gamma^{-1})^{-1}(\alpha_0\gamma\alpha_0\gamma^{-1}\alpha_1^2)$.

smaller regular covers than these universal covers for the remaining Archimedean tilings. As mentioned in Section 2, given a polyhedron $\mathcal{P}$, universal regular cover $\{p, q\}$ and a base flag Φ of $\mathcal{P}$, by Theorems 5.2 and 5.3 of [Har99a] $\mathcal{P} \cong \{p, q\}/N$ where N is the stabilizer in $[p, q]$ of Φ. Moreover, as seen in Theorem 2, in the case of polyhedra, the minimal regular cover of $\mathcal{P}$ is determined by $[p, q]/Core([p, q], N)$. To show that $[p, q]$ is not the minimal cover for the tilings $(3.3.3.3.6), (3.3.3.4.4), (3.3.4.3.4), (3.4.6.4)$ and $(4.6.12)$ it suffices to show that $Core([p, q], N)$ is non-trivial in each case. We do this by observing that each of the elements given in Table 6 are nontrivial in $[p, q]$, and fix each flag of the corresponding tiling, and so are in $Core([p, q], N)$, so $Core([p, q], N)$ is nontrivial in all five cases. We intend to address the structure of minimal covers of the remaining Archimedean tilings in a subsequent paper.

6.2 Open Questions and Concluding Remarks

The current work suggests a number of questions that do not, as yet, appear to have been fully addressed in the literature.

While it is known that the monodromy groups of polyhedra are string C-groups [MPW], there exists a polytope $\mathcal{T}$ (the "Tomotope") of higher rank where $\mathrm{Mon}(\mathcal{T})$ is

Table 6: Nontrivial elements of $Core([p,q],N)$ for each of the remaining Archimedean tilings.

Tiling	Universal Cover	Nontrivial core element
(3.3.3.3.6)	$\{6,5\}$	$[r_1r_2(r_1r_0)^2]^4$
(3.3.3.4.4)	$\{12,5\}$	$(r_0r_1r_2r_1)^8[r_1r_2(r_1r_0)^5]^8$
(3.3.4.3.4)	$\{12,5\}$	$[r_1r_2(r_1r_0)^5]^4$
(3.4.6.4)	$\{12,4\}$	$(r_1r_0r_1r_2)^{12}$
(4.6.12)	$\{12,3\}$	$[(r_1r_0)^5r_1r_2(r_1r_0)^3r_1r_2]^6$

not a string C-group [MPW12]. Is it is possible to determine a broadly interesting class of polytopes whose monodromy groups are all string C-groups? In particular, are the monodromy groups of all chiral polytopes string C-groups?

A related question is the range of possible group structures of monodromy groups, especially in the case of polyhedra. The monodromy group of a polytope is isomorphic to a subgroup of S_n where n is the number of flags, what then is the smallest index of this subgroup in S_n for polytopes of rank d with n flags? In general, what groups can be the monodromy groups of abstract polytopes? Note that this includes all of the string C-groups since the monodromy group of a regular polytope is isomorphic to its automorphism group.

There remains much to be understood about the structure of quotients of regular abstract polytopes, and about the structure of less symmetric polytopes more generally. The concepts and results discussed in this paper provide some of the necessary framework for approaching these questions in the context of non-finite polyhedra.

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