Locally Restricted Compositions IV. Nearly Free Large Parts and Gap-Freeness

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Abstract

We define the notion of asymptotically free for locally restricted compositions, which means roughly that large parts can often be replaced by any larger parts. Two well-known examples are Carlitz and alternating compositions. We show that large parts have asymptotically geometric distributions. This leads to asymptotically independent Poisson variables for numbers of various large parts. Based on this we obtain asymptotic formulas for the probability of being gap free and for the expected values of the largest part, number of distinct parts and number of parts of multiplicity k, all accurate to $o(1)$.

Dedicated to the memory of Herb Wilf.

1 Introduction

Various authors have considered aspects of unrestricted compositions and Carlitz compositions (unequal adjacent parts) that require knowledge about the large parts. The results

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include information about largest part, number of distinct parts, gap-freeness and number of parts of multiplicity k. We extend these results to a broad class of compositions, drawing on earlier work on *locally restricted compositions* [3] by defining a subclass of locally restricted compositions for which we can show that the large parts are asymptotically independent geometric random variables. This leads to asymptotically independent Poisson random variables for numbers of various large parts. Our main goal is to prove Theorem 1. Although a full understanding of the theorem requires some definitions, it can be read now. Among the compositions included in our definition are unrestricted, Carlitz and alternating up-down.

Although it was not possible to compute generating functions in [3], various properties were established, including the following.

(a) The number of compositions of n is $Ar^{-n}(1+O(\delta^n))$ for some $0<\delta<1$ because of a simple pole in the generating function. Since the convergence to Ar^{-n} is exponentially fast, the values of r and A can be estimated fairly easily if one can count compositions for relatively small values of n. [3, Theorem 3]

(b) If a subcomposition can occur arbitrarily often, the number of times it occurs in a random composition of n has a distribution that is asymptotically normal with mean and variance asymptotically proportional to n. The same is true for the total number of parts in a random composition. [3, Theorem 4]

(c) In many cases, the largest part and number of distinct parts in a random composition is asymptotic to $\log_{1/r} n$. [3, Section 9]

Various special cases were considered [2, 4], where more could be said about the generating functions. In none of these papers was the behavior of the large parts addressed beyond that in (c).

Definition 1 (Composition terminology). $\mathbb{N}$ *and* $\mathbb{N}_0$ *denote the positive integers and the non-negative integers, respectively.*

A composition is written $\vec{\mathbf{c}} = c_1 \cdots c_k$ *where* $c_i \in \mathbb{N}$. *We use the same notation to denote concatenation of compositions as in* $\vec{\mathbf{a}}_1 \cdots \vec{\mathbf{a}}_m$. *The length of* $\vec{\mathbf{c}} = c_1 \cdots c_k$ *is denoted by* $\mathrm{len}(\vec{\mathbf{c}}) = k$ *and the sum of the parts by* $\Sigma(\vec{\mathbf{c}})$.

A subcomposition *of* $\vec{\mathbf{c}}$ *is a sequence of one or more consecutive parts of* $\vec{\mathbf{c}}$. *The ordered* k*-tuple* $(L_1, \ldots, L_k)$ *is a* subsequence of $\vec{\mathbf{c}}$ *if for some increasing sequence of indices* $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant \mathrm{len}(\vec{\mathbf{c}})$ *we have* $c_{j_i} = L_i, 1 \leqslant i \leqslant k$. *A subsequence of a composition is* marked *if the elements of the subsequence are distinguished in some manner. For example, in the composition* $abacb$ *there is no marked subsequence whereas* $\dot{a}ba\dot{c}b$ *and* $ab\dot{a}\dot{c}b$ *each contain the marked subsequence* (a, c).

Definition 2 (Local restriction function). *Let* $m, p \in \mathbb{N}$. *A* local restriction function of type (m, p) *is a function*

$$\phi : \{0, 1, \ldots, m-1\} \times (\mathbb{N}_0)^{p+1} \to \{0, 1\}$$

with $\phi(i; 0, \ldots, 0) = 1$ *for all* i. *The integers* m *and* p *are called, respectively, the* modulus *and* span *of* ϕ.

Definition 3 (Class of compositions determined by local restrictions)**.** *Let ϕ be a local restriction function. The* class of compositions determined by ϕ *is*

$$\mathcal{C}_\phi = \{\vec{\mathbf{c}} : \vec{\mathbf{c}} \text{ is a composition, and } \phi(i \bmod m; c_i, c_{i-1}, \ldots, c_{i-p}) = 1 \text{ for } i \in \mathbb{Z}\}.$$

If an index j refers to a part before the first part ($j < 1$) or after the last part (j greater than the number of parts), we set $c_j = 0$.

A class $\mathcal{C}$ of compositions is locally restricted *if $\mathcal{C} = \mathcal{C}_\phi$ for some local restriction function ϕ.*

If Φ is a set of local restriction functions, we define $\mathcal{C}_\Phi = \cup_{\phi\in\Phi}\mathcal{C}_\phi$.

The number m determines a periodicity. The number p determines a "window"—by looking at parts c_j with $0 < |i-j| \leqslant p$, we can determine what values, if any, are allowed for c_i. We could replace m by any multiple of itself, p by any larger value, and redefine ϕ to get the same class of compositions.

Example 1 (Alternating compositions)**.** Up-down compositions $c_1 \leqslant c_2 \geqslant c_3 \leqslant \cdots$ can be described as follows. Set $m = 2$, $p = 1$,

$$\phi(1; a, 0) = 1, \quad \phi(1; a, b) = 1, \quad \phi(0; 0, a) = 1 \quad \text{and} \quad \phi(0; b, a) = 1, \tag{1}$$

whenever $0 < a \leqslant b$. Otherwise set $\phi = 0$, except that $\phi(i; 0, 0) = 1$ as required by Definition 2.

The function ϕ describes alternating compositions that start by going up (because $\phi(1; a, b) = 1$) and have an odd number of parts (because $\phi(0; 0, a) = 1$ permits the first zero after the composition to be in an even position; but $\phi(1; 0, a) = 0$ forbids it to be in an odd position). We could have included an even number of parts as well by defining $\phi(1; 0, a) = 1$.

We cannot extend the definition of ϕ to include compositions that begin by going down. These can be defined by switching 0 and 1 in the first argument of ϕ to give a new function ϕ'. With the extension to ϕ (and hence ϕ') noted in the previous paragraph, $\mathcal{C}_{\{\phi,\phi'\}}$ consists of all alternating compositions.

Suppose we require that the inequalities be strict. This can be done by simply changing $0 < a \leqslant b$ to $0 < a < b$ in (1). Now, however, we can include all strict alternating compositions in one ϕ instead of using $\mathcal{C}_{\{\phi,\phi'\}}$. Set $m = 1$, $p = 2$,

$$\phi(0; 0, 0, a) = \phi(0; a, 0, 0) = 1, \quad \phi(0; 0, a, b) = \phi(0; a, b, 0) = 1 \quad \text{and} \quad \phi(0; a, b, c) = 1,$$

when $a, b, c \in \mathbb{N}$ and either $a < b > c$ or $a > b < c$. It may appear at first that $m = 1$ causes periodicity to be lost; however, by looking at the two previous parts we can determine which of $c_{i-1} > c_{i-2}$ and $c_{i-1} < c_{i-2}$ holds. This will not work with weakly alternating compositions since they can have arbitrarily long strings of equal parts.

Definition 4 (Recurrent compositions)**.** *Let $\mathcal{C}$ be a class of locally restricted compositions with span p and modulus m.*

We say that a subcomposition $\vec{\mathbf{s}}$ *is* recurrent at j modulo m *if, for every* k *and every* $\vec{\mathbf{a}}\,\vec{\mathbf{x}}\,\vec{\mathbf{z}} \in \mathcal{C}$ *with* $\operatorname{len}(\vec{\mathbf{a}}) \geqslant p$ *and* $\operatorname{len}(\vec{\mathbf{z}}) \geqslant p$*, there is a composition* $\vec{\mathbf{a}} \cdots \vec{\mathbf{z}} \in \mathcal{C}$ *containing at least* k *copies of* $\vec{\mathbf{s}}$ *starting at positions congruent to* j *modulo* m*. Furthermore we require that for at least one such composition* $\vec{\mathbf{a}} \cdots \vec{\mathbf{z}} \in \mathcal{C}$*, if* S *is the set of indices where* $\vec{\mathbf{s}}$ *starts, then* $\gcd(i-k \mid i,k \in S) = m$*.*

- *If* $\vec{\mathbf{s}}$ *is recurrent for some* j*, we say* $\vec{\mathbf{s}}$ *is* recurrent*.*
- *If a recurrent subcomposition has length 1, we call it a* recurrent part*.*
- *A class* $\mathcal{C}_\phi$ *(and* ϕ*) is* recurrent *if every subcomposition* $c_i \cdots c_j$ *of* $\vec{\mathbf{c}} \in \mathcal{C}$ *with* $i > p$ *and* $j + p \leqslant \operatorname{len}(\vec{\mathbf{c}})$ *is recurrent.*
- *A class* $\mathcal{C}_\Phi$ *(and* Φ*) is* recurrent *if* ϕ *is recurrent for every* $\phi \in \Phi$*.*

It is a consequence of these definitions that if $\vec{\mathbf{r}}$ and $\vec{\mathbf{s}}$ are recurrent subcompositions, $\operatorname{len}(\vec{\mathbf{a}}) \geqslant p$, $\operatorname{len}(\vec{\mathbf{z}}) \geqslant p$, and $\vec{\mathbf{a}}\,\vec{\mathbf{x}}\,\vec{\mathbf{z}} \in \mathcal{C}$, then there is a composition $\vec{\mathbf{a}} \cdots \vec{\mathbf{r}} \cdots \vec{\mathbf{s}} \cdots \vec{\mathbf{z}}$ in $\mathcal{C}_\phi$. (We get $\vec{\mathbf{a}} \cdots \vec{\mathbf{r}}\vec{\mathbf{y}}\vec{\mathbf{z}}$ for some $\vec{\mathbf{y}}$. Replace $\vec{\mathbf{a}}$ with $\vec{\mathbf{a}} \cdots \vec{\mathbf{r}}$ and $\vec{\mathbf{x}}$ with $\vec{\mathbf{y}}$ in the definition.)

For the first ϕ in Example 1, the 2-part subcomposition ab is recurrent at 1 modulo 2 whenever $0 < a < b$ and is recurrent at 0 modulo 2 whenever $0 < b < a$. The part 1 is recurrent at 1 modulo 2 but is not recurrent at 0 modulo 2. The formulation at the end of Example 1 with $m = 1$ does not satisfy the gcd condition in Definition 4, precisely because "up" and "down" alternate. (The gcd condition in the definition is needed to insure that the transfer matrix in [3] has a required positivity property, which the matrix arising from $m = 1$ lacks.)

Remark (Ignoring nonrecurrent parts)**.** Since nonrecurrent parts can only appear in the first or last p parts, and since almost all compositions of n have $\Theta(n)$ parts, we can usually ignore the nonrecurrent parts in our asymptotic estimates.

Definition 5 (Similar restrictions)**.** *Suppose* ϕ *and* ϕ' *are local restriction functions with the same modulus and span. Suppose* $\mathcal{C}_\phi$ *and* $\mathcal{C}_{\phi'}$ *are recurrent and there is a* k *such that* $\vec{\mathbf{s}}$ *is recurrent at* j *mod* m *in* $\mathcal{C}_\phi$ *if and only if it is recurrent at* $(j+k)$ *mod* m *in* $\mathcal{C}'_\phi$*. We then say that* $\mathcal{C}_\phi$ *and* $\mathcal{C}_{\phi'}$ *are* similar *and write* $\mathcal{C}_\phi \approx \mathcal{C}_{\phi'}$ *as well as* $\phi \approx \phi'$*.*

Clearly $\approx$ is an equivalence relation.

Example 2 (Alternating compositions again)**.** In Example 1, $\phi \approx \phi'$ and so it turns out that Theorem 1 will apply to all alternating compositions. This remains true if we make either one or both of the inequalities $x < y$ and $x > y$ weak. However, weak and strong inequalities give restrictions which are not similar. For example, if the restrictions in ϕ' were changed to weak giving ϕ'', we would not have $\phi \approx \phi''$ and so we could not apply Theorem 1 to $\{\phi, \phi''\}$.

Remark (Some asymptotics)**.** We refer to (a) and (b) near the start of this section. Since the radius of convergence r in (a) depends only on the recurrent subcompositions, it will follow that the form $A(r^{-n}(1 + O(\delta^n))$ still holds for $\mathcal{C}_\Phi$ when Φ is a *finite* set of similar restrictions. For essentially the same reason, the normality in (b) continues to hold. (See Section 4 for details.)

Definition 6 (Asymptotically free)**.** *Let $\mathcal{C}_\phi$ be a set of locally restricted compositions of span p. If $\mathcal{C}_\phi$ is recurrent and the following hold, we say that ϕ and the compositions in $\mathcal{C}_\phi$ are* asymptotically free.

(a) Suppose j and r_i are such that $\vec{\mathbf{r}}(x) = r_1 \cdots r_p x r_{p+2} \cdots r_{2p+1}$ is recurrent at j modulo m for infinitely many values of x. Then there is an M (depending on j and the r_i) such that, if $\vec{\mathbf{r}}(x)$ occurs at a position j mod m in a composition, we may replace that x by any $x' \geqslant M$.

(b) There is at least one set of values j and r_i of the sort described in (a).

Let Φ be a finite set of similar *local restriction functions. If $\phi \approx \phi'$ and ϕ is asymptotically free, then clearly ϕ' is asymptotically free. Hence we say that Φ and the compositions in $\mathcal{C}_\Phi$ are* asymptotically free *if $\mathcal{C}_\phi$ is asymptotically free for some $\phi \in \Phi$.*

It is fairly easy to verify that asymptotically free $\mathcal{C}_\phi$ are special cases of the regular $\mathcal{C}_\phi$ studied in [3]. Note that, since ϕ has span p, no parts other than the r_i impose restrictions on x. We arrived at the notion of asymptotically free as a concept succinctly stated, fairly intuitive, and inclusive of a number of known examples, for which the results of Theorem 1 hold. It would be of interest to extend these results to more classes of compositions.

Example 3 (A bad definition)**.** We could have attempted to define asymptotically free Φ by simply insisting that (a) and (b) hold for $\mathcal{C}_\Phi$, however this is insufficient. Consider $\Phi = \{\phi, \phi'\}$ and ϕ (resp. ϕ') requires that parts in odd (resp. even) positions be odd. Then large odd parts will tend to be more common than large even parts and so the conclusion in Theorem 1(a) would be false.

Example 4 (Generalized Carlitz compositions)**.** Carlitz compositions are defined by the restriction $c_i \neq c_{i-1}$. They were generalized to restricted differences in [2] by requiring that $c_i - c_{i-1} \notin \mathcal{N}$ where $\mathcal{N}$ is a fixed set of integers. (Carlitz compositions correspond to $\mathcal{N} = \{0\}$.) These compositions are recurrent with modulus 1 and span 1. If $\mathcal{N}$ is finite, we have asymptotically free compositions. For the generalized Carlitz compositions studied in [2], $\mathcal{N}$ was the same for all c_{i-1}. We can generalize further by letting $\mathcal{N}$ depend on the value of c_{i-1}, say $\mathcal{N}(c_{i-1})$. If all the $\mathcal{N}(c)$ are finite, we still have asymptotically free compositions; however, they cannot be studied by the method in [2]. Instead, [3] must be used.

Example 5 (Some periodic conditions)**.** Up-down compositions have constraints of modulus 2. General periodic inequality constraints were studied in [4]. These are all asymptotically free provided they allow parts to both increase and decrease. As in the preceding example, we could require that the change between adjacent parts be dependent on the parts. For example, we could require that the ratio of adjacent parts be at least 2 ($c_i/c_{i-1} \geqslant 2$ for an increase and $c_{i-1}/c_i \geqslant 2$ for a decrease).

For fixed k, k-rowed compositions $a_{i,j}$ in which differences of adjacent parts avoid a finite set are asymptotically free. One interleaves the parts to produce a one-rowed composition: If $a_{i,j}$ are the parts of a k-rowed composition of n, then $c_{i+k(j-1)} = a_{i,j}$ for

$1 \leqslant i \leqslant k$ and $j = 1, 2, \cdots$ gives a bijection with one-rowed compositions $\vec{\mathbf{c}}$ of n. We can take the modulus and span to be k.

Definition 7 (Gap free). *A composition with largest part M is called* gap free *if it contains all recurrent parts less than M.*

The restriction of gap-free to recurrent parts is used to rule out classes such as the following. Let $\mathcal{C}$ be all compositions subject to the restriction that 2 and 3 can appear only as the first part of a composition. Since almost all compositions contain 1 and no composition in $\mathcal{C}$ can contain both 2 and 3, almost no compositions in $\mathcal{C}$ would be gap-free if we required that the support of the parts be an interval in $\mathbb{N}$.

Conventions. We use the following conventions in this paper.

- When we talk about something random, we always mean that it is chosen uniformly at random from the set in question. We say that a property holds *asymptotically almost surely* (a.a.s) if the probability that the property holds tends to 1 as the size of the set goes to infinity, and we also say that *almost all* objects in the set have the property.
- Expectation is denoted by E.
- After a class of compositions has been defined, we usually omit the modifiers (e.g. asymptotically free) and refer to elements of the class simply as compositions.
- The number of compositions of n in the class $\mathcal{C}$ is asymptotically Ar^{-n}. We will always use A and r for these parameters.
- All logarithms are to the base $1/r$ except the natural logarithm ln.

Remember that we call $\mathcal{C}_\Phi$ asymptotically free if and only if Φ is a finite set of similar asymptotically free local restriction functions.

Theorem 1 (Main theorem). *Let $\gamma \doteq 0.577216$ be Euler's constant and let*

$$P_k(x) \;=\; \log e \sum_{\ell \neq 0} \Gamma(k + 2i\pi\ell \log e) \exp(-2i\ell\pi \log x). \tag{2}$$

(This is a periodic function of $\log x$. For $1/2 < r < 1$ and $k = 0$ the amplitude is less than 10^{-6}.)

Let Φ be asymptotically free and let r be the radius of convergence of the generating function for $\mathcal{C}_\Phi$. The following are true for some $C > 0$, which has the same value in all parts of the theorem.

(a) Select a composition of n uniformly at random. Let $X_0(n)$ be the number of parts and $X_k(n)$ the number of parts of size k. For recurrent k and $\epsilon > 0$,

$$\operatorname{Prob}\left(\left|\frac{X_k(n)}{X_0(n)} - \frac{\mathsf{E}(X_k(n))}{\mathsf{E}(X_0(n))}\right| > \epsilon\right) \to 0 \quad \textit{as } n \to \infty. \tag{3}$$

Furthermore, the limit

$$u_k = \lim_{n\to\infty} \frac{\mathsf{E}(X_k(n))}{\mathsf{E}(X_0(n))} \tag{4}$$

exists, and $u_k \sim Br^k$ as $k \to \infty$ for some positive constant B.

(b) *Let the random variable M_n be the size of the maximum part in a random composition of n. For any function $\omega_b(n)$ such that $\omega_b(n) \to \infty$ as $n \to \infty$, $|M_n - \log n| < \omega_b(n)$ a.a.s. Furthermore*

$$\mathsf{E}(M_n) \;=\; \log\left(\frac{Cn}{1-r}\right) + \gamma \log e - \frac{1}{2} + P_0\left(\frac{Cn}{1-r}\right) + o(1),$$

where $C = B \lim_{n\to\infty} \mathsf{E}(X_0(n))/n$.

(c) *Let ν be the number of nonrecurrent parts. (Since the compositions are asymptotically free, ν is finite.) Let the random variable D_n be the number of distinct* <u>*recurrent*</u> *parts in a random composition of n. For any function $\omega_c(n)$ such that $\omega_c(n) \to \infty$ as $n \to \infty$, $|D_n - \log n| < \omega_c(n)$ a.a.s. Furthermore*

$$\mathsf{E}(D_n) + \nu \;=\; \log(Cn) + \gamma \log e - \frac{1}{2} + P_0(Cn) + o(1).$$

(d) *Let $q_n(k)$ be the fraction of compositions of n which are gap-free and have largest part k. There is a function $\omega_d(n) \to \infty$ as $n \to \infty$ such that*

$$q_n(k) \;\sim\; \exp\left(\frac{-Cnr^{k+1}}{1-r}\right) \prod_{j\leqslant k} \left(1 - \exp\left(-Cnr^j\right)\right) \tag{5}$$

uniformly for $|k - \log n| < \omega_d(n)$. Furthermore, for any constant D, the minimum of $q_n(k)$ over $|k - \log n| < D$ is bounded away from zero.

(e) *Let q_n be the fraction of compositions of n which are gap-free. Then q_n is asymptotic to the sum of the right side of (5), where the sum may be restricted to $|k - \log n| < \omega_d(n)$ for any $\omega_d(n) \to \infty$ as $n \to \infty$. Furthermore, $q_n \sim p_m$ where $m = \left\lfloor \frac{Cn}{1-r} \right\rfloor$ and*

$$p_m \;=\; \begin{cases} 1 & \text{if } \; m = 0; \\ \displaystyle\sum_{k=0}^{m-1} p_k \binom{m}{k} r^k (1-r)^{m-k} & \text{if } \; m > 0. \end{cases} \tag{6}$$

(f) *Let $g_n(k)$ be the fraction of compositions of n that have exactly k parts of maximum size. Then for each fixed k and as $n \to \infty$,*

$$g_n(k) \;\sim\; \frac{(1-r)^k}{k!} P_k\left(\frac{Cn}{1-r}\right) + \frac{(1-r)^k \log e}{k}.$$

(g) *Let $D_n(k)$ be the number of distinct recurrent parts that appear exactly k times in a random composition of n. For fixed $k > 0$*

$$\mathsf{E}(D_n(k)) \;=\; \frac{P_k(Cn)}{k!} + \frac{\log e}{k} + o(1).$$

Let $m_n(k)$ be the probability that a randomly chosen recurrent part size in a random composition of n has multiplicity k. For fixed k, $m_n(k) \sim \mathsf{E}(D_n(k))/\log n$.

(h) *Let Φ' be a finite set of local restriction functions similar to those in Φ. The values of r, B and C are the same for $\mathcal{C}_\Phi$ and $\mathcal{C}_{\Phi'}$.*

We recall that $\Gamma(a + iy)$ goes to zero exponentially fast as $y \to \pm\infty$. Thus the sum (2) is dominated by the terms with small ℓ.

Parts (b) and (c) of the theorem can be thought of in terms of the weak law of large numbers. For example, (b) tells us that, for all $\epsilon > 0$,

$$\Pr\left(\left|\frac{M_n}{\mathsf{E}(M_n)} - 1\right| > \epsilon\right) \to 0 \text{ as } n \to \infty,$$

and the condition on $\omega_b(n)$ provides a bound on the rate of convergence.

Since estimating C is generally harder than estimating A, the following theorem is sometimes useful.

Theorem 2 (Sometimes $A = C$). *Let $\mathcal{C}$ be a class of asymptotically free compositions and let the number of compositions of n be asymptotic to Ar^{-n}. Suppose that there is some ℓ such that, whenever the number of parts in each of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is at least ℓ, we have that $\vec{\mathbf{c}} = \vec{\mathbf{a}}x\vec{\mathbf{b}}$ is in $\mathcal{C}$ for infinitely many x if and only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are in $\mathcal{C}$. Then $C = A$, where C is the constant in Theorem 1.*

Theorem 3 (Asymptotically Poisson). *Let ζ_j be the number of parts of size j in a random composition in $\mathcal{C}$ of size n. Then there is a function $\omega(n) \to \infty$ such that the random variables $\{\zeta_j : \log n - \omega(n) \leqslant j \leqslant n\}$ are asymptotically independent Poisson random variables with means $\mu_j = Cnr^j$.*

2 Discussion and Examples

Remark (Some previous results). We review some results that involve the study of parts of large size.

Most results deal with unrestricted compositions. As far as we know, the first result is due to Odlyzko and Richmond [22]. For $a(n, m)$, the number of compositions of n with largest part m, they prove the sequence is unimodal for each n and show that the m which maximizes $a(n, m)$ is always one of the two integers closest to $\log_2 n$. The fact that the largest part M_n is strongly concentrated is well known. For example, it appears as an exercise in [7]. Hwang and Yeh [15] studied the distinct parts in a random composition, obtaining asymptotics for the expected value of their number and sum as well as other results. Hitczenko and Stengle [14] also studied the expected number of distinct parts. The asymptotic probability that a composition is gap-free was obtained by Hitczenko and Knopfmacher [10]. They based their proof on a gap free result they obtained for samples of iid geometric random variables, which we also use in our study of gap-freeness. Wilf asked about $m_n(k)$, the probability that a randomly chosen part size in a random composition of n had multiplicity k. This problem was studied by Hitczenko, Rousseau and Savage [13, 12]. Louchard [20] studied $D_n(k)$, obtaining information about its moments. Archibald and Knopfmacher [1] studied the largest missing part in compositions that are not gap free.

Fewer results have been obtained for Carlitz compositions. Using [6], Knopfmacher and Prodinger [19] obtained asymptotics for the largest part in Carlitz compositions and observed that there was oscillatory behavior. The expected number of distinct parts, $\mathsf{E}(D_n)$, was studied by Hitczenko and Louchard [11] who required an independence assumption that was eliminated by Goh and Hitczenko [9]. Kheyfets [18] obtains results for parts of multiplicity k that parallel those mentioned in the previous paragraph for $D_n(k)$ and $m_n(k)$ in the unrestricted case. Louchard and Prodinger [21] study the distribution of part sizes.

Theorem 1 extends most of these results to asymptotically free compositions. One exception is [1] which came to our attention when this paper was essentially complete. It is likely that our methods can generalize their results, although with less accuracy than they obtain. Most of the known results for unrestricted and Carlitz compositions have greater accuracy than our results which typically have $o(1)$ error rather than more explicit estimates. Also, we do not have formulas for the two constants C and r appearing in our results, whereas they are known for unrestricted and Carlitz compositions. However, since the number of compositions is Ar^{-n} with an exponentially small relative error the more important r is easily estimated if one can count compositions for moderate values of n efficiently.

An earlier version of this paper appeared, without proofs, as the extended abstract [5]. The present paper considers a more general class of compositions and contains some additional results.

Example 6 ($A = C$)**.** It is easily seen that Theorem 2 applies to the following classes of compositions

(a) unrestricted compositions (so $C = 1/2$);

(b) compositions where the value of c_i is restricted only by c_{i-1} and c_{i+1} and may be arbitrarily large;

(c) alternating compositions ($c_{2i-1} < c_{2i} > c_{2i+1}$) where the number of parts must be odd.

We note that (b) includes Carlitz compositions and so $C \doteq 0.4563634741$ for Carlitz compositions [21]. The inequality conditions in (c) can be generalized: we may require that $c_{2i} - c_{2i-1}$ and $c_{2i} - c_{2i+1}$ belong to some subset of $\mathbb{Z}$ that contains arbitrarily large positive values and the subset may depend on i modulo some period.

Although (c) gives $A = C$ for only one type of alternating compositions, it follows from Theorem 1(h) that the value obtained for r, B and C in this case are the same for the various types of alternating compositions discussed in Example 1 even though they have differing values of A.

Example 7 (Gap-free)**.** The numbers p_m in (e) were studied by Hitczenko and Knopfmacher [10] who showed that they oscillated with the same period as (2) when $r > 1/2$. They showed that, for $r = 1/2$, there is no oscillation. Their Figure 7 shows that the amplitude of oscillation of p_m is less than 10^{-6}. Consequently, if r is known, one can determine the asymptotic value of p_m and hence q_n to within 10^{-6}. The following are the values of p_m for three families of compositions, correct up to the sixth decimal place.

- For Carlitz compositions, it is known $r \doteq .57134979$. It follows from (6) that $p_m \doteq 0.372000$ for $m \geqslant 25$.
- For strictly alternating compositions ($c_{2i-1} < c_{2i} > c_{2i+1}$), $r \doteq 0.63628175$ by [4]. It follows from (6) that $p_m \doteq 0.252277$ for $m \geqslant 25$.
- For weakly alternating compositions ($c_{2i-1} \leqslant c_{2i} \geqslant c_{2i+1}$), $r \doteq .57614877$ by [4]. It follows from (6) that $p_m \doteq 0.363144$ for $m \geqslant 25$.

Here is an alternative definition of gap-free based on the literature: A composition is gap-free if, whenever it contains two recurrent parts, say a and b, it contains all recurrent parts between a and b. This definition does not alter the conclusions of Theorem 1(d,e) because, by Lemma 1(b) below, the fraction of compositions of n that omit the smallest recurrent part is exponentially small.

Example 8 (Conjectures of Jaklič, Vitrih and Žagar)**.** Let $\mathrm{Max}_k(n)$ (resp. $\mathrm{Min}_k(n)$) denote the number of all compositions of n such that there are more than k copies of the maximal (resp. minimal) part. Jaklič et al. [16] conjectured that, when $k = 1$

$$\lim_{n\to\infty} \frac{\mathrm{Min_k}(n+1)}{\mathrm{Min_k}(n)} = 2 \tag{7}$$

$$\lim_{n\to\infty} \frac{\mathrm{Max_k}(n+1)}{\mathrm{Max_k}(n)} = 2. \tag{8}$$

In fact, the conjectures hold for the compositions studied in this paper and all $k \geqslant 1$ provided 2 is replaced with $1/r$ and Min is restricted to recurrent parts. The number of occurrences of any given recurrent part is $\Theta(n)$ for almost all recurrent locally restricted compositions of n by [3]. Thus (7) follows immediately from the fact that the number of compositions of n is asymptotic to Ar^{-n}. We now prove (8). Note that

$$\mathrm{Max}_k(n) \sim Ar^{-n}\Big(1 - \sum_{i \leqslant k} g_n(i)\Big).$$

By Theorem 1(f), $g_n(i) \sim g_{n+1}(i)$ and $g_n(i)$ is bounded away from zero as $n \to \infty$. Thus

$$\frac{\mathrm{Max_k}(n+1)}{\mathrm{Max_k}(n)} \sim \frac{\left(1-\sum_{i\leqslant k} g_{n+1}(i)\right)Ar^{-n-1}}{\left(1-\sum_{i\leqslant k} g_n(i)\right)Ar^{-n}} \sim \frac{1-\sum_{i\leqslant k} g_{n+1}(i)}{1-\sum_{i\leqslant k} g_n(i)}\frac{1}{r} \sim \frac{1}{r}.$$

One can change the definition of Max_k to mean exactly k copies of the maximal part and a similar proof will hold.

Example 9 (Counterexamples without freeness)**.** It was shown in Theorem 1(f) of [2] that when differences of adjacent parts are restricted to a finite set, the largest part is asymptotically almost surely of order $\sqrt{\log n}$, so the bound in Theorem 1(a) fails.

3 Statement of Lemmas

The following six lemmas are used in our proofs of Theorems 1, 2 and 3.

Lemma 1 (Normality and tails)**.** *Let $\mathcal{C}_\Phi$ be a class of asymptotically free compositions and let d be arbitrary. Let $\mathcal{R}$ be a possibly infinite nonempty set of recurrent subcompositions each of which contains at most d parts. Assume that if we alter Φ to forbid the elements of $\mathcal{R}$, the resulting class of compositions is still recurrent. Let the random variable X_n be either the number of occurrences of elements of $\mathcal{R}$ in a random composition of n or the number of parts in a random composition of n. The following are true.*

(a) The distribution of X_n is asymptotically normal with mean and variance asymptotically proportional to n.

(b) There are constants $C_i > 0$ depending on what X_n counts such that

$$\Pr(X_n < C_1 n) \ < \ C_2(1+C_3)^{-n} \ \text{ for all } n.$$

(c) Let $\vec{\mathbf{s}}$ be a subcomposition. There is a constant B dependent only on $\mathcal{C}$ such that the probability that a random composition contains at least one copy of $\vec{\mathbf{s}}$ is at most $Bnr^{\Sigma(\vec{\mathbf{s}})}$.

Definition 8 (The function φ)**.** *As in Definition 6 let $\vec{\mathbf{r}}(x) = r_1 \cdots r_p x r_{p+2} \cdots r_{2p+1}$ where $\vec{\mathbf{r}} = \vec{\mathbf{r}}(0) = r_1 \cdots r_p 0 r_{p+2} \cdots r_{2p+1}$. For an asymptotically free class the set*

$$S(\vec{\mathbf{r}}) = \{x : \vec{\mathbf{r}}(x) \text{ is recurrent } \}$$

is either finite or co-finite. So, there is a smallest integer $q(\vec{\mathbf{r}})$ such that

$$\text{either } [q(\vec{\mathbf{r}}),\infty) \subseteq S(\vec{\mathbf{r}}) \ \text{ or } \ [q(\vec{\mathbf{r}}),\infty) \subseteq \overline{S(\vec{\mathbf{r}})}.$$

Define

$$\varphi(P) = \max\{q(\vec{\mathbf{r}}) : r_i \leqslant P \text{ for } 1 \leqslant i \leqslant 2p+1\}.$$

It follows from the definition that if $\max(\vec{\mathbf{r}}) \leqslant P$ and $x \geqslant \varphi(P)$ and $\vec{\mathbf{r}}(x)$ is recurrent at j modulo m, then $\vec{\mathbf{r}}$ is asymptotically free at j modulo m.

Definition 9 (P-isolated)**.** *Suppose $\vec{\mathbf{c}} = c_{i-p} \cdots c_i \cdots c_{i+p}$ is a recurrent subcomposition. If no c_j, except possibly c_i, exceeds P, we call c_i P-*isolated.

A consequence of the definitions is that, whenever $x \geqslant \varphi(P)$ is P-isolated we are free to replace x by any part that is of size $\varphi(P)$ or greater.

Lemma 2 (Large part separation)**.** *Let $\mathcal{C}_\Phi$ be a class of asymptotically free compositions. Suppose $\delta > 0$. There is a $P = P(\delta)$ and $N = N(\delta)$ such that the following holds for every $m \geqslant \varphi(P)$. Let $\mathcal{M}(n)$ be the set of compositions of n in which a part of size m has been marked. For all $n > N + m$ the subset of $\mathcal{M}(n)$ in which the marked part is not P-isolated has size less than $\delta|\mathcal{M}(n)|$.*

The following lemma proves most of Theorem 1(a).

Lemma 3 (Geometric probabilities)**.** *We use the notation of Theorem 1(a).*

(a) Equations (3) and (4) are true.

(b) Recall that u_k is the limit (on n) of the ratio $\mathsf{E}(X_k(n))/\mathsf{E}(X_0(n))$. For all sufficiently large parts k and ℓ depending on $\delta > 0$, we have

$$\left| \frac{u_k\, r^{-k}}{u_\ell\, r^{-\ell}} - 1 \right| \;<\; \delta.$$

(c) We have $u_k \sim Br^k$ for some positive constant B.

Lemma 4 (Marked compositions)**.** *Fix k and a class $\mathcal{C}$ of asymptotically free compositions. Let A be such that the number of compositions of n is asymptotic to Ar^{-n} and let C be as in Theorem 1. If $\mathbf{L}(n) = (L_1(n), \ldots, L_k(n))$ is a sequence of k-tuples of integers with*

$$\max(L_i) = o(n) \quad \textit{and} \quad \min(L_i) \to \infty \ \textit{ as } n \to \infty,$$

then the number of compositions $\vec{\mathbf{c}}$ of n having $\mathbf{L} = (L_1 \cdots L_k)$ as a marked subsequence is

$$(A + o(1)) \frac{(Cn)^k r^{s-n}}{k!} \ \textit{ as } n \to \infty, \ \textit{ where } s = L_1 + \cdots + L_k. \tag{9}$$

Lemma 5 (Characterization of Poisson)**.** *Let $(m)_k := m(m-1)\cdots(m-k+1)$ denote the falling factorial. Suppose that $\zeta_1, \ldots, \zeta_n = \zeta_1(n), \ldots, \zeta_n(n)$ is a set of non-negative integer variables on a probability space Λ_n, $n = 1, 2, \ldots,$ and there is a sequence of positive reals $\gamma(n)$ and constants $0 < \alpha < 1$ and $0 < c < 1$ such that*

(i) $\gamma(n) \to \infty$ and $n - \gamma(n) \to \infty$;

(ii) for any fixed positive integers ℓ, $m_1, \ldots, m_\ell$, and sequences $k_1(n) < k_2(n) < \cdots < k_\ell(n)$ with $|k_i(n) - \gamma(n)| = O(1)$, $1 \leqslant i \leqslant \ell$, we have

$$\mathsf{E}\left((\zeta_{k_1(n)})_{m_1}(\zeta_{k_2(n)})_{m_2}\cdots(\zeta_{k_\ell(n)})_{m_\ell}\right) \sim \prod_{j=1}^{\ell} \alpha^{(k_j(n)-\gamma(n))m_j}, \tag{10}$$

(iii) $\Pr(\zeta_{k(n)} > 0) = O\left(c^{k(n)-\gamma(n)}\right)$ *uniformly for all* $k(n) > \gamma(n)$.

Then there exists a function $\omega(n) \to \infty$ so that for $k = \lfloor \gamma(n) - \omega(n) \rfloor$, the total variation distance between the distribution of $(\zeta_k, \zeta_{k+1}, \ldots, \zeta_n)$, and that of $(Z_k, Z_{k+1}, \ldots, Z_n)$ tends to 0, where the $Z_j = Z_j(n)$ are independent Poisson random variables with $\mathsf{E}Z_j = \alpha^{j-\gamma(n)}$.

Remark. The preceding lemma, Lemma 5, is applied to obtain the Poisson result for large parts stated as Theorem 3. The latter, in turn, is used with Mellin transforms to prove Theorem 1(b-d); and, with a result of Hitczenko and Knopfmacher [10] on sequences of geometric i.i.d. random variables, to prove Theorem 1(e,f).

Lemma 6 (Plentitude of recurrent parts)**.** *Let ζ_j be the number of occurrences of j in a random composition of n, and let $k > 0$ be arbitrary and fixed. If $\omega(n) \to \infty$, then*

$$\sum_{\substack{j < \log n - \omega(n) \\ j \text{ recurrent}}} \Pr(\zeta_j < k) \;=\; o(1).$$

4 The Transfer Matrix and Sets of Functions

Before embarking on the proofs, we summarize some facts from [3] which will be used and reduce the study of a finite set Φ to a single ϕ since only single ϕ's were considered in [3].

We may replace the span p by any larger value without altering the set of compositions, provided we adjust the definition of ϕ. Thus *we will assume that the span is a multiple of the modulus m.* (Refer back to Definition 2 for terminology.)

Let $C(n)$ be the number of compositions of n in a regular, locally restricted class $\mathcal{C}_\phi$, and let $F(x) = \sum C(n)x^n$ be the ogf (ordinary generating function). Then, as proven in Theorem 2 of that paper,

$$F(x^2) = \varphi(x) + F_{NR}(x^2), \tag{11}$$

where

$$\varphi(x) = \mathbf{s}(x)^{\mathrm{t}} \left(\sum_{k=0}^{\infty} T(x)^k \right) \mathbf{f}(x). \tag{12}$$

We now explain the various parts of (11). Here a "small number of parts" is at most some small multiple of p.

The transfer matrix $T(x)$ is defined in terms of a certain sequence of words $\vec{\nu}_1, \vec{\nu}_2, \ldots$, where by a *word* we mean a recurrent subcomposition of length p, the span, (see Definition 2) whose first part is at j mod p where j is the same for all words indexing T (and thus $\mathbf{s}$ and $\mathbf{f}$ as well). The list contains all such recurrent words and

$$T(x)_{ij} \;=\; \begin{cases} x^{\Sigma(\vec{\nu}_i)+\Sigma(\vec{\nu}_j)} & \text{if } \vec{\nu}_j \text{ can follow } \vec{\nu}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Except for parts near the ends, every composition is a concatenation of such words. A single application of the transfer matrix corresponds to the adjunction of p additional parts to the composition.

The infinite vectors $\mathbf{s}(x), \mathbf{f}(x)$ have analytic entries corresponding to compositions with a small number of parts. The component $s_i(x)$ of $\mathbf{s}(x)$ deals with the generating function for the beginning of compositions where the last p parts in the beginning are $\vec{\nu}_i$. Similarly, $f_j(x)$ deals with the generating function for endings whose first p parts are $\vec{\nu}_j$.

The function $F_{NR}(x)$ is the ogf for the subclass of compositions not counted in $\varphi(x)$. These compositions have at most some small number of parts. The ogf $F_{NR}(x)$ has radius of convergence 1. (This is slightly different from the definition of F_{NR} in [3]; however, all that matters for the theory is that F_{NR} has radius of convergence 1 and that (11) counts all compositions exactly once.)

To assure that $T(x)$ satisfies certain useful technical conditions, it is necessary to have the arguments x^2 and x as indicated in (a). See the latter part of this section and [3] for more details on $T(x)$.

4.1 Reduction to a single ϕ and Theorem 1(h)

Before discussing the more technical issues related to asymptotics, we explain why it suffices to consider one ϕ instead of an entire finite set Φ of similar ϕ. This discussion will also prove Theorem 1(h).

Suppose $\phi \approx \phi'$ and let T be the transfer matrix for ϕ. Since $T_{ij} \neq 0$ if and only if $\nu_i\nu_j$ is recurrent, we can use the same transfer matrix for ϕ'; however, the vectors $\mathbf{s}$ and $\mathbf{f}$ will be different. In fact, if k is as in Definition 5, the number of parts in the subcompositions of the vectors $\mathbf{s}$ for ϕ and ϕ' will differ by k mod m. Nearly all results in [3] depend on T but not on $\mathbf{s}$ or $\mathbf{f}$. The exception is the constant A in the asymptotic estimate Ar^{-n} for the number of compositions of n.

It follows that, if the sets $\mathcal{C}_\phi$, $\phi \in \Phi$, were pairwise disjoint we could simply obtain results for one $\phi \in \Phi$ and combine the results where, whenever A is present we simply sum the values of A for the various $\phi \in \Phi$. We now show that this can, in principle, be done. There is no need to do this in practice since analytic methods for obtaining reasonable estimates of A are seldom available even for a single ϕ.

Fix temporarily a $\phi \in \Phi$. Partition the elements ϕ' of Φ into m sets $\Phi_0, \ldots, \Phi_{m-1}$ according to the value of k in Definition 5. We now focus on these sets, first considering functions in different sets and then functions in the same set.

Suppose $\phi \in \Phi_i$ and $\phi' \in \Phi_j$ where $i \neq j$. Consider the compositions in $\mathcal{C}_\phi \cap \mathcal{C}_{\phi'}$. Let the value of $\phi\phi'$ be simply the product of ϕ and ϕ'. We note that $\mathcal{C}_\phi \cap \mathcal{C}_{\phi'} = \mathcal{C}_{\phi\phi'}$ since a composition is in the intersection if and only if it satisfies both local restriction functions. Since $i \neq j$, it follows that the transfer matrix for the intersection will be the same as that for ϕ with some nonzero entries replaced by zeroes. By Lemma 2(f) of [3] and the realization that the spectral radius determines the growth rate (see below) it follows that the number of compositions of n in the intersection grows at an exponentially smaller rate than the number in $\mathcal{C}_\phi$. Hence, for asymptotic purposes, we may treat the m sets $\mathcal{C}_{\Phi_i}$ as if they are disjoint. It follows that, except for Theorem 2, we may assume we are dealing with just one Φ_i.

We now consider a single Φ_k. Suppose $\vec{\mathbf{c}} \in \mathcal{C}_{\Phi_k}$ is counted by (12). We can write it in the form $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{z}}$, where $\vec{\mathbf{b}}$ is a sequence of words $\vec{\nu}$ that index T, $\vec{\mathbf{s}}$ and $\vec{\mathbf{f}}$, $\vec{\mathbf{a}}$ ends with one of these ν and $\vec{\mathbf{z}}$ starts with one of them. By absorbing a recursive word or two in $\vec{\mathbf{a}}$ and $\vec{\mathbf{z}}$ if needed, we can insure that the following two assumptions hold for some ℓ_i.

(i) Since all compositions come from the same $\mathcal{C}_{\Phi_k}$ we can assume $\operatorname{len}(\vec{\mathbf{a}}) = \ell_0$, the same value for all compositions in $\mathcal{C}_{\Phi_k}$.

(ii) Since multiplication by T adds p parts to the compositions, we can assume that the longest and shortest values of $\operatorname{len}(\vec{\mathbf{z}})$, say ℓ_1 and ℓ_2, differ by less than p.

It follows that each composition in $\mathcal{C}_{\Phi_k}$ with at least $\ell_0 + \ell_1$ parts is counted by $\mathbf{s}T^k\mathbf{f}$ for some k and has uniquely determined $\vec{\mathbf{a}}$ and $\vec{\mathbf{z}}$. We can limit attention to compositions with at least $\ell_0 + \ell_1$ parts since the generating function for those with fewer parts has radius of convergence at least 1.

With each $\phi_i \in \Phi_k$ we associate two sets $\mathcal{S}_i$ and $\mathcal{F}_i$ as follows. $\vec{\mathbf{a}} \in \mathcal{S}_i$ and $\vec{\mathbf{z}} \in \mathcal{F}_i$ if and only if they satisfy (i) and (ii) above and $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{z}} \in \mathcal{C}_{\phi_i}$ for some $\vec{\mathbf{z}}$. The set $\mathcal{S}_i$ determines $\mathbf{s}$ as follows. If $\vec{\mathbf{a}} \in \mathcal{S}_i$ ends with ν_j, then a generating function obtained from $\vec{\mathbf{a}}$ is added to s_j. A similar construction holds for $\vec{\mathbf{z}}$ and $\mathbf{f}$. Thus $\mathcal{S}_i \times \mathcal{F}_i$ determines the compositions in $\mathcal{C}_{\phi_i}$. If we had $(\mathcal{S}_i \times \mathcal{F}_i) \cap (\mathcal{S}_j \times \mathcal{F}_j) = \emptyset$, it would follow that $\mathcal{C}_{\phi_i} \cap \mathcal{C}_{\phi_j}$ would contain at most some compositions shorter than $\ell_0 + \ell_1$. Thus, we need only prove that a union of Cartesian products $\cup_{\phi_i \in \Phi_k}(\mathcal{S}_i \times \mathcal{F}_i)$ can always be written as a disjoint union of such products. This is done by considering the given terms $\mathcal{S}_i \times \mathcal{F}_i$ one at the time, and using the identity

$$(A \times B) \cap (C \times D)^c \;=\; \Big(A \times (B \setminus D)\Big) \cup \Big((A \setminus C) \times (B \cap D)\Big),$$

where the union is disjoint. (Think of $C \times D$ as the latest $\mathcal{S}_i \times \mathcal{F}_i$, and $A \times B$ as one of the pairwise disjoint components of the previously processed $(i-1)$ products. We keep $C \times D$ as a new component, and each previously existing component is replaced by two disjoint pieces.) For each product in the resulting disjoint union, we construct a ϕ, and their sum is the generating function for $\mathcal{C}_k$, with the possible exception of short compositions.

4.2 Analytic aspects of $T(x)$ from [3]

By Lemma 3 of [3], at each $x_0 \in (0,1)$ we have a neighborhood and functions $\lambda(x)$, $E(x)$ and $B(x)$ analytic in that neighborhood such that

$$T(x) = \lambda(x)E(x) + B(x), \tag{13}$$

where $E(x)$ is the projection onto the one-dimensional eigenspace of eigenvalue $\lambda(x)$, the spectral radius of $B(x)$ is less than $\lambda(x)$, and $E(x)B(x) = B(x)E(x) = 0$. The proof of Lemma 3 relies heavily on results and methods from [17]. If we choose for x_0 the point $r^{1/2}$, $0 < r < 1$, where $\lambda(r^{1/2}) = 1$, it follows from (13) that

$$\sum_{i \geqslant 0} T(x)^i \;=\; \frac{\lambda(x)}{1-\lambda(x)} E(x) + (I - B(x))^{-1}, \tag{14}$$

in a punctured neighborhood $0 < |x - r^{1/2}| < \delta$. (We use $r^{1/2}$ so that r is the radius of convergence for the ogf $F(x)$, and we are consistent with our convention that $C(n) \sim Ar^{-n}$.)

The neighborhood $|x - r^{1/2}| < \delta$, which we shall refer to as the δ-neighborhood, plays a key role in our proof of Lemma 4. At any point in this neighborhood, except the center $x = r^{1/2}$, the relation (14) holds. At any x_0 with $|x_0| = r^{1/2}$, except $x_0 = \pm r^{1/2}$, the spectral radius of $T(x_0)$ is strictly less than 1 by Lemma 1 of [3]. Near such an x_0, the sum $S(x) = \sum_{i \geqslant 0} T(x)^i$ converges and $S(x)$ is analytic in a neighborhood of x_0. On the other hand, near $x_0 = r^{1/2}$ the equation (14) shows that still $S(x)$ is analytic except for an isolated singularity at $x = r^{1/2}$. It is also shown in [3] that the root of $\lambda(x) = 1$ at $x = r^{1/2}$ is a simple root; thus near $r^{1/2}$ we have an analytic $\beta(x)$ with $\lambda(x) = 1 - \beta(x)(1 - x/r^{1/2})$, and $\beta(r^{1/2}) \neq 0$.

5 Proof of Lemma 1

Part (a) follows from [3, Thm. 4] with just one random variable $Y_1(n) = X_n$. (The definition of "unrelated events" for that theorem is somewhat technical. The condition in Lemma 1 that altered Φ be recurrent insures that it holds.)

We now prove (b). Into the transfer matrix $T(x)$ of [3], introduce a new variable $0 < s \leqslant 1$ that keeps track of the number of occurrences of elements of $\mathcal{R}$ or simply the number of parts. If d exceeds the span of ϕ, d behaves like a new span and it will be necessary to change T so that one application of T adds more parts to the composition. Call the new matrix $T(x,s)$ and call the largest eigenvalue $\lambda(x,s)$. This leads to the asymptotics $A(s)r(s)^{-n}$ where $r(s)$ is the solution to $\lambda(r^{1/2}(s), s) = 1$. The case $s = 1$ corresponds to the asymptotics for $C(n)$, the number of compositions of n. In the general case, we have asymptotics for $\sum_k C(n,k)s^k$ where $C(n,k)$ is the number of compositions of n with exactly k copies of elements of $\mathcal{R}$. It follows that

$$\sum_{k<\delta n} C(n,k) \;\leqslant\; s^{-\delta n} \sum_k C(n,k)s^k \;\sim\; A(s)(s^\delta r(s))^{-n}.$$

Hence it suffices to show that

$$s^\delta r(s) > r(1) \tag{15}$$

for some s and δ. By Lemma 2(f) of [3], $\lambda(x,s) < \lambda(x,1)$ for $x > 0$ and $0 < s < 1$. Since λ is monotonically increasing in x, $r(s) > r(1)$ and so (15) holds for all sufficiently small δ depending on s. This completes the proof of (b).

The proof of (c) is essentially the same as that given in Section 9 of [3] for large part size. Since there are slight changes, we repeat it here for completeness.

Let p be the span of ϕ. Consider an expanded class $\mathcal{C}_\Psi$ where Ψ is the same as ϕ except that the first p and last p parts of compositions are unrestricted. The transition matrix $T(x)$ is unchanged. Therefore Ψ has the same radius of convergence r as ϕ. Hence the number of compositions of n in $\mathcal{C}_\Psi$ is bounded above by Cr^{-n} for some C. Hence the generating function for $\mathcal{C}_\Psi$ compositions by sum of parts is bounded coefficient-wise by $C(1-x/r)^{-1}$.

Imagine marking a copy of $\vec{\mathbf{s}}$ in each composition in $\mathcal{C}_\phi$. By the previous paragraph, the generating function for such compositions of n is bounded coefficient-wise by

$$\frac{C}{1-x/r}x^s\frac{C}{1-x/r} \quad \text{where } s=\Sigma(\vec{\mathbf{s}}).$$

Hence the number of such compositions of n is bounded above by nC^2r^{s-n}.

The previous paragraph overcounts the number of compositions containing $\vec{\mathbf{s}}$. For some $C'>0$, the total number of compositions of n is at least $C'r^{-n}$ for large n. Taking the ratio gives (c) with $B=C^2/C'$. $\square$

6 Proof of Lemma 2

Throughout the proof, whenever a new condition is imposed on P or N it is understood that the implied values must be at least as large as those already chosen. All implied limits, as in $o(1)$, are as $n\to\infty$.

Let $\mathcal{M}^*(n)$ be the subset of $\mathcal{M}(n)$ in which the marked part is not P-isolated. Let $M^*(n)$ and $M(n)$ be the cardinalities of these two sets. We will overestimate $M^*(n)$ and underestimate $M(n)$ and show that their ratio can be made arbitrarily small provided P and $n-m$ are sufficiently large.

For both counts, we consider compositions of the form $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}$ where $\vec{\mathbf{b}}$ contains a special sequence of parts.

We then sum over a. The composition $\vec{\mathbf{a}}$ will be like compositions in the class $\mathcal{C}_\phi$ except that there will be conditions on the last p parts. Since $T(x)$ is unchanged, the radius of convergence is unchanged and so the number of $\vec{\mathbf{a}}$ is $\Theta(r^{-a})$ as $a\to\infty$. A similar result holds for $\vec{\mathbf{c}}$. We refer to this below as "theta". Let $a=\Sigma(\vec{\mathbf{a}})$ and $c=\Sigma(\vec{\mathbf{c}})$.

We start with the underestimate of $M(n)$. Let $\vec{\mathbf{r}}(x)$ be as in Definition 6 and let b be the sum of its parts *excluding* x. Choose P so that x is P-isolated in $\vec{\mathbf{r}}(x)$. Thus b is fixed as $n\to\infty$, but we may increase P as necessary later. Since $m\geqslant\varphi(P)$, we may replace x with a marked part m. Let $\vec{\mathbf{b}}=\vec{\mathbf{r}}(m)$. To underestimate $M(n)$, we will obtain a lower bound on the number of occurrences of $\vec{\mathbf{b}}$. Since the r_i are fixed and the span is p, the choices for $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$ such that $\vec{\mathbf{a}}\vec{\mathbf{b}}\vec{\mathbf{c}}\in\mathcal{C}$ are independent of m. By theta there are B and s such that there are at least Br^{-a} choices for $\vec{\mathbf{a}}$ and Br^{-c} for $\vec{\mathbf{c}}$ when $a\geqslant s$ and $c\geqslant s$.

Thus the total number of occurrences of $\vec{\mathbf{b}}$ is at least

$$\sum_{a=s}^{n-(b+m)-s} B^2 r^{-s} r^{-(n-s-(b+m))} \;=\; (n-2s-b-P+1)B^2 r^{-n+b+m},$$

and so for sufficiently large n and some constant $C_0 < B^2 r^b$,

$$M(n) \;\geqslant\; nC_0 r^{-n+m}.$$

For the non-isolated overcount, let $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$ be compositions where we put no restrictions on how they begin or end. By theta the number of such compositions of ℓ is bounded above by $Dr^{-\ell}$ for some D. The composition $\vec{\mathbf{b}}$ will contain at most $p+1$ parts. It will either begin or end with the marked part m and the other ending part will be at least P so that the marked part is not P-isolated. Let b be the sum of the parts in $\vec{\mathbf{b}}$, omitting the marked part m. It follows that $b \geqslant P$. We bound the number of $\vec{\mathbf{b}}$ as follows. Ignore the part m. Choose a first part b_1 in b ways. Choose an additional $p-1$ parts, allowing parts of size zero, which will be ignored when constructing $\vec{\mathbf{b}}$. Since the remaining parts sum to $b - b_1 \leqslant b-1$, each of them has at most b values. Hence we have the bound $2 \cdot b \cdot b^{p-1}$, where the factor of 2 is arises from the choice of which end to place m. Thus, for some constant C_1,

$$M^*(n) \;\leqslant\; C_1 \sum_{b=P}^{n} \sum_{a=0}^{n-b-m} r^{-a} b^p r^{-(n-a-(m+b))} \;<\; C_1 n r^{-n+m} \sum_{b\geqslant P} b^p r^b.$$

Combining our two estimates, we have for some constant C_2 and sufficiently large n

$$\frac{M^*(n)}{M(n)} \;<\; C_2 \sum_{i\geqslant P} i^p r^i + o(1).$$

By choosing P sufficiently large, we can make this arbitrarily small. □

7 Proof of Lemma 3

By Lemma 1(a), the total number of parts and the number of parts of size k are asymptotically normally distributed with means and variances proportional to n. Thus (a) follows.

We now prove (b). Let p be the span of ϕ. Apply Lemma 2 to obtain $P = P(\delta')$, where δ' is sufficiently small and depends on the value of δ in (b).

Choose k and ℓ larger than $\varphi(P)$. Later we let P and hence k and ℓ tend to infinity slowly.

Consider compositions of $n+k$ with a marked part of size k. By changing a part of size k into one of size ℓ we obtain a composition of $n+\ell$ with a marked part of size ℓ. This is a bijection between compositions containing a marked P-isolated part of size k and those containing a marked P-isolated part of size ℓ, the marked part being the one that is changed. By Lemma 2, we can ignore those compositions with marked parts that are not P-isolated. Since the number of compositions of m is asymptotic to Ar^{-m}, the number of compositions with such P-isolated marked parts is asymptotic to both $\mathsf{E}(X_k(n+k))Ar^{-n-k}$ and $\mathsf{E}(X_\ell(n+\ell))Ar^{-n-\ell}$. Since $\mathsf{E}(X_0(n+k)) \sim \mathsf{E}(X_0(n)) \sim \mathsf{E}(X_0(n+\ell))$ as $n \to \infty$ with $k = o(n)$ and $\ell = o(n)$, (b) follows.

Part (c) follows by letting $\ell \to \infty$ in (b): Since δ can be made arbitrarily small by choosing k sufficiently large, it follows that $\lim u_\ell r^{-\ell}$ must exist and be nonzero. □

8 Proof of Lemma 4

Let $Q(n) = \varphi(P(n))$ where $P(n)$ is some unspecified value that we will allow to increase "sufficiently slowly" with n. When referring to P, Q and the L_i in the statement of the lemma, we will omit "(n)". P must increase so slowly that $\min(L_i) \geqslant Q$. Let $s = L_1 + \cdots + L_k$ and $m = n - s + kQ$. Since $L_i = o(n)$, we have $m \sim n$.

Let $\mathbf{R} = (R_1, \ldots, R_k)$ denote an arbitrary k-tuple of positive integers.

Denote the k-tuple $\mathbf{R}$ with $R_i = Q$ for $1 \leqslant i \leqslant k$ by Q^k.

Let $\mathcal{M}(n, \mathbf{R})$ be the set of compositions of n that have $\mathbf{R}$ as a marked subsequence and let $M(n, \mathbf{R}) = |\mathcal{M}(n, \mathbf{R})|$. We would like to establish a bijection between $\mathcal{M}(n, \mathbf{L})$ and $\mathcal{M}(m, Q^k)$ by simply replacing the elements of one marked subsequence with those of the other. Unfortunately this may fail if any of the following hold:

(a) an element of the marked subsequence has a part exceeding P within distance p;

(b) an element of the marked subsequence occurs within the first p parts;

(c) an element of the marked subsequence occurs within the last p parts.

(The reason for (b) and (c) is that φ applies only to the recurrent parts of the composition and the ends may not be recurrent.) Let the subscript $*$ refer to those compositions for which none of (a)–(c) hold, *except that p is replaced by $2p$ in (c).* Note that the proposed bijection is actually a bijection when restricted to $\mathcal{M}_*(n, \mathbf{L})$ and $\mathcal{M}_*(m, Q^k)$. We will show that

$$M_*(m, Q^k) \sim M(m, Q^k) \quad \text{and} \quad M_*(n, \mathbf{L}) \sim M(n, \mathbf{L}). \tag{16}$$

It then follows that

$$M(n, \mathbf{L}) \sim M_*(n, \mathbf{L}) = M_*(m, Q^k) \sim M(m, Q^k) \tag{17}$$

and so it suffices to estimate the size of any set that contains $\mathcal{M}_*(m, Q^k)$ and is contained in $\mathcal{M}(m, Q^k)$.

Overcounting compositions in $\mathcal{M} \backslash \mathcal{M}_*$. The idea is to allow the parts within distance p of a marked part to be arbitrary. We separate the composition into a sequence of

$k+1$ possibly empty subcompositions by removing the k marked parts. Each of the subcompositions so obtained may have been shifted with regard to its modulus and may have beginning and ending subsequences that are not allowed by the local restriction ϕ. We overcount them by counting compositions with arbitrary shifts in their moduli and with the initial and final p parts arbitrary. The generating function for these possible subcompositions has the form (11); however, the values of $\mathbf{s}(x)$ and $\mathbf{f}(x)$ will be different. Since $T(x)$ is unchanged, it is a consequence of the results in [3] that the number of such subcompositions of i is bounded above by $K_1 r^{-i}$ for some K_1 and so the generating function is bounded coefficient-wise by $K_1(1-x/r)^{-1}$.

The generating function for compositions that have a nonisolated marked part Q is bounded coefficient-wise by

$$\left(\frac{K_1}{1-x/r}\right)^{k+1} k\,(2px^P)x^{kQ},$$

where

- the first factor bounds the subcompositions,
- k chooses a marked part,
- $2p$ bounds the choices of a part near the chosen marked part,
- x^P increases that part by P to insure that a nearby part exceeds P, and
- x^{kQ} inserts the marked parts Q^k.

The coefficient of x^m is bounded by $K_2 m^k r^{-m+kQ+P}$ for some constant K_2. This takes care of all cases except the occurrence of two nearby marked parts Q.

Suppose there are two nearby marked parts. Modify the previous argument by removing these two nearby parts and all parts between them. The termwise bound is now given by the generating function

$$\left(\frac{K_1}{1-x/r}\right)^{k}(k-1)\left(\sum_{j=0}^{p}\frac{x}{1-x}\right)x^{kQ},$$

where the $k-1$ chooses a position to insert the pair of nearby marked Q's and the summation inserts arbitrary parts between these two Q's. Since the summation has radius of convergence 1 and $r<1$, the coefficient of x^m is bounded by $K_3 m^{k-1} r^{-m+kQ}$. Thus

$$\begin{aligned} M(m,Q^k)-M_*(m,Q^k) &= m^k r^{-m+kQ}\Big(O(r^P)+O(1/m)\Big) \\ &= n^k r^{-n+s}\Big(O(r^P)+O(1/n)+o(1)\Big) \qquad (18) \end{aligned}$$

When a composition in $\mathcal{M}(n,\mathbf{L})$ is transformed by replacing $\mathbf{L}$ by Q^k and the result is an illegal composition, it must be of the form we have just bounded and so the bound in (18) is also a bound for $M(n,\mathbf{L})-M_*(n,\mathbf{L})$.

Building marked compositions. We now build marked compositions that form a set between $\mathcal{M}_*(m, Q^k)$ and $\mathcal{M}(m, Q^k)$. Define the transfer matrix $A_Q(x)$ by

$$A_Q(x)_{i,j} \;=\; \begin{cases} T(x)_{i,j} & \text{if } \nu_i \text{ has exactly one } Q; \\ 0 & \text{otherwise.} \end{cases}$$

Let $S(x) = \sum_{i \geqslant 0} T(x)^i$. Define the power-series $f_{k,Q}(x)$ by

$$f_{k,Q}(x^2) = \mathbf{s}(x)^{\mathrm{t}} \left(S(x)A_Q(x)\right)^k S(x)\,\mathbf{f}(x). \tag{19}$$

Since $[x^n]f_{k,Q}(x)$ equals the number of compositions counted by $M(n, Q^k)$ having at most one marked Q per word and no marked Q's near the ends of the composition, it follows that

$$M_*(n, Q^k) \leqslant [x^n]f_{k,Q}(x) \leqslant M(n, Q^k).$$

Using (14) in (19):

$$\begin{aligned} f_{k,Q}(x^2) \;&=\; \mathbf{s}(x)^{\mathrm{t}} \left(\frac{\lambda(x)}{1-\lambda(x)} E(x)A_Q(x) + B_Q(x) \right)^k \\ &\qquad \times \left(\frac{\lambda(x)}{1-\lambda(x)} E(x) + (I - B(x))^{-1} \right) \mathbf{f}(x), \end{aligned} \tag{20}$$

where $B_Q(x) = (I - B(x))^{-1} A_Q(x)$. When the products in (20) are expanded, we obtain something of the form

$$f_{k,Q}(x^2) \;=\; \left(\frac{\lambda(x)}{1-\lambda(x)} \right)^{k+1} \mathbf{s}(x)^{\mathrm{t}} (E(x)A_Q(x))^k E(x)\mathbf{f}(x) + \frac{h_Q(x)}{(1-\lambda(x))^k},$$

where $h_Q(x)$ is analytic in a neighborhood of $r^{1/2}$ because everything in (20) except $\frac{\lambda(x)}{1-\lambda(x)}$ is. The first term determines the leading asymptotic behavior of the coefficients. Recalling that $E(x)$ is a projection onto the 1-dimensional eigenspace of $\lambda(x)$, define the functions $\mathbf{v}(x)$ and $\alpha_Q(x)$ by the equations

$$\mathbf{v}(x) = E(x)\mathbf{f}(x), \quad \text{and} \quad E(x)A_Q(x)\mathbf{v}(x) = \alpha_Q(x)\mathbf{v}(x),$$

which are analytic in a neighborhood of $r^{1/2}$. Thus

$$[x^{2m}]\, f_{k,Q}(x^2) \;=\; D \frac{(mD_Q)^k}{k!} r^{-m} + o(M^k r^{-n}) \;\text{ for some } D, D_Q. \tag{21}$$

The constants in (21). Since $M(m, Q^0)$ counts compositions with no marked parts, it equals $C(m)$, the number of compositions of m. Since $C(m) \sim Ar^{-m}$, we have $D = A$.

Note that $M(m, Q^1)/C(m)$ is the average number of parts of size Q in a composition of m. We use the notation of Theorem 1(a), the results in Lemma 3, and the fact [3] that $X_0(m)$ is asymptotically normal with mean and variance asymptotically proportional to

m. By Lemma 3, if we let P (and hence Q) go to infinity sufficiently slowly with m, then $\mathsf{E}(X_Q(m))/\mathsf{E}(X_0(m)) \sim Br^Q$. In this case

$$mD_Q \;\sim\; \frac{M(m,Q^1)}{M(m,Q^0)} \;=\; \mathsf{E}(X_Q(m)) \;=\; \frac{\mathsf{E}(X_Q(m))}{\mathsf{E}(X_0(m))}\mathsf{E}(X_0(m)) \;\sim\; Br^Q\mathsf{E}(X_0(m)).$$

Recalling the defining relationship

$$C = B\lim_{m\to\infty}\frac{\mathsf{E}(X_0(m))}{m}$$

in Theorem 1(b), we have $D_Q \sim Cr^Q$ as $Q \to \infty$.

It follows from (18) that, if $P \to \infty$ sufficiently slowly with m, then (16) holds and so (21) provides the asymptotics for $M(m, Q^k)$ and $M(n, \mathbf{L})$. □

9 Proofs of Lemmas 5 and 6

Proof of Lemma 5: This is Lemma 12 of [8]. □

Proof of Lemma 6: Fix $\vec{\mathbf{r}}(x)$ as in Definition 6, let $P = \max(r_i)$ and $Q \geqslant \varphi(P)$. By Lemma 1(a), $\Pr(\zeta_j < k) = o(1)$ for every fixed recurrent j. Thus

$$\sum_{j<Q}\Pr(\zeta_j < k) \;=\; o(1).$$

We now consider $j \geqslant Q$. Let $C(n)$ be the number of compositions of n and let $C_j(n)$ be the number of those having fewer than k copies of the part j. For some $\delta > 0$ to be specified later, let $C_j^+(n)$ be the number of those containing at least δn copies of $\vec{\mathbf{r}}(x)$ and $C_j^-(n)$ be the remainder.

Let $C^-(n)$ count compositions with fewer than δn copies of $\vec{\mathbf{r}}(x)$. By Lemma 1(b) with $\delta = C_1$ sufficiently small, $C^-(n)/C(n)$ goes to zero exponentially fast as $n \to \infty$. Since $C_j^-(n) \leqslant C^-(n)$, it follows that $C_j^-(n) < C_2(1+C_3)^{-n}C(n)$ where the constants do not depend on j.

In each composition counted by $C_j^+(n)$ replace x by j in k of the $\vec{\mathbf{r}}(x)$. This can be done in at least $\binom{\delta n}{k}$ ways, giving a composition of $n+k(j-x)$. Since the resulting composition can have at most $2k-1$ parts of size j, it could have arisen by this replacement process in at most $\binom{2k-1}{k}$ ways. Thus

$$C_j^+(n)\binom{\delta n}{k} \;\leqslant\; C(n+k(j-x))\binom{2k-1}{k}$$

and so

$$\frac{C_j^+(n)}{C(n)} \;\leqslant\; \frac{\binom{2k-1}{k}}{\binom{\delta n}{k}}\,\frac{C(n+k(j-x))}{C(n)} < \frac{Br^{-kj}}{n^k}$$

for some $B = B(k, \delta, x)$ independent of j. Thus

$$C_j(n)/C(n) \; < \; Br^{-kj}/n^k + C_2(1+C_3)^{-n},$$

where all constants are independent of n and j. Summing the right side over $Q \leqslant j \leqslant \log n - \omega(n)$, we obtain the bound

$$C_4 r^{k\omega(n)} + C_2(1+C_3)^{-n}\log n \; = \; o(1)$$

for some constants C_i. □

10 Proofs of Theorems 2 and 3

Proof of Theorem 2: There is an increasing function $A_\ell(k, P)$ with supremum A such that the number of compositions of $n \geqslant k$ with end parts at most P is at least $A_\ell(k, P)r^{-n}$. There is a decreasing function $A_u(k)$ with infimum A such that the number of compositions of $n \geqslant k$ is at most $A_u(k)r^{-n}$.

Using the construction $\vec{\mathbf{c}} = \vec{\mathbf{a}}x\vec{\mathbf{b}}$ in the statement of Theorem 2 together with the idea and notation in the above proof of Lemma 4, we construct a composition with one marked part. If $k \leqslant t \leqslant n-k-Q$, the number of compositions $\vec{\mathbf{a}}Q\vec{\mathbf{b}}$ with $\Sigma(\vec{\mathbf{a}}) = t$ and $\Sigma(\vec{\mathbf{b}}) = n-t-Q$ is between $A_\ell(k,P)^2 r^{Q-n}$ and $A_u(k)^2 r^{Q-n}$. Sum over all t in the interval $k \leqslant t \leqslant n-k-Q$. Let $k \to \infty$ sufficiently slowly with n. This shows that C_P in the proof of Lemma 4 satisfies

$$A^2 \; = \; \Big(\lim_{k\to\infty} A_\ell(k,P)\Big)^2 \leqslant \; C_P A \; \leqslant \; \Big(\lim_{k\to\infty} A_u(k)\Big)^2 \; = \; A^2.$$

The theorem follows. □

Proof of Theorem 3: We will show that the three hypotheses (i)-(iii) of Lemma 5 are satisfied with the choices $\gamma(n) = \log(Cn)$, $\alpha = r$, and $c = r$. The first, (i), is obvious.

For (ii), let $\ell, m_1, \ldots, m_\ell$ be fixed and let $k_1(n) < k_2(n) < \cdots < k_\ell(n)$ be sequences satisfying $k_i(n) = \log n + O(1)$. The expectation $\mathsf{E}\prod(\zeta_{k_i})_{m_i}$, when multiplied by $C(n)$, equals the number of compositions in which m_i parts of size k_i have been marked and linearly ordered, $1 \leqslant i \leqslant \ell$. Let $m = \sum_i m_i$ and let $(L_1, \ldots, L_m)$ be one of the $\binom{m}{m_1,\ldots,m_\ell}$ possible linear orders of m_1 k_1's, etc. Given a marked composition counted by Lemma 4, the linear orders may be imposed on the marked parts in $\prod m_i!$ ways. Hence,

$$C(n)\,\mathsf{E}\prod(\zeta_{k_i})_{m_i} \;\sim\; \binom{m}{m_1,\ldots,m_\ell}\prod m_i!\,\frac{A(Cn)^m r^{s-n}}{m!}$$

where $s = \sum_i m_i k_i$. Dividing both sides by $C(n) \sim Ar^{-n}$ and noting $Cn = r^{-\gamma(n)}$ completes the confirmation of hypothesis (ii).

Finally, the third hypothesis (iii) is given by Lemma 1 (c). □

11 Proof of Theorem 1

We recall that all logarithms are to the base $1/r$.

Proof of Theorem 1(a): Assertion (a) was proved in Lemma 3, except for the formula relating B and C. That relation was proved in the last part of the proof of Lemma 4. $\square$

Proof of Theorem 1(d): The final claim in (d) is easily proved by bounding the right side of (5). One can also show that the sum of the right side of (5) over $|k-\log n| > \omega_d(n)$ tends to zero. Thus it suffices to prove (5).

By Theorem 3 there is some $\omega(n) \to \infty$ such that

$$\begin{aligned} q_n(k) &\sim p(n)\left(\prod_{j=f(n)}^{k}\left(1-\exp\left(-r^{j-\log(Cn)}\right)\right)\right)\left(\prod_{j=k+1}^{n}\exp\left(-r^{j-\log(Cn)}\right)\right) \\ &= p(n)\left(\prod_{j=f(n)}^{k}\left(1-\exp\left(-Cnr^{j}\right)\right)\right)\exp\left(-Cn\sum_{j=k+1}^{n} r^{j}\right), \end{aligned}$$

where $f(n) = \lfloor \log(Cn) - \omega(n) \rfloor$ and $p(n)$ is the probability that a random composition of n contains all recurrent parts less than $f(n)$. With a little calculation, we see that Theorem 1(d) is equivalent to $p(n) \sim 1$. With ζ_j as in Lemma 6, we have

$$p(n) \ \geqslant\ 1 - \sum_{j\leqslant f(n)} \Pr(\zeta_j = 0) \ =\ 1 - o(1).$$

We now prove the final claim about strong concentration. Since the probability of being gap-free is bounded away from zero by previous claim in (d), it suffices to prove the validity of the statement for all compositions. The result now follows from (b). This proves Theorem 1(d). $\square$

Proof of Theorem 1(b,c): We first turn to the formula in (c). Since ζ_j in Theorem 3 is irrelevant for small j, we let ζ_j be as in Lemma 6 and $\omega(n)$ be any function that goes to infinity. Note that

$$\mathsf{E}(D_n) + \nu \ =\ \sum_{j=1}^{n} \Pr(\zeta_j \neq 0)$$

and so, by Lemma 6,

$$\begin{aligned} \mathsf{E}(D_n) + \nu \ &=\ o(1) \ +\sum_{j=1}^{\lfloor \log n + \omega(n)\rfloor - 1} 1 && (22) \\ &\quad + \sum_{j=\lfloor \log n - \omega(n)\rfloor}^{\lfloor \log n + \omega(n)\rfloor} \Pr(\zeta_j \neq 0) && (23) \\ &\quad + \sum_{j=\lfloor \log n + \omega(n)\rfloor + 1}^{n} \Pr(\zeta_j \neq 0). && (24) \end{aligned}$$

By Lemma 1(c)

$$\Pr(\zeta_j \neq 0) \;=\; O(nr^j) \;\text{ provided } j \to \infty \text{ with } n.$$

Thus the sum in (24) is $O(r^{\omega(n)}) = o(1)$. By the Poisson distribution, the terms in the sum (23) are $1 - \exp(-Cnr^j) + o(1)$ and so, if $\omega(n) \to \infty$ sufficiently slowly, that sum is

$$o(1) + \sum\Big(1 - \exp\big(-Cnr^j\big)\Big).$$

Since the sum of $\exp(-Cnr^j)$ over $j < \lfloor \log n - \omega(n) \rfloor$ is $o(1)$, we may replace 1 in the sum (22) with $1 - \exp(-Cnr^j)$. Finally,

$$\sum_{j > \lfloor \log n + \omega(n) \rfloor} \Big(1 - \exp\big(-Cnr^j\big)\Big) \;=\; o(1)$$

and so

$$\mathsf{E}(D_n) + \nu \;= \sum_{j \geqslant 0} \Big(1 - \exp\big(-Cnr^j\big)\Big) - 1 + o(1).$$

Let $f(x) = \sum_{j \geqslant 0}\Big(1 - \exp\big(-xr^j\big)\Big)$. Then $\mathsf{E}(D_n) = f(Cn) - 1 + o(1)$. We use the standard Mellin transform. (See [7, p.765], and also their Example B.5 which treats $r = 1/2$). It follows that

$$f(x) \;=\; \log x + \gamma \log e + \frac{1}{2} + P_0(x) + o(1),$$

where $P_0(x)$ is given by (2). This proves Theorem 1(c).

For the maximum part size M_n, we proceed in a similar manner:

$$\begin{aligned}
\mathsf{E}(M_n) \;&=\; \sum_{j=1}^{n} \Pr(M_n \geqslant j) \\
&=\; \sum_{j=1}^{\lfloor \log n - \omega(n) \rfloor - 1} 1 \;+ \sum_{j=\lfloor \log n - \omega(n) \rfloor}^{n} \Big(1 - \Pr(\wedge_{i \geqslant j}\{\zeta_j = 0\})\Big) + o(1) \\
&=\; \sum_{j=1}^{\lfloor \log n - \omega(n) \rfloor - 1} 1 \;+ \sum_{j=\lfloor \log n - \omega(n) \rfloor}^{n} \left(1 - \exp\left(-\frac{Cn}{1-r} r^j\right)\right) + o(1) \\
&=\; \sum_{j \geqslant 0}^{n} \left(1 - \exp\left(-\frac{Cn}{1-r} r^j\right)\right) - 1 + o(1) = f\left(\frac{Cn}{1-r}\right) - 1 + o(1) \\
&=\; \log\left(\frac{Cn}{1-r}\right) + \gamma \log e - \frac{1}{2} + P_0\left(\frac{Cn}{1-r}\right) + o(1).
\end{aligned}$$

It remains to prove the claims about $|D_n - \log n|$ and $|M_n - \log n|$. Since $D_n \leqslant M_n$, it suffices to establish the lower bound for D_n and the upper for M_n. The upper bound on M_n was proved in [3, Section 9]. In the proof of Theorem 1(d) we showed that $p(n) \sim 1$, which establishes the lower bound on D_n. $\square$

Proof of Theorem 1(e–g): Let $\Gamma := (\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$ be a sequence of i.i.d. geometric random variables with parameter $p = 1 - r$. Hitczenko and Knopfmacher [10] showed that the probability the sequence Γ is gap-free is given by the p_m in our (6) and they established the oscillation of p_m when $p \neq 1/2$.

Let $\omega(m)$ go to infinity arbitrarily slowly with m. Let M_m be the largest Γ_i.

By Theorem 1(b) $|M_m - \log m| < \omega(m)$, and, as was shown in the proof of Theorem 1(d), all recurrent parts less than $\log m - \omega(m)$ are asymptotically almost surely present in Γ. Let

$$\begin{aligned} \zeta_j &:= |\{i : \Gamma_i = j\}|, & \lambda_j &:= m(1-r)r^{j-1}, \\ k^- &:= \lfloor \log m - \omega(m) \rfloor, & k^+ &:= \lfloor \log m + \omega(m) \rfloor. \end{aligned}$$

When $k^- \leqslant k \leqslant k^+$,

$$\Pr(\zeta_j = k) \ \sim \ e^{-\lambda_j} \lambda_j^k / k!$$

by the standard Poisson approximation for i.i.d. rare random variables. It should be well-known that $\{\zeta_j : k^- \leqslant j \leqslant k^+\}$ are asymptotically independent, but we include a proof since we lack a reference. For all fixed positive integers $m_1, \ldots, m_j$, we have

$$\begin{aligned} \Pr\left(\wedge_{k^- \leqslant j \leqslant k^+} \{\zeta_j = m_j\}\right) &= \frac{m! (\lambda_{k^-}/m)^{m_{k^-}} \cdots (\lambda_{k^+}/m)^{m_{k^+}}}{(m_{k^-})! \cdots (m_{k^+})! (m - m_{k^-} + \cdots + m_{k^+})!} \\ &\quad \times \left(1 - \frac{\lambda_{k^-} + \ldots + \lambda_{k^+}}{m}\right)^{m - (m_{k^-} + \ldots + m_{k^+})} \\ &\sim \frac{\lambda_{k^-}^{m_{k^-}} \cdots \lambda_{k^+}^{m_{k^+}}}{(m_{k^-})! \cdots (m_{k^+})!} \exp(-(\lambda_{k^-} + \ldots + \lambda_{k^+})). \end{aligned}$$

Thus, with k the largest part,

$$\begin{aligned} p_m &\sim \sum_{k=k^-}^{k^+} \left(\prod_{j=k+1}^{k^+} e^{-\lambda_j} \right) \left(\prod_{j=k^-}^{k} (1 - e^{-\lambda_j}) \right) \\ &\sim \sum_{k=k^-}^{k^+} \exp\left(-m r^k\right) \prod_{j=k^-}^{k} \left(1 - \exp\left(-m(1-r)r^{j-1}\right)\right). \end{aligned} \tag{25}$$

Equation (25) is the same as the sum of (5) if $m = Cn/(1-r)$. However, (25) was derived under the assumption that m is an integer. We now treat (25) as a function of real variable m, say $f(m)$, and show that $f'(m) = o(1)$ as $m \to \infty$. It then follows that $f(x) \sim f(\lfloor x \rfloor)$ as $x \to \infty$ and we will be done. Call the terms in the sum (25) $T_k(m)$. We have

$$\begin{aligned} |T_k'(m)| &< \left| \frac{T_k'(m)}{T_k(m)} \right| = \left| (\ln(T_k(m))' \right| \leqslant r^k + \sum_{j=k^-}^{k} \frac{(1-r)r^{j-1}}{\exp\left(m(1-r)r^{j-1}\right) - 1} \\ &< r^k + \sum_{j=k^-}^{k} \frac{(1-r)r^{j-1}}{m(1-r)r^{j-1}} \leqslant r^{k^-} + \frac{k - k^- + 1}{m} < \frac{\omega_1(m)}{m} \end{aligned}$$

for some $\omega_1(m) \to \infty$ much slower than m. Since there are only $2\omega(m)$ values for k, $f'(m) = o(1)$.

The oscillation is associated with the imaginary poles, which are at $2k\pi i/\ln(1/q)$ in the notation of [10]. When the result is translated back from m to n, we obtain the same period as P in (2).

We now prove (f). It follows from Theorem 3 that

$$g_n(k) \;\sim \sum_{j>\log(Cn)-\omega(n)} \Pr(\zeta_j = k, \zeta_{j+1} = \zeta_{j+2} = \cdots = 0).$$

Setting $j = \ell + \lfloor \log(Cn) \rfloor$ and $\delta(n) = Cn - \lfloor \log(Cn) \rfloor$,

$$g_n(k) \sim \sum_{\ell=-\infty}^{\infty} \frac{r^{k(\ell-\delta(n))}}{k!} \prod_{i \geqslant \ell} \exp\left(-r^{i-\delta(n)}\right) \sim \sum_{\ell=-\infty}^{\infty} \frac{r^{k(\ell-\delta(n))}}{k!} \exp\left(\frac{-r^{\ell-\delta(n)}}{1-r}\right).$$

It follows from Poisson's summation formula [23] that

$$g_n(k) \;\sim\; \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k!} \exp(-2\pi i \ell t) r^{k(t-\delta(n))} \exp\left(\frac{-r^{t-\delta(n)}}{1-r}\right) dt.$$

Setting $z = \frac{r^{t-\delta(n)}}{1-r}$,

$$\begin{aligned}
g_n(k) \;&\sim\; \frac{(1-r)^k \log e}{k!} \sum_{\ell=-\infty}^{\infty} \exp(-2\pi i \ell(\delta(n) - \log(1-r))) \int_0^{\infty} e^{-z} z^{k-1+2\pi i \ell \log e} dz \\
&\sim\; \frac{(1-r)^k \log e}{k!} \sum_{\ell=-\infty}^{\infty} \Gamma\left(k + 2\pi i \ell \log e\right) \exp\left(-2\pi i \ell \log \frac{Cn}{1-r}\right) \\
&\sim\; \frac{(1-r)^k}{k!} P_k\left(\frac{Cn}{1-r}\right) + \frac{(1-r)^k \log e}{k}.
\end{aligned}$$

This completes the proof of (f).

We now prove (g). By Lemma 6 and Theorem 1(b) we may limit our attention to parts j for which $|j - \log n| \leqslant \omega(n)$. By Theorem 3, the probability that part j appears with multiplicity k is asymptotically $e^{-\mu_j} \mu_j^k / k!$ where $\mu_j = Cnr^j$. Using the Poisson summation formula as in the proof of (f), the expected number of parts of multiplicity k is asymptotic to

$$\begin{aligned}
&\frac{1}{k!} \sum_j \exp\left(-Cnr^j\right) \left(Cnr^j\right)^k \\
\sim\;& \frac{1}{k!} \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-2i\pi \ell t - r^{t-\delta(n)}\right) r^{k(t-\delta(n))} dt \\
\sim\;& \frac{\log e}{k!} \sum_{\ell=-\infty}^{\infty} \exp\left(-2i\pi \ell \log(Cn)\right) \Gamma\left(k + 2i\pi \ell \log e\right) \\
\sim\;& \frac{P_k(Cn)}{k!} + \frac{\log e}{k}.
\end{aligned}$$

The claim about $m_n(k)$ follows from the fact that $m_n(k) = \mathsf{E}(D_n(k)/D_n)$ and the tight concentration of D_n in (c)—an argument used by Louchard [20] for unrestricted compositions. $\square$

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