# New proofs of determinant evaluations related to plane partitions

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**Abstract**

We give a new proof of a determinant evaluation due to Andrews, which has been used to enumerate cyclically symmetric and descending plane partitions. We also prove some related results, including a $q$-analogue of Andrews's determinant.

**Keywords:** Plane partitions; Determinant evaluations; Orthogonal polynomials

## 1 Introduction

In 1979, George Andrews [1] managed to evaluate the determinant

$$\det_{0\leqslant m,n\leqslant N-1}\left(\delta_{mn}+\binom{x+m+n}{n}\right). \tag{1}$$

This allowed him to enumerate so called cyclically symmetric plane partitions (using the case $x=0$) and descending plane partitions ($x=2$). Andrews's proof, which takes up most of his 33 pages paper, amounts to partially working out the LU-factorization of the underlying matrix. This requires both clever guess-work and creative use of hypergeometric series identities. Later, Andrews and Stanton [3] found a shorter proof, using what Krattenthaler [13] has called "a magnificient factorization theorem" due to Mills, Robbins and Rumsey [18]. This proof can be simplified further, see [5, 12, 20]. It is our purpose to present a new and simple method for evaluating (1), using orthogonal polynomials. Roughly speaking, we compute (1) by viewing each matrix element as the

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scalar product of two Meixner–Pollaczek polynomials, with respect to the orthogonality measure for certain Wilson polynomials, see §3.

Our method can be used to prove further results. Ciucu, Eisenkölbl, Krattenthaler and Zare [4] found that

$$\det_{0\leqslant m,n\leqslant N-1}\left(\delta_{mn}+t\binom{x+m+n}{n}\right) \tag{2}$$

can be evaluated in closed form when $t^6=1$. Up to conjugation, this gives four cases, $t=\pm 1$ and $t=\pm e^{2\mathrm{i}\pi/3}$, the case $t=1$ being (1). Our proof of (1) can be modified to include the remaining three cases, see §4.

We will also obtain some new variations of (1). Note that evaluating (2) is equivalent to evaluating

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}+t(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};1\right)\right). \tag{3}$$

Indeed, by the Chu–Vandermonde summation, the ${}_2F_1$ equals $(b)_{m+n}/(b)_m(b)_n$. Dividing the $n$th column by $n!(b)_n$ then gives (2), with $x=b-1$. Using continuous dual Hahn polynomials rather than Wilson polynomials, we will evaluate

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}+t\,2^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{2}\right)\right) \tag{4}$$

whenever $t^4=1$ (giving three non-equivalent cases, $t=\pm 1$ and $t=\mathrm{i}$) and

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}+t\,3^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{3}\right)\right) \tag{5}$$

whenever $t^3=-1$ (giving two non-equivalent cases, $t=-1$, $t=e^{\mathrm{i}\pi/3}$), see §5. These results are related to weighted enumeration of alternating sign matrixes. Indeed, as we explain further below, the case $b=1$, $t=e^{2\mathrm{i}\pi/3}$ of (3) relates to the famous problem of enumerating alternating sign matrices of fixed size. Similarly, it follows from the work of Colomo and Pronko [6] that the case $b=1$, $t=\mathrm{i}$ of (4) is related to the 2-enumeration of alternating sign matrices and the case $b=1$, $t=e^{i\pi/3}$ of (5) to the 3-enumeration.

Another problem, already discussed in [1], is to obtain a $q$-analogue of Andrews's determinant. In the combinatorially most interesting cases, $x=0$ and $x=2$, such $q$-analogues were proved by Mills, Robbins and Rumsey [17], thereby settling conjectures of Macdonald [16] and Andrews [1]. However, until now nobody has found a $q$-analogue for the case of general $x$. We propose such an identity in Theorem 14 where, roughly speaking, the summable ${}_2F_1$ in (3) is replaced by a non-summable ${}_4\phi_3$. However, our Theorem 14 does not contain the $q$-analogues found by Mills, Robbins and Rumsey. It would be interesting to prove those results using the method of the present work. Further identities that one should look at can be found in [11, 14]. As an example, Guoce Xin conjectured an evaluation of

$$\det_{0\leqslant m,n\leqslant N-1}\left(\delta_{mn}-\binom{x+m+n}{n+1}\right),$$

which was published as [14, Conj. 35] and recently proved in [11].

We would like to acknowledge that our main idea is contained in the work of Colomo and Pronko [6, 7] on the six-vertex model. In [6], these authors found a new determinant formula for the partition function of the homogeneous six-vertex model with domain wall boundary conditions. At the "ice point", the Colomo–Pronko formula expresses the number of states of the model (on an $N \times N$ lattice) as

$$\det_{0 \leqslant m,n \leqslant N-1} \left( -e^{2i\pi/3} \delta_{mn} + e^{\mathrm{i}\pi/3} \binom{m+n}{n} \right), \tag{6}$$

which is essentially the case $x = 0$, $t = e^{2\mathrm{i}\pi/3}$ of (2). On the other hand, by the alternating sign matrix theorem [15, 22], the number of states is

$$\frac{1!4!7! \cdots (3N-2)!}{N!(N+1)!(N+2)! \cdots (2N-1)!}. \tag{7}$$

If we want to prove directly that (6) equals (7) we can proceed as follows. Let

$$\langle f, g \rangle_{\pm} = \pm \, \mathrm{PV} \int_{-\infty}^{\infty} f(x) g(x) \, \frac{e^{\pm \pi x}}{\sinh(3\pi x)} \, dx$$

and

$$\langle f, g \rangle = \langle f, g \rangle_+ + \langle f, g \rangle_- = 2 \int_{-\infty}^{\infty} f(x) g(x) \, \frac{\sinh(\pi x)}{\sinh(3\pi x)} \, dx.$$

Consider the determinant

$$D = \det_{0 \leqslant m,n \leqslant N-1} \left( \langle p_m, p_n \rangle \right),$$

where $p_n$ is a monic polynomial of degree $n$. By linearity in rows and columns, $D$ does not depend on the choice of $p_n$. Choosing $p_n$ as orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle_+$, $D$ essentially reduces to (6) [6]. On the other hand, choosing $p_n$ as orthogonal with respect to $\langle \cdot, \cdot \rangle$, $D$ becomes diagonal and can thus be evaluated [7]. (Choosing $p_n(x) = x^n$ gives a Hankel determinant, which is a limit case of the Izergin–Korepin formula [9] used by Kuperberg [15] in his proof of (7).) Essentially, our results are obtained by variations of this idea.

## 2 Preliminaries on orthogonal polynomials

For the benefit of the reader, we collect some facts on Wilson, continuous dual Hahn, Meixner–Pollaczek and Askey–Wilson polynomials, see [10]. We refer to [2] or [8] for the standard notation for hypergeometric and basic hypergeometric series used throughout the paper.

The *Wilson polynomials* are defined by

$$\begin{aligned} W_n \left( x^2; a_1, a_2, a_3, a_4 \right) &= (a_1 + a_2)_n (a_1 + a_3)_n (a_1 + a_4)_n \\ &\quad \times {}_4F_3 \left( \begin{matrix} -n, a_1 + a_2 + a_3 + a_4 + n - 1, a_1 + \mathrm{i}x, a_1 - \mathrm{i}x \\ a_1 + a_2, a_1 + a_3, a_1 + a_4 \end{matrix} ; 1 \right). \end{aligned} \tag{8}$$

This is a polynomial of degree $n$ in $x^2$ with leading coefficient

$$(-1)^n(a_1+a_2+a_3+a_4+n-1)_n.$$

If the parameters $a_k$ are all positive, Wilson polynomials satisfy the orthogonality relation

$$\frac{\Gamma(a_1+a_2+a_3+a_4)}{2\pi\prod_{1\leqslant j<k\leqslant 4}\Gamma(a_j+a_k)}\int_0^\infty\left|\frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)\Gamma(a_4+\mathrm{i}x)}{\Gamma(2ix)}\right|^2 \\ \times W_m\left(x^2;a_1,a_2,a_3,a_4\right)W_n\left(x^2;a_1,a_2,a_3,a_4\right)\,dx=h_n\,\delta_{mn}, \quad (9)$$

where

$$h_n=h_n^{\mathrm{W}}(a_1,a_2,a_3,a_4)=\frac{a_1+a_2+a_3+a_4-1}{a_1+a_2+a_3+a_4-1+2n}\frac{n!\prod_{1\leqslant j<k\leqslant 4}(a_j+a_k)_n}{(a_1+a_2+a_3+a_4-1)_n}.$$

Later, we will choose $a_1=0$. Then, the pole of the factor $\Gamma(a_1+ix)$ at $x=0$ is cancelled by the pole of $\Gamma(2ix)$. Thus, (9) remains valid for $a_1=0$ as long as the other parameters are positive.

The *continuous dual Hahn polynomials* are defined by

$$S_n\left(x^2;a_1,a_2,a_3\right)=(a_1+a_2)_n(a_1+a_3)_n\,{}_3F_2\left(\begin{matrix}-n,a_1+\mathrm{i}x,a_1-\mathrm{i}x\\a_1+a_2,a_1+a_3\end{matrix};1\right).$$

This is a polynomial of degree $n$ in $x^2$ with leading coefficient $(-1)^n$. If all $a_k$ are positive, then

$$\frac{1}{2\pi\prod_{1\leqslant j<k\leqslant 3}\Gamma(a_j+a_k)}\int_0^\infty\left|\frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)}{\Gamma(2ix)}\right|^2 \\ \times S_m\left(x^2;a_1,a_2,a_3\right)S_n\left(x^2;a_1,a_2,a_3\right)\,dx=h_n\,\delta_{mn}, \quad (10)$$

where

$$h_n=h_n^{\mathrm{CDH}}(a_1,a_2,a_3)=n!(a_1+a_2)_n(a_1+a_3)_n(a_2+a_3)_n.$$

Similarly as for (9), (10) holds also for $a_1=0$ as long as the other parameters are positive.

The *Meixner–Pollaczek polynomials* are defined by

$$P_n^{(\lambda)}(x;\phi)=\frac{(2\lambda)_n}{n!}\,e^{\mathrm{i}n\phi}\,{}_2F_1\left(\begin{matrix}-n,\lambda+\mathrm{i}x\\2\lambda\end{matrix};1-e^{-2\mathrm{i}\phi}\right). \quad (11)$$

This is a polynomial in $x$ of degree $n$ with leading coefficient

$$\frac{(2\sin\phi)^n}{n!}.$$

For $\lambda>0$ and $0<\phi<\pi$,

$$\frac{(2\sin\phi)^{2\lambda}}{2\pi\Gamma(2\lambda)}\int_{-\infty}^{\infty}e^{(2\phi-\pi)x}\left|\Gamma(\lambda+\mathrm{i}x)\right|^2P_m^{(\lambda)}(x;\phi)P_n^{(\lambda)}(x;\phi)\,dx=h_n\delta_{mn}, \quad (12)$$

where

$$h_n = h_n^{\mathrm{MP}}(\lambda) = \frac{(2\lambda)_n}{n!}.$$

We will need the expansion formula

$$P_n^{(\lambda)}\left(x; \frac{\pi}{2} + \phi\right) = (-1)^n \frac{(2\lambda)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(2\lambda)_k} (2\sin\phi)^{n-k} P_k^{(\lambda)}\left(x; \frac{\pi}{2} - \phi\right), \tag{13}$$

which can be proved by inserting (11), changing the order of summation and using the binomial theorem.

Finally, the *Askey–Wilson polynomials* are defined by

$$p_n(\cos\theta; a_1, a_2, a_3, a_4|q) = \frac{(a_1a_2, a_1a_3, a_1a_4; q)_n}{a_1^n} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, a_1a_2a_3a_4q^{n-1}, a_1e^{\mathrm{i}\theta}, a_1e^{-\mathrm{i}\theta} \\ a_1a_2, a_1a_3, a_1a_4 \end{matrix}; q, q\right).$$

This is a polynomial in $\cos\theta$ of degree $n$ with leading coefficient

$$2^n(a_1a_2a_3a_4q^{n-1}; q)_n.$$

We write the orthogonality using $e^{\mathrm{i}\theta}$ rather than $\cos\theta$ as integration variable. Assuming

$$|q|, |a_1|, |a_2|, |a_3|, |a_4| < 1, \tag{14}$$

we have

$$\frac{(q;q)_\infty \prod_{1\leqslant j<k\leqslant 4}(a_ja_k;q)_\infty}{2(a_1a_2a_3a_4;q)_\infty} \oint \frac{(z^2, z^{-2}; q)_\infty}{(a_1z, a_1z^{-1}, a_2z, a_2z^{-1}, a_3z, a_3z^{-1}, a_4z, a_4z^{-1}; q)_\infty} \times p_m\left(\frac{z+z^{-1}}{2}; a_1, a_2, a_3, a_4|q\right) p_n\left(\frac{z+z^{-1}}{2}; a_1, a_2, a_3, a_4|q\right) \frac{dz}{2\pi \mathrm{i} z} = h_n\delta_{mn}, \tag{15}$$

where the integral is over the positively oriented unit circle and

$$h_n = h_n^{\mathrm{AW}}(a_1, a_2, a_3, a_4; q) = \frac{1 - a_1a_2a_3a_4q^{-1}}{1 - a_1a_2a_3a_4q^{2n-1}} \frac{(q;q)_n \prod_{1\leqslant j<k\leqslant 4}(a_ja_k;q)_n}{(a_1a_2a_3a_4q^{-1};q)_n}.$$

We will need the fact that (15) remains valid when $a_1 = 1$, as long as the other conditions in (14) hold. The reason is that the double zero of the factor $(a_1z, a_1z^{-1}; q)_\infty$ at $z = 1$ is cancelled by the double zero of $(z^2, z^{-2}; q)_\infty$.

## 3 Proof of Andrews's determinant

We first explain the main idea behind our proof in general terms. Suppose we are given three symmetric bilinear forms $\langle\cdot,\cdot\rangle_k$, $k = -1, 0, 1$, which are defined on polynomials and related by

$$\langle f, g\rangle_0 = \langle f, g\rangle_1 + \langle f, g\rangle_{-1}. \tag{16}$$

In the generic situation, there exist monic polynomials $p_n^{(k)}$ of degree $n$, with $\langle p_m^{(k)}, p_n^{(k)}\rangle_k = h_n^{(k)}\delta_{mn}$. We assume that this is the case for $k=0$ and $k=1$.

Consider the determinant

$$D = \det_{0\leqslant m,n\leqslant N-1}\left(\langle p_m, p_n\rangle_0\right), \tag{17}$$

with $p_n$ a monic polynomial of degree $n$. By linearity in rows and columns, $D$ is independent of the choice of $p_n$. In particular, choosing $p_n = p_n^{(0)}$ we find that $D = h_0^{(0)} h_1^{(0)} \cdots h_{N-1}^{(0)}$. Choosing $p_n = p_n^{(1)}$ then gives the key identity

$$\det_{0\leqslant m,n\leqslant N-1}\left(h_m^{(1)}\delta_{mn} + \langle p_m^{(1)}, p_n^{(1)}\rangle_{-1}\right) = \prod_{n=0}^{N-1} h_n^{(0)}. \tag{18}$$

In the cases that we will consider, the bilinear forms will be defined by

$$\langle f, g\rangle_k = \int_{-\infty}^{\infty} f(x)g(x)w_k(x)\,dx, \qquad k=-1,0,1,$$

where $w_0 = w_1 + w_{-1}$. In particular, we will show that if we take

$$w_{\pm 1}(x) = \frac{3^{(b+2)/2}}{4\pi\Gamma(b)}\, e^{\pm\pi x}\left|\Gamma\left(\frac{b}{2} + 3\mathrm{i}x\right)\right|^2, \tag{19a}$$

$$w_0(x) = \frac{3^{(b+2)/2}}{2\pi\Gamma(b)}\, \cosh(\pi x)\left|\Gamma\left(\frac{b}{2} + 3\mathrm{i}x\right)\right|^2, \tag{19b}$$

where $b > 0$, then (18) becomes Andrews's determinant evaluation (1), with $x = b-1$.

Let us first compute the polynomials $p_n^{(0)}$. Since $w_0$ is even, we can write $p_{2n}^{(0)}(x) = q_n(x^2)$, $p_{2n+1}^{(0)}(x) = x\,r_n(x^2)$, where $q_n$ and $r_n$ are monic orthogonal polynomials on the positive half-line with weight $w_0$ and $x^2 w_0$, respectively.

Recall that the gamma function satisfies the duplication formula

$$(2\pi)^{1/2}\Gamma(2x) = 2^{2x-1/2}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right),$$

the triplication formula

$$2\pi\Gamma(3x) = 3^{3x-1/2}\Gamma(x)\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right)$$

and the reflection formula, which we write as

$$\Gamma\left(\frac{1}{2}+\mathrm{i}x\right)\Gamma\left(\frac{1}{2}-\mathrm{i}x\right) = \frac{\pi}{\cosh(\pi x)}.$$

Combining these identities, one readily writes

$$\begin{aligned} w_0(x) &= \frac{3^{3b/2}}{32\pi^3\Gamma(b)} \left| \frac{\Gamma(\mathrm{i}x)\Gamma(\mathrm{i}x+b/6)\Gamma(\mathrm{i}x+b/6+1/3)\Gamma(\mathrm{i}x+b/6+2/3)}{\Gamma(2\mathrm{i}x)} \right|^2 \\ &= \frac{\Gamma(a_1+a_2+a_3+a_4)}{4\pi\prod_{1\leqslant j<k\leqslant 4}\Gamma(a_j+a_k)} \left| \frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)\Gamma(a_4+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)} \right|^2, \end{aligned}$$

with $(a_1,a_2,a_3,a_4) = (0, b/6, b/6+1/3, b/6+2/3)$. Comparing this with (9), we find that

$$p_{2n}^{(0)}(x) = \frac{(-1)^n}{(b/2+n)_n} W_n\left(x^2; 0, \frac{b}{6}, \frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}\right) \tag{20a}$$

and that

$$h_{2n}^{(0)} = \frac{1}{(b/2+n)_n^2} h_n^{\mathrm{W}}\left(0, \frac{b}{6}, \frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}\right) = \frac{n!(b/2)_n(b/2)_{3n}(b+1)_{3n}}{3^{6n}(b/2)_{2n}(b/2+1)_{2n}}. \tag{20b}$$

Since $x^2|\Gamma(ix)|^2 = |\Gamma(ix+1)|^2$, we can also write

$$x^2 w_0(x) = C\frac{\Gamma(b_1+b_2+b_3+b_4)}{4\pi\prod_{1\leqslant j<k\leqslant 4}\Gamma(b_j+b_k)} \left| \frac{\Gamma(b_1+\mathrm{i}x)\Gamma(b_2+\mathrm{i}x)\Gamma(b_3+\mathrm{i}x)\Gamma(b_4+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)} \right|^2,$$

with $(b_1,b_2,b_3,b_4) = (1, b/6, b/6+1/3, b/6+2/3)$ and

$$C = \frac{b_2b_3b_4}{b_2+b_3+b_4} = \frac{b(b+4)}{2^2\cdot 3^3}.$$

It follows that

$$p_{2n+1}^{(0)}(x) = \frac{(-1)^n}{(b/2+n+1)_n} x\, W_n\left(x^2; 1, \frac{b}{6}, \frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}\right) \tag{20c}$$

and

$$\begin{aligned} h_{2n+1}^{(0)} &= \frac{b(b+4)}{2^2\cdot 3^3(b/2+n+1)_n^2} h_n^{\mathrm{W}}\left(1, \frac{b}{6}, \frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}\right) \\ &= \frac{b(b+4)}{4}\frac{n!(b/2+1)_n(b/2+3)_{3n}(b+1)_{3n}}{3^{6n+3}(b/2+1)_{2n}(b/2+2)_{2n}}. \end{aligned} \tag{20d}$$

As for the polynomials $p_n^{(\pm1)}$, it follows from (12) that

$$p_n^{(\pm1)}(x) = \frac{n!}{3^{3n/2}} P_n^{(b/2)}\left(3x, \frac{\pi}{2} \pm \frac{\pi}{6}\right)$$

and that

$$h_n^{(\pm1)} = \langle p_m^{(\pm1)}, p_n^{(\pm1)}\rangle_{\pm1} = \frac{(n!)^2}{2\cdot 3^{3n}} h_n^{\mathrm{MP}}(b/2) = \frac{n!\,(b)_n}{2\cdot 3^{3n}}. \tag{21}$$

To compute $\langle p_m^{(1)}, p_n^{(1)}\rangle_{-1}$, we use (13) to expand

$$p_n^{(1)} = (-1)^n (b)_n \sum_{k=0}^{n} \frac{(-n)_k}{k!(b)_k}\, 3^{3(k-n)/2} p_k^{(-1)}.$$

It follows that

$$\langle p_m^{(1)}, p_n^{(1)}\rangle_{-1} = \frac{(-1)^{m+n}(b)_m(b)_n}{2\cdot 3^{3(m+n)/2}} \sum_{k=0}^{\min(m,n)} \frac{(-m)_k(-n)_k}{k!(b)_k} = \frac{(-1)^{m+n}(b)_{m+n}}{2\cdot 3^{3(m+n)/2}}, \qquad (22)$$

where the final step is the Chu–Vandermonde summation.

By (21) and (22), the general determinant identity (18) is now reduced to

$$\det_{0\leqslant m,n\leqslant N-1}\left(\frac{m!\,(b)_m}{2\cdot 3^{3m}}\,\delta_{mn} + (-1)^{m+n}\frac{(b)_{m+n}}{2\cdot 3^{3(m+n)/2}}\right) = \prod_{n=0}^{[(N-1)/2]} h_{2n}^{(0)} \prod_{n=0}^{[(N-2)/2]} h_{2n+1}^{(0)},$$

with $h_n^{(0)}$ as in (20). Multiplying the $n$th row and $n$th column with $(-1)^n 2^{1/2} 3^{3n/2}$, for each $n$, we arrive at the following result.

**Theorem 1** (Andrews)**.** *The following determinant evaluation holds:*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!\,(b)_m\delta_{mn} + (b)_{m+n}\right) = 2^N\left(\frac{b(b+4)}{4}\right)^{\left[\frac{N}{2}\right]}$$
$$\times \prod_{n=0}^{[(N-1)/2]} \frac{n!(b/2)_n(b/2)_{3n}(b+1)_{3n}}{(b/2)_{2n}(b/2+1)_{2n}} \prod_{n=0}^{[(N-2)/2]} \frac{n!(b/2+1)_n(b/2+3)_{3n}(b+1)_{3n}}{(b/2+1)_{2n}(b/2+2)_{2n}}.$$

Dividing the $n$th column by $n!\,(b)_n$ and writing

$$\frac{(b)_{m+n}}{n!\,(b)_n} = \binom{b+m+n-1}{n},$$

we see that Theorem 1 is indeed equivalent to the evaluation of (1).

## 4 The CEKZ variations

We will now modify our proof to cover the three variations of Andrews's determinant discovered by Ciucu, Eisenkölbl, Krattenthaler and Zare [4].

For the first variation, we take

$$w_{\pm 1}(x) = \frac{3^{(b+2)/2}}{4\pi\Gamma(b)}\, e^{\pm\pi x}\Gamma\left(\frac{b}{2}+3\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-3\mathrm{i}x\right), \qquad (23a)$$

$$w_0(x) = \frac{3^{(b+2)/2}}{2\pi\Gamma(b)}\, \cosh(\pi x)\Gamma\left(\frac{b}{2}+3\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-3\mathrm{i}x\right). \qquad (23b)$$

In other words, $w_k$ are obtained by multiplying the weights in (19) with

$$\frac{\Gamma(b/2+3\mathrm{i}x+1)}{\Gamma(b/2+3\mathrm{i}x)} = 3\mathrm{i}\left(x-\frac{\mathrm{i}b}{6}\right).$$

Recall that, in general, if $p_n$ are monic orthogonal polynomials with

$$\int p_m(x)p_n(x)\,d\mu(x) = h_n\delta_{mn},$$

then

$$\tilde{p}_n(x) = \frac{p_{n+1}(x) - \frac{p_{n+1}(a)}{p_n(a)}\,p_n(x)}{x-a}$$

are monic orthogonal polynomials with

$$\int \tilde{p}_m(x)\tilde{p}_n(x)\,(x-a)\,d\mu(x) = \tilde{h}_n\delta_{mn},$$

where

$$\tilde{h}_n = -\frac{p_{n+1}(a)}{p_n(a)}\,h_n.$$

In the case at hand, it follows that

$$h_n^{(0)}\Big|_{w_0 \text{ as in (23b)}} = -3\mathrm{i}\frac{p_{n+1}^{(0)}(\mathrm{i}b/6)}{p_n^{(0)}(\mathrm{i}b/6)}\,h_n^{(0)}\Big|_{w_0 \text{ as in (19b)}},$$

where the quantities on the right-hand side are given in (20). Applying the explicit formula (8) with $a_1$ and $a_2$ interchanged, both ${}_4F_3$:s reduce to a single term, and we find that

$$\begin{aligned} p_{2n}^{(0)}\left(\frac{\mathrm{i}b}{6}\right)\Big|_{w_0 \text{ as in (19b)}} &= \frac{(-1)^n(b/6)_n(b/3+1/3)_n(b/3+2/3)_n}{(b/2+n)_n}, \\ p_{2n+1}^{(0)}\left(\frac{\mathrm{i}b}{6}\right)\Big|_{w_0 \text{ as in (19b)}} &= \frac{\mathrm{i}b}{6}\frac{(-1)^n(b/6+1)_n(b/3+1/3)_n(b/3+2/3)_n}{(b/2+n+1)_n}. \end{aligned}$$

After simplification, this gives

$$h_{2n}^{(0)}\Big|_{w_0 \text{ as in (23b)}} = \frac{b}{2}\frac{n!(b/2+1)_n(b/2+1)_{3n}(b+1)_{3n}}{3^{6n}(b/2+1)_{2n}^2}, \tag{24a}$$

$$h_{2n+1}^{(0)}\Big|_{w_0 \text{ as in (23b)}} = \frac{b(b+1)(b+4)}{2}\frac{n!(b/2+1)_n(b/2+3)_{3n}(b+3)_{3n}}{3^{6n+4}(b/2+2)_{2n}^2}. \tag{24b}$$

The remaining quantities that we need can be obtained from the following Lemma. We formulate it so as to cover also some cases needed in §5.

**Lemma 2.** *For $b$ and $t$ positive and $-\pi/2 < \phi < \pi/2$, define the pairing*

$$\langle p, q\rangle_\phi = \frac{t(2\cos\phi)^{b+1}}{2\pi\Gamma(b+1)} \int_{-\infty}^{\infty} p(x)q(x)\, e^{2\phi t x}\Gamma\left(\frac{b}{2}+t\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-t\mathrm{i}x\right)\, dx.$$

*Then, the rescaled Meixner–Pollaczek polynomials*

$$p_n(x) = \frac{n!}{(2t\cos\phi)^n}\, P_n^{((b+1)/2)}\left(tx-\frac{\mathrm{i}}{2};\frac{\pi}{2}+\phi\right)$$

*are monic and satisfy the orthogonality relation*

$$\langle p_m, p_n\rangle_\phi = \frac{e^{\mathrm{i}\phi} n!(b+1)_n}{(2t\cos\phi)^{2n}}\,\delta_{mn} \tag{25}$$

*as well as*

$$\langle p_m, p_n\rangle_{-\phi} = e^{-\mathrm{i}\phi}\left(-\frac{\tan\phi}{t}\right)^{m+n}(b+1)_m(b+1)_n\, {}_2F_1\left(\begin{matrix}-m,-n\\ b+1\end{matrix};\frac{1}{4\sin^2\phi}\right). \tag{26}$$

*Proof.* Consider integrals of the form

$$\frac{t(2\cos\phi)^{b+1}}{2\pi\Gamma(b+1)} \int_{-\infty}^{\infty} p(x)\, e^{2\phi t x}\Gamma\left(\frac{b}{2}+t\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-t\mathrm{i}x\right)\, dx \tag{27}$$

with $p$ a polynomial. If we replace $x \mapsto x + \mathrm{i}/2t$ and then shift the contour of integration back to the real line, the value of the integral does not change. This is true since the contour does not cross any poles of the integrand and since, by [2, Cor. 1.4.4], for large values of $|\operatorname{Re} x|$ one may estimate

$$\left|\Gamma\left(\frac{b}{2}+1+t\mathrm{i}x\right)\Gamma\left(\frac{b}{2}-t\mathrm{i}x\right)\right| \leqslant C|\operatorname{Re} x|^b e^{-\pi t|\operatorname{Re} x|}$$

uniformly in any vertical strip. Thus, making also a further change of variables $x \mapsto x/t$, we find that (27) equals

$$\frac{(2\cos\phi)^{b+1}e^{\mathrm{i}\phi}}{2\pi\Gamma(b+1)} \int_{-\infty}^{\infty} p\left(\frac{x+\mathrm{i}/2}{t}\right) e^{2\phi x}\left|\Gamma\left(\frac{b+1}{2}+\mathrm{i}x+1\right)\right|^2 dx.$$

The orthogonality (25) then follows from (12). Moreover, (13) gives

$$p_n^{(\phi)} = (-1)^n(b+1)_n \sum_{k=0}^{n} \frac{(-n)_k}{k!(b+1)_k}\left(\frac{\tan\phi}{t}\right)^{n-k} p_k^{(-\phi)},$$

where we indicate the $\phi$-dependence of the polynomials $p_n$. Combining this with (25), with $\phi$ replaced by $-\phi$, gives (26). □

In the case at hand, it follows from Lemma 2 that the monic orthogonal polynomials with respect to $w_1$ are given by

$$p_n^{(1)}(x) = \frac{n!}{3^{3n/2}}\, P_n^{((b+1)/2)}\left(3x - \frac{\mathrm{i}}{2}, \frac{2\pi}{3}\right)$$

and that

$$h_n^{(1)} = \frac{e^{\mathrm{i}\pi/6}(b)_{n+1} n!}{2\cdot 3^{3n+1/2}}, \tag{28}$$

$$\langle p_m^{(1)}, p_n^{(1)}\rangle_{-1} = (-1)^{m+n}\frac{e^{-\mathrm{i}\pi/6}(b)_{m+n+1}}{2\cdot 3^{(3m+3n+1)/2}}. \tag{29}$$

Plugging (24), (28) and (29) into (18), replacing $b$ by $b-1$ and simplifying, we recover the following result.

**Theorem 3** (Ciucu, Eisenkölbl, Krattenthaler and Zare)**.** *One has*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!\,(b)_m\delta_{mn} - e^{2\mathrm{i}\pi/3}(b)_{m+n}\right) = \left(e^{-\mathrm{i}\pi/6}\sqrt{3}\right)^N \left(\frac{b(b+3)}{3}\right)^{\left[\frac{N}{2}\right]}$$
$$\times \prod_{n=0}^{[(N-1)/2]} \frac{n!\left(\frac{b+1}{2}\right)_n \left(\frac{b+1}{2}\right)_{3n}(b)_{3n}}{\left(\frac{b+1}{2}\right)_{2n}^2} \prod_{n=0}^{[(N-2)/2]} \frac{n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+5}{2}\right)_{3n}(b+2)_{3n}}{\left(\frac{b+3}{2}\right)_{2n}^2}.$$

The second variation is obtained by choosing $w_1$ as in (23a) but replacing $w_{-1}$ by its negative. Then,

$$w_0(x) = \frac{3^{(b+2)/2}}{2\pi\Gamma(b)}\,\sinh(\pi x)\Gamma\left(\frac{b}{2}+3\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-3\mathrm{i}x\right). \tag{30}$$

Since $(x+\mathrm{i}b/6)w_0(x)$ is odd, the monic orthogonal polynomials with respect to $w_0$ can be constructed as $p_{2n}^{(0)}(x) = s_n(x^2)$, $p_{2n+1}^{(0)}(x) = (x+\mathrm{i}b/6)\,t_n(x^2)$, where $s_n$ are orthogonal on the positive half-line with respect to $(w_0(x)+w_0(-x))/2 = xw_0(x)/(x-\mathrm{i}b/6)$ and $t_n$ are orthogonal with respect to $x(x+\mathrm{i}b/6)w_0(x)$.

To identify these polynomials, we write

$$\frac{x}{x-\mathrm{i}b/6}\,w_0(x) = \frac{\mathrm{i}b}{2\cdot 3^{1/2}}\frac{\Gamma(a_1+a_2+a_3+a_4)}{4\pi\prod_{1\leqslant j<k\leqslant 4}\Gamma(a_j+a_k)}$$
$$\times\left|\frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)\Gamma(a_4+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$
$$x\left(x+\frac{\mathrm{i}b}{6}\right)w_0(x) = \frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{7/2}}\frac{\Gamma(b_1+b_2+b_3+b_4)}{4\pi\prod_{1\leqslant j<k\leqslant 4}\Gamma(b_j+b_k)}$$
$$\times\left|\frac{\Gamma(b_1+\mathrm{i}x)\Gamma(b_2+\mathrm{i}x)\Gamma(b_3+\mathrm{i}x)\Gamma(b_4+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$

where

$$(a_1,a_2,a_3,a_4)=\left(\frac{1}{2},\frac{b}{6},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3}\right),$$
$$(b_1,b_2,b_3,b_4)=\left(\frac{1}{2},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3},\frac{b}{6}+1\right).$$

It follows that

$$p_{2n}^{(0)}(x)=\frac{(-1)^n}{(b/2+n+1/2)_n}\,W_n\left(x^2;\frac{1}{2},\frac{b}{6},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3}\right),$$
$$p_{2n+1}^{(0)}(x)=\frac{(-1)^n}{(b/2+n+3/2)_n}\left(x+\frac{\mathrm{i}b}{6}\right)W_n\left(x^2;\frac{1}{2},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3},\frac{b}{6}+1\right)$$

and that

$$\begin{aligned}h_{2n}^{(0)}&=\frac{\mathrm{i}b}{2\cdot 3^{1/2}(b/2+n+1/2)_n^2}\,h_n^{\mathrm{W}}\left(\frac{1}{2},\frac{b}{6},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3}\right)\\&=\frac{\mathrm{i}b}{2}\frac{n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+3}{2}\right)_{3n}(b+1)_{3n}}{3^{6n+1/2}\left(\frac{b+1}{2}\right)_{2n}\left(\frac{b+3}{2}\right)_{2n}},\end{aligned}\tag{31a}$$

$$\begin{aligned}h_{2n+1}^{(0)}&=\frac{1}{(b/2+n+3/2)_n^2}\frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{7/2}}\,h_n^{\mathrm{W}}\left(\frac{1}{2},\frac{b}{6}+\frac{1}{3},\frac{b}{6}+\frac{2}{3},\frac{b}{6}+1\right)\\&=\frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{6n+7/2}}\frac{n!(b/2+3/2)_n(b/2+5/2)_{3n}(b+3)_{3n}}{(b/2+3/2)_{2n}(b/2+5/2)_{2n}}.\end{aligned}\tag{31b}$$

Since (28) is still valid and (29) holds up to a change of sign, we conclude that

$$\det_{0\leqslant m,n\leqslant N-1}\left(\frac{e^{\mathrm{i}\pi/6}m!\,(b)_{m+1}}{2\cdot 3^{(3m+1)/2}}\,\delta_{mn}+(-1)^{m+n+1}\frac{e^{-\mathrm{i}\pi/6}(b)_{m+n+1}}{2\cdot 3^{(3m+3n+1)/2}}\right)=\prod_{n=0}^{N-1}h_n^{(0)},$$

with $h_n^{(0)}$ as in (31). Replacing $b$ with $b-1$ and simplifying, we arrive at the following result.

**Theorem 4** (Ciucu, Eisenkölbl, Krattenthaler and Zare)**.** *One has*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!\,(b)_m\delta_{mn}+e^{2\mathrm{i}\pi/3}(b)_{m+n}\right)$$
$$=e^{\mathrm{i}\pi N/3}\big(b(b+1)\big)^{\left[\frac{N}{2}\right]}\prod_{n=0}^{[(N-1)/2]}\frac{n!\left(\frac{b}{2}\right)_n\left(\frac{b+2}{2}\right)_{3n}(b)_{3n}}{\left(\frac{b}{2}\right)_{2n}\left(\frac{b+2}{2}\right)_{2n}}\prod_{n=0}^{[(N-2)/2]}\frac{n!\left(\frac{b+2}{2}\right)_n\left(\frac{b+4}{2}\right)_{3n}(b+2)_{3n}}{\left(\frac{b+2}{2}\right)_{2n}\left(\frac{b+4}{2}\right)_{2n}}.$$

For the final variation, we choose $w_1$ as in (19a), but replace $w_{-1}$ by its negative. Then,

$$w_0(x)=\frac{3^{(b+2)/2}}{2\pi\Gamma(b)}\sinh(\pi x)\left|\Gamma\left(\frac{b}{2}+3\mathrm{i}x\right)\right|^2.\tag{32}$$

Since $\langle 1,1\rangle_0 = 0$, there does not exist a system of orthogonal polynomials with respect to $w_0$. Thus, (18) is not applicable. However, we can compute the determinant (17) using orthogonal polynomials with respect to the weight $xw_0(x)$.

More generally, consider the determinant (17), when the scalar product is given by integration against an odd weight function $w_0$. Suppose there exist monic orthogonal polynomials $q_n$ with

$$\int_{-\infty}^{\infty} q_m(x^2)q_n(x^2)\, xw_0(x)\, dx = 2\int_0^{\infty} q_m(x^2)q_n(x^2)\, xw_0(x)\, dx = c_n\delta_{mn}.$$

Then, the monic polynomials $p_{2n}(x) = q_n(x^2)$, $p_{2n+1}(x) = xq_n(x^2)$ satisfy

$$\langle p_m, p_n\rangle_0 = \begin{cases} c_k, & \{m,n\} = \{2k, 2k+1\}, \\ 0, & \text{else.} \end{cases}$$

Choosing $p_n$ in this way, $D$ reduces to the block-diagonal determinant

$$\begin{vmatrix} 0 & c_0 & 0 & 0 & \dots & 0 \\ c_0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & c_1 & & \\ 0 & 0 & c_1 & 0 & & \\ \vdots & & & & \ddots & \\ 0 & & & & & 0 \end{vmatrix}.$$

Thus, as a substitute for (18) we have

$$\det_{0\leqslant m,n\leqslant N-1}\left(h_m^{(1)}\delta_{mn} + \langle p_m^{(1)}, p_n^{(1)}\rangle_{-1}\right) = \begin{cases} (-1)^{N/2}\left(c_0c_1\cdots c_{(N-2)/2}\right)^2, & N \text{ even}, \\ 0, & N \text{ odd}. \end{cases} \tag{33}$$

In the case at hand, we observe that

$$xw_0(x)\Big|_{w_0 \text{ as in (32)}} = \frac{1}{3\mathrm{i}}\frac{x}{x - \mathrm{i}b/6}\, w_0(x)\Big|_{w_0 \text{ as in (30)}},$$

which gives

$$c_n = \frac{1}{3\mathrm{i}}\, h_{2n}^{(0)}\Big|_{\text{as in (31a)}} = \frac{b}{2}\cdot\frac{n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+3}{2}\right)_{3n}(b+1)_{3n}}{3^{6n+3/2}\left(\frac{b+1}{2}\right)_{2n}\left(\frac{b+3}{2}\right)_{2n}}.$$

Since (21) holds and (22) holds up to a change of sign, (33) can be simplified to the following form.

**Theorem 5** (Ciucu, Eisenkölbl, Krattenthaler and Zare)**.** *When $N$ is even,*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!\,(b)_m\delta_{mn} - (b)_{m+n}\right) = (-1)^{\frac{N}{2}} b^N \prod_{n=0}^{(N-2)/2}\left(\frac{n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+3}{2}\right)_{3n}(b+1)_{3n}}{\left(\frac{b+1}{2}\right)_{2n}\left(\frac{b+3}{2}\right)_{2n}}\right)^2,$$

*whereas if $N$ is odd the determinant vanishes.*

## 5 Further variations

It is natural to look for further interesting specializations of (18). We have not found any more cases that are as nice as Andrews's determinant in the sense that the quantities $h_n^{(0)}$, $h_n^{(1)}$ and $\langle p_m^{(1)}, p_n^{(1)} \rangle_{-1}$ all factor completely. However, from the viewpoint of orthogonal polynomials, there are five particularly natural cases based on continuous Hahn polynomials rather than Wilson polynomials. As we mentioned in the introduction, some of the resulting determinant evaluations are related to weighted enumeration of alternating sign matrices. Since the computations are completely parallel to those in §3–4, we will be rather brief.

In the first of these five cases, we choose the weight functions as

$$\begin{aligned} w_{\pm 1}(x) &= \frac{2^{b/2}}{2\pi\Gamma(b)}\, e^{\pm\pi x} \left|\Gamma\left(\frac{b}{2} + 2\mathrm{i}x\right)\right|^2, \\ w_0(x) &= \frac{2^{b/2}}{\pi\Gamma(b)}\, \cosh(\pi x) \left|\Gamma\left(\frac{b}{2} + 2\mathrm{i}x\right)\right|^2, \end{aligned} \tag{34}$$

with $b > 0$.

With $(a_1, a_2, a_3) = (0, b/4, b/4 + 1/2)$ and $(b_1, b_2, b_3) = (1, b/4, b/4 + 1/2)$,

$$\begin{aligned} w_0(x) &= \frac{1}{4\pi\Gamma(a_1+a_2)\Gamma(a_1+a_3)\Gamma(a_2+a_3)} \left|\frac{\Gamma(a_1+ix)\Gamma(a_2+ix)\Gamma(a_3+ix)}{\Gamma(2ix)}\right|^2, \\ x^2 w_0(x) &= \frac{b(b+2)}{16} \cdot \frac{1}{4\pi\Gamma(b_1+b_2)\Gamma(b_1+b_3)\Gamma(b_2+b_3)} \left|\frac{\Gamma(b_1+ix)\Gamma(b_2+ix)\Gamma(b_3+ix)}{\Gamma(2ix)}\right|^2. \end{aligned}$$

Exactly as in the proof of Theorem 1, it follows that

$$\begin{aligned} h_{2n}^{(0)}(x) &= h_n^{\mathrm{CDH}}\left(0, \frac{b}{4}, \frac{b}{4} + \frac{1}{2}\right) = \frac{n!\left(\frac{b+1}{2}\right)_n \left(\frac{b}{2}\right)_{2n}}{4^n}, \\ h_{2n+1}^{(0)}(x) &= \frac{b(b+2)}{16}\, h_n^{\mathrm{CDH}}\left(1, \frac{b}{4}, \frac{b}{4} + \frac{1}{2}\right) = b(b+2)\frac{n!\left(\frac{b+1}{2}\right)_n \left(\frac{b+4}{2}\right)_{2n}}{4^{n+2}}, \\ p_n^{(\pm 1)}(x) &= \frac{n!}{2^{3n/2}}\, P_n^{(b/2)}\left(2x, \frac{\pi}{2} \pm \frac{\pi}{4}\right), \\ h_n^{(1)} &= \frac{n!(b)_n}{2^{3n+1}}, \end{aligned} \tag{35}$$

$$\langle p_m^{(1)}, p_n^{(1)} \rangle_{-1} = \frac{(-1)^{m+n}(b)_m(b)_n}{2^{m+n+1}}\, {}_2F_1\left(\begin{matrix} -m, -n \\ b \end{matrix}; \frac{1}{2}\right). \tag{36}$$

After simplification, (18) then reduces to the following new identity.

**Theorem 6.** *The following determinant evaluation holds:*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}+2^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{2}\right)\right)$$
$$=2^{N^2}\left(\frac{b(b+2)}{8}\right)^{[N/2]}\prod_{n=0}^{[(N-1)/2]}n!\left(\frac{b+1}{2}\right)_n\left(\frac{b}{2}\right)_{2n}\prod_{n=0}^{[(N-2)/2]}n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+4}{2}\right)_{2n}.$$

Next, we take

$$w_{\pm 1}(x)=\pm\frac{2^{b/2}}{2\pi\Gamma(b)}\,e^{\pm\pi x}\Gamma\left(\frac{b}{2}+2\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-2\mathrm{i}x\right),$$
$$w_0(x)=\frac{2^{b/2}}{\pi\Gamma(b)}\,\sinh(\pi x)\Gamma\left(\frac{b}{2}+2\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-2\mathrm{i}x\right).$$

Similarly as in the proof of Theorem 4, we write

$$\frac{x}{x-\mathrm{i}b/4}\,w_0(x)=\frac{\mathrm{i}b}{2}\frac{1}{4\pi\prod_{1\leqslant j<k\leqslant 3}\Gamma(a_j+a_k)}\left|\frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$
$$x\left(x+\frac{\mathrm{i}b}{4}\right)w_0(x)=\frac{\mathrm{i}b(b+1)(b+2)}{16}\frac{1}{4\pi\prod_{1\leqslant j<k\leqslant 3}\Gamma(b_j+b_k)}\times\left|\frac{\Gamma(b_1+\mathrm{i}x)\Gamma(b_2+\mathrm{i}x)\Gamma(b_3+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$

where $(a_1,a_2,a_3)=(1/2,b/4,b/4+1/2)$ and $(b_1,b_2,b_3)=(1/2,b/4+1/2,b/4+1)$, and conclude that

$$h_{2n}^{(0)}=\frac{\mathrm{i}b}{2}\,h_n^{\mathrm{CDH}}\left(\frac{1}{2},\frac{b}{4},\frac{b}{4}+\frac{1}{2}\right)=\frac{\mathrm{i}b}{2^{2n+1}}\,n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+2}{2}\right)_{2n},\tag{37}$$

$$h_{2n+1}^{(0)}=\frac{\mathrm{i}b(b+1)(b+2)}{16}\,h_n^{\mathrm{CDH}}\left(\frac{1}{2},\frac{b}{4}+\frac{1}{2},\frac{b}{4}+1\right)=\frac{\mathrm{i}b(b+1)(b+2)}{2^{2n+4}}\,n!\left(\frac{b+3}{2}\right)_n\left(\frac{b+4}{2}\right)_{2n}.\tag{38}$$

Moreover, it follows from Lemma 2 that

$$h_n^{(1)}=\frac{e^{\mathrm{i}\pi/4}n!(b)_{n+1}}{2^{3n+3/2}},$$
$$\langle p_m^{(1)},p_n^{(1)}\rangle_{-1}=(-1)^{m+n+1}e^{-\mathrm{i}\pi/4}\frac{b(b+1)_m(b+1)_n}{2^{m+n+3/2}}\,{}_2F_1\left(\begin{matrix}-m,-n\\ b+1\end{matrix};\frac{1}{2}\right).$$

In the resulting instance of (18), we replace $b$ by $b-1$ and simplify to obtain the following result.

**Theorem 7.** *The following determinant evaluation holds:*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}+\mathrm{i}2^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{2}\right)\right)=e^{\frac{\mathrm{i}\pi N}{4}}2^{\frac{N(2N-1)}{2}}$$
$$\times\left(\frac{b(b+1)}{4}\right)^{[N/2]}\prod_{n=0}^{[(N-1)/2]}n!\left(\frac{b}{2}\right)_n\left(\frac{b+1}{2}\right)_{2n}\prod_{n=0}^{[(N-2)/2]}n!\left(\frac{b+2}{2}\right)_n\left(\frac{b+3}{2}\right)_{2n}.$$

Next, we choose $w_1$ as in (34) but replace $w_{-1}$ by its negative. Then,

$$w_0(x)=\frac{2^{b/2}}{\pi\Gamma(b)}\,\sinh(\pi x)\left|\Gamma\left(\frac{b}{2}+2\mathrm{i}x\right)\right|^2.$$

Exactly as in the proof of Theorem 5, we find that (33) holds with

$$c_n=\frac{1}{2\mathrm{i}}\,h_{2n}^{(0)}\Big|_{\text{as in (37)}}=\frac{b}{2^{2n+2}}\,n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+2}{2}\right)_{2n},$$

$h_n^{(1)}$ as in (35) and $\langle p_m^{(1)},p_n^{(1)}\rangle_{-1}$ as in (36) apart from a change of sign. After simplification, we obtain the following determinant evaluation.

**Theorem 8.** *When $N$ is even,*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}-2^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{2}\right)\right)$$
$$=(-1)^{\frac{N}{2}}2^{\frac{N(2N-3)}{2}}b^N\prod_{n=0}^{(N-2)/2}\left(n!\left(\frac{b+1}{2}\right)_n\left(\frac{b+2}{2}\right)_{2n}\right)^2,$$

*whereas if $N$ is odd, the determinant vanishes.*

We now turn to determinant evaluations of the form (5). Let

$$w_{\pm1}(x)=\pm\frac{3}{4\pi\Gamma(b)}\,e^{\pm2\pi x}\Gamma\left(\frac{b}{2}+3\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-3\mathrm{i}x\right),$$
$$w_0(x)=\frac{3}{2\pi\Gamma(b)}\,\sinh(2\pi x)\Gamma\left(\frac{b}{2}+3\mathrm{i}x+1\right)\Gamma\left(\frac{b}{2}-3\mathrm{i}x\right).$$

We then have

$$\frac{x}{x-\mathrm{i}b/6}\,w_0(x)=\frac{\mathrm{i}3^{1/2}b}{2}\frac{1}{4\pi\prod_{1\leqslant j<k\leqslant 3}\Gamma(a_j+a_k)}\left|\frac{\Gamma(a_1+\mathrm{i}x)\Gamma(a_2+\mathrm{i}x)\Gamma(a_3+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$
$$x\left(x+\frac{\mathrm{i}b}{6}\right)w_0(x)=\frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{3/2}}\frac{1}{4\pi\prod_{1\leqslant j<k\leqslant 3}\Gamma(b_j+b_k)}$$
$$\times\left|\frac{\Gamma(b_1+\mathrm{i}x)\Gamma(b_2+\mathrm{i}x)\Gamma(b_3+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2,$$

where $(a_1, a_2, a_3) = (b/6, b/6+1/3, b/6+2/3)$, $(b_1, b_2, b_3) = (b/6+1/3, b/6+2/3, b/6+1)$. As in the proof of Theorem 4, it follows that

$$\begin{aligned} h_{2n}^{(0)} &= \frac{\mathrm{i}3^{1/2}b}{2}\, h_n^{\mathrm{CDH}}\left(\frac{b}{6}, \frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}\right) = \frac{\mathrm{i}b}{2\cdot 3^{3n-1/2}}\, n!(b+1)_{3n}, \qquad (39)\\ h_{2n+1}^{(0)} &= \frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{3/2}}\, h_n^{\mathrm{CDH}}\left(\frac{b}{6}+\frac{1}{3}, \frac{b}{6}+\frac{2}{3}, \frac{b}{6}+1\right)\\ &= \frac{\mathrm{i}b(b+1)(b+2)}{2\cdot 3^{3n+3/2}}\, n!(b+3)_{3n} \end{aligned}$$

and, using Lemma 2,

$$\begin{aligned} h_n^{(1)} &= \frac{e^{\mathrm{i}\pi/3} n!(b)_{n+1}}{2\cdot 3^{2n}},\\ \langle p_m^{(1)}, p_n^{(1)}\rangle_{-1} &= (-1)^{m+n+1} e^{-\mathrm{i}\pi/3} \frac{b(b+1)_m(b+1)_n}{2\cdot 3^{(m+n)/2}}\, {}_2F_1\left(\begin{matrix}-m,-n\\ b+1\end{matrix}; \frac{1}{3}\right). \end{aligned}$$

After replacing $b$ by $b-1$, (18) can be simplified to the following form.

**Theorem 9.** *The following determinant evaluation holds:*

$$\begin{aligned} &\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn} + e^{\mathrm{i}\pi/3}3^{(m+n)/2}(b)_m(b)_n\, {}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix}; \frac{1}{3}\right)\right)\\ &\qquad = e^{\frac{\mathrm{i}\pi N}{6}} 3^{\frac{N(N+1)}{4}} \left(\frac{b(b+1)}{\sqrt{3}}\right)^{[N/2]} \prod_{n=0}^{[(N-1)/2]} n!(b)_{3n} \prod_{n=0}^{[(N-2)/2]} n!(b+2)_{3n}. \end{aligned}$$

Finally, we choose

$$w_{\pm 1}(x) = \pm\frac{3}{4\pi\Gamma(b)}\, e^{\pm 2\pi x}\left|\Gamma\left(\frac{b}{2}+3\mathrm{i}x\right)\right|^2,$$

$$w_0(x) = \frac{3}{2\pi\Gamma(b)}\, \sinh(2\pi x)\left|\Gamma\left(\frac{b}{2}+3\mathrm{i}x\right)\right|^2.$$

As in the proof of Theorem 5, we find that (33) holds with

$$\begin{aligned} c_n &= \frac{1}{3\mathrm{i}}\, h_{2n}^{(0)}\Big|_{\text{as in (39)}} = \frac{b}{2\cdot 3^{3n+1/2}}\, n!(b+1)_{3n},\\ h_n^{(1)} &= \frac{1}{2\cdot 3^{2n}}\, n!(b)_n,\\ \langle p_m^{(1)}, p_n^{(1)}\rangle_{-1} &= \frac{(-1)^{m+n+1}(b)_m(b)_n}{2\cdot 3^{(m+n)/2}}\, {}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix}; \frac{1}{3}\right), \end{aligned}$$

which gives the following identity after simplification.

**Theorem 10.** *When $N$ is even,*

$$\det_{0\leqslant m,n\leqslant N-1}\left(m!(b)_m\delta_{mn}-3^{(m+n)/2}(b)_m(b)_n\,{}_2F_1\left(\begin{matrix}-m,-n\\ b\end{matrix};\frac{1}{3}\right)\right)$$
$$=(-1)^{N/2}3^{N^2/4}b^N\prod_{n=0}^{(N-2)/2}\left(n!(b+1)_{3n}\right)^2$$

*whereas if $N$ is odd the determinant vanishes.*

It may be instructive to summarize the results obtained so far. We have considered weight functions

$$w_{\pm 1}(x)=\frac{l(2\cos(\pi k/2l))^b}{4\pi\Gamma(b)}(\pm 1)^\delta e^{\pm k\pi x}\Gamma\left(\frac{b}{2}+lix+\varepsilon\right)\Gamma\left(\frac{b}{2}-lix\right),$$

which are normalized so that $w_0=w_1+w_{-1}$ has total mass 1 when $\delta=\varepsilon=0$, and the parameters are as in the following table:

| | $k$ | $l$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|---|
| Theorem 1 | 1 | 3 | 0 | 0 |
| Theorem 3 | 1 | 3 | 0 | 1 |
| Theorem 4 | 1 | 3 | 1 | 1 |
| Theorem 5 | 1 | 3 | 1 | 0 |
| Theorem 6 | 1 | 2 | 0 | 0 |
| Theorem 7 | 1 | 2 | 1 | 1 |
| Theorem 8 | 1 | 2 | 1 | 0 |
| Theorem 9 | 2 | 3 | 1 | 1 |
| Theorem 10 | 2 | 3 | 1 | 0 |

The case $(k,l,\delta,\varepsilon)=(1,2,0,1)$, which may appear to be missing, merely gives the complex conjugate of Theorem 7.

## 6 A $q$-analogue of Andrews's determinant

To find a $q$-analogue of Andrews's determinant, it is natural to replace the Wilson polynomials in (20) by Askey–Wilson polynomials. More precisely, a natural starting point would be to combine the polynomials

$$p_n(x;1,b,bq,bq^2|q^3),\qquad p_n(x;q^3,b,bq,bq^2|q^3)$$

to a single orthogonal system. This is indeed possible, within the framework of orthogonal Laurent polynomials on the unit circle.

Throughout this section, we write

$$\omega=e^{2\pi i/3}.$$

**Lemma 11.** *For $|q|, |b| < 1$, let*

$$\langle f, g\rangle_0 = \frac{(q^3;q^3)_\infty(b,b^2q;q)_\infty}{(b^3q^3;q^3)_\infty}\oint f(z)g(z)\frac{1}{1+z}\frac{(z^2,z^{-2};q^3)_\infty}{(z,z^{-1};q^3)_\infty(bz,bz^{-1};q)_\infty}\frac{dz}{2\pi \mathrm{i}z},$$

*with integration over the positively oriented unit circle. Then, the Laurent polynomials*

$$p_{2n}^{(0)}(z) = \frac{1}{(b^3q^{3n};q^3)_n}\,p_n\left(\frac{z+z^{-1}}{2};1,b,bq,bq^2|q^3\right),$$

$$p_{2n+1}^{(0)}(z) = \frac{z-1}{(b^3q^{3n+3};q^3)_n}\,p_n\left(\frac{z+z^{-1}}{2};q^3,b,bq,bq^2|q^3\right)$$

*satisfy the orthogonality relations*

$$\langle p_m^{(0)}, p_n^{(0)}\rangle_0 = h_n^{(0)}\delta_{mn},$$

*where*

$$h_{2n}^{(0)} = \frac{(q^3,b^3;q^3)_n(b,b^2q;q)_{3n}}{(b^3,b^3q^3;q^3)_{2n}}, \tag{40a}$$

$$h_{2n+1}^{(0)} = -\frac{(1-b)(1-bq^2)}{(1-\omega bq)(1-\omega^2bq)}\frac{(q^3,b^3q^3;q^3)_n(bq^3,b^2q;q)_{3n}}{(b^3q^3,b^3q^6;q^3)_{2n}}. \tag{40b}$$

*Proof.* In the integral defining $\langle p_m^{(0)}, p_n^{(0)}\rangle_0$, write

$$\oint f(z)\,\frac{dz}{2\pi\mathrm{i}z} = \frac{1}{2}\oint \big(f(z)+f(z^{-1})\big)\,\frac{dz}{2\pi\mathrm{i}z}.$$

When $m$ and $n$ are both even, the integrand is invariant under $z\mapsto z^{-1}$, including the factor

$$\frac{1}{1+z}+\frac{1}{1+z^{-1}} = 1.$$

The integral then reduces to (15), and we obtain the desired orthogonality with

$$h_{2n}^{(0)} = \frac{h_n^{\mathrm{AW}}(1,b,bq,bq^2|bq^3)}{(b^3q^{3n};q^3)_n^2},$$

which agrees with (40a). When $m$ and $n$ have different parity, we get an integral containing

$$\frac{z-1}{1+z}+\frac{z^{-1}-1}{1+z^{-1}} = 0,$$

so the orthogonality is obvious. Finally, when $m$ and $n$ are both odd, we encounter the factor

$$\frac{(z-1)^2}{1+z}+\frac{(z^{-1}-1)^2}{1+z^{-1}} = -(1-z)(1-z^{-1}).$$

We now observe that $(1-z)(1-z^{-1})$ times the orthogonality measure (normalized as in (15)) for $p_n(x;1,b,bq,bq^2|q^3)$ is the corresponding measure for

$$p_n(x;q^3,b,bq,bq^2|q^3),$$

apart from the multiplier

$$\left.\frac{(1-b_2)(1-b_3)(1-b_4)}{1-b_2b_3b_4}\right|_{b_2=b,\,b_3=bq,\,b_4=bq^2}=\frac{(1-b)(1-bq^2)}{(1-\omega bq)(1-\omega^2 bq)},$$

and conclude that

$$h_{2n+1}^{(0)}=-\frac{(1-b)(1-bq^2)}{(1-\omega bq)(1-\omega^2 bq)}\frac{h_n^{\mathrm{AW}}(q^3,b,bq,bq^2|bq^3)}{(b^3q^{3n+3};q^3)_n^2},$$

which agrees with (40b). □

Note that $p_k^{(0)}$ is a linear combination of the first $k+1$ terms in the sequence

$$1,\ z,\ z^{-1},\ z^2,\ z^{-2},\dots.$$

Moreover, the coefficient of the $(k+1)$st term is 1. If we let these two properties define a *monic Laurent polynomial of degree* $k$, then the discussion leading to (18) remains valid if "polynomial" is replaced throughout by "Laurent polynomial".

To apply this modified version of (18) we must split the orthogonality measure for $p_n^{(0)}$ in two parts. This will be achieved by the following version of Watson's quintuple product identity [21]. The fact that this fundamental result is applicable is a strong indication that we are doing something natural.

**Lemma 12.** *The following identity holds:*

$$\frac{1}{1+z}\frac{(z^2,z^{-2};q^3)_\infty}{(z,z^{-1};q^3)_\infty}=\frac{1-\omega}{3}\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\big((qz\omega,\omega^2/z;q)_\infty-\omega^2(qz\omega^2,\omega/z;q)_\infty\big). \tag{41}$$

*Proof.* The left-hand side of (41) can be expressed as

$$\frac{1}{z}(-z,-q/z;q^3)_\infty(q^3z^2,q^3/z^2;q^6)_\infty.$$

By the quintuple product identity, as given in [8, Ex. 5.6], the Laurent expansion of this function is

$$\frac{1}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{3n}{2}}z^{3n-1}(1+zq^{3n}). \tag{42}$$

On the other hand, by the triple product identity [8, Eq. (1.6.1)], the right-hand side of (41) has Laurent expansion

$$\frac{1-\omega}{3(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n+1}{2}}z^n(\omega^n-\omega^{2n+2}). \tag{43}$$

It is easily verified that (42) and (43) agree. □

Let us introduce the notation

$$\mu_{a,b}(f) = \frac{(q,b^2;q)_\infty}{(qab,b/a;q)_\infty} \oint f(z) \frac{(qaz,1/az;q)_\infty}{(bz,b/z;q)_\infty} \frac{dz}{2\pi \mathrm{i} z}.$$

Then, using Lemma 12, the bilinear form introduced in Lemma 11 splits as in (16), with

$$\begin{aligned} \langle f,g\rangle_1 &= \frac{(1-\omega)(1-b\omega^2)}{3(1+b)}\, \mu_{\omega,b}(fg), \\ \langle f,g\rangle_{-1} &= \frac{(1-\omega^2)(1-b\omega)}{3(1+b)}\, \mu_{\omega^2,b}(fg). \end{aligned}$$

To proceed, we need the following result.

**Proposition 13.** *Let*

$$P_n^{(a,b;q)}(z) = z^{-\left[\frac{n}{2}\right]}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, b/a \\ q^{1-n}/ab \end{matrix}; q, \frac{qz}{b}\right)$$

*and let*

$$C_n = C_n^{(a,b;q)} = \begin{cases} 1, & n \text{ even}, \\ a^n (b/a;q)_n/(ab;q)_n, & n \text{ odd}. \end{cases}$$

*Then, $P_n^{(a,b;q)}/C_n$ is a monic Laurent polynomial of degree $n$. For $|b|$, $|q|<1$ we have the orthogonality relation*

$$\mu_{a,b}\left(P_m^{(a,b;q)} P_n^{(a,b;q)}\right) = h_n \delta_{mn}, \tag{44}$$

*where*

$$h_n = (-1)^n a^{2\left[\frac{n}{2}\right]} \frac{(q,b^2,b/a;q)_n}{(ab;q)_n (abq;q)_{2[n/2]} (b/a;q)_{2[(n+1)/2]}}.$$

*Moreover,*

$$\begin{aligned} \mu_{c,b}\left(P_m^{(a,b;q)} P_n^{(a,b;q)}\right) &= a^{\left[\frac{m}{2}\right]+\left[\frac{n}{2}\right]} \frac{(b^2;q)_m (b^2;q)_n (qc/a;q)_{\left[\frac{m}{2}\right]+\left[\frac{n}{2}\right]} (a/c;q)_{\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]}}{(ab;q)_m (ab;q)_n (qbc;q)_{\left[\frac{m}{2}\right]+\left[\frac{n}{2}\right]} (b/c;q)_{\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]}} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} q^{-m}, q^{-n}, ab, b/a \\ b^2, q^{-\left[\frac{m}{2}\right]-\left[\frac{n}{2}\right]}a/c, q^{1-\left[\frac{m+1}{2}\right]-\left[\frac{n+1}{2}\right]}c/a \end{matrix}; q, q\right). \end{aligned} \tag{45}$$

Before proving Proposition 13, we point out that (44) is equivalent to a result of Pastro [19]. Namely, if we let $p_n(z) = z^{[n/2]} P_n^{(a,b;q)}(z)$, then $p_n$ is a monic polynomial of degree $n$. Moreover, (44) means that

$$\mu_{a,b}(z^{-k} p_n(z)) = 0, \qquad k = 0,1,\dots,n-1.$$

It follows that

$$\oint p_m(z)\, p_n(1/z) \frac{(qaz,1/az;q)_\infty}{(bz,b/z;q)_\infty} \frac{dz}{2\pi \mathrm{i} z} = 0, \qquad m \neq n$$

or, if the parameters are such that $p_n$ have real coefficients,

$$\oint p_m(z)\,\overline{p_n(z)}\,\frac{(qaz,1/az;q)_\infty}{(bz,b/z;q)_\infty}\frac{dz}{2\pi \mathrm{i} z}=0,\qquad m\neq n.$$

It is easily seen that this orthogonal system on the unit circle is equivalent to the one introduced by Pastro.

*Proof of* Proposition 13. It is straight-forward to check that $P_n^{(a,b;q)}/C_n$ is a monic Laurent polynomial of degree $n$. To prove (45), we use [8, Eq. (III.6–7)] to write

$$\begin{aligned}P_n^{(a,b;q)}(z)&=z^{-\left[\frac{n}{2}\right]}\left(\frac{a}{b}\right)^n\frac{(b^2;q)_n}{(ab;q)_n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},b/a,bz\\ b^2,0\end{matrix};q,q\right)\\&=\frac{z^{\left[\frac{n+1}{2}\right]}}{b^n}\frac{(b^2;q)_n}{(ab;q)_n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},ab,b/z\\ b^2,0\end{matrix};q,q\right).\end{aligned}$$

These expressions also clarify the relation to Meixner–Pollaczek polynomials (11). Expressing $P_m^{(a,b;q)}$ using the first of these identities and $P_n^{(a,b;q)}$ using the second one gives

$$\begin{aligned}\mu_{c,b}\left(P_m^{(a,b;q)}P_n^{(a,b;q)}\right)&=\frac{a^m}{b^{m+n}}\frac{(q,b^2;q)_\infty(b^2;q)_m(b^2;q)_n}{(qbc,b/c;q)_\infty(ab;q)_m(ab;q)_n}\\&\quad\times\sum_{k=0}^m\sum_{l=0}^n\frac{(q^{-m},b/a;q)_k}{(q,b^2;q)_k}\frac{(q^{-n},ab;q)_l}{(q,b^2;q)_l}q^{k+l}\oint z^{\left[\frac{n+1}{2}\right]-\left[\frac{m}{2}\right]}\frac{(qcz,1/cz;q)_\infty}{(bq^kz,bq^l/z;q)_\infty}\frac{dz}{2\pi\mathrm{i}z}.\end{aligned}$$

By Ramanujan's summation [8, Eq. (II.29)],

$$\frac{(q,b^2,qcz,1/cz;q)_\infty}{(qbc,b/c,bq^kz,bq^l/z;q)_\infty}=\frac{(b^2;q)_{k+l}}{(b/c;q)_k(qbc;q)_l}\sum_{x=-\infty}^{\infty}\frac{(q^{1-k}c/b;q)_x}{(q^{l+1}bc;q)_x}\left(q^kbz\right)^x$$

in the annulus $|q^{k+l}b|<|z|<|b^{-1}|$, which is consistent with our conditions on the parameters. Since only the term with $x=[m/2]-[(n+1)/2]$ contributes to the integral, we conclude that

$$\begin{aligned}\mu_{c,b}\left(P_m^{(a,b;q)}P_n^{(a,b;q)}\right)&=\frac{a^m}{b^{\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]+n}}\frac{(b^2;q)_m(b^2;q)_n(qc/b;q)_{\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}}{(ab;q)_m(ab;q)_n(qbc;q)_{\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}}\\&\quad\times\sum_{k=0}^m\sum_{l=0}^n\frac{(b^2;q)_{k+l}}{(b^2;q)_k(b^2;q)_l}\frac{(q^{-m},b/a;q)_k}{(q,q^{-\left[\frac{m}{2}\right]+\left[\frac{n+1}{2}\right]}b/c;q)_k}\frac{(q^{-n},ab;q)_l}{(q,q^{\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]+1}bc;q)_l}q^{k+l}.\end{aligned}$$

Applying [8, Eq. (II.7)] in the form

$$\frac{(b^2;q)_{k+l}}{(b^2;q)_k(b^2;q)_l}=\sum_{x=0}^{\min(k,l)}\frac{(q^{-k},q^{-l};q)_x}{(q,b^2;q)_k}\left(b^2q^{k+l}\right)^x,$$

replacing $(k,l)$ by $(k+x, l+x)$ and changing the order of summation gives after simplification

$$\mu_{c,b}\left(P_m^{(a,b;q)}P_n^{(a,b;q)}\right) = \frac{a^m}{b^{\left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]+n}} \frac{(b^2;q)_m(b^2;q)_n(qc/b;q)_{\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}}{(ab;q)_m(ab;q)_n(qbc;q)_{\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}}$$
$$\times \sum_{x=0}^{\min(m,n)} \frac{(q^{-m},q^{-n},ab,b/a;q)_x}{(q,b^2,q^{-\left[\frac{m}{2}\right]+\left[\frac{n+1}{2}\right]}b/c, q^{1+\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}bc;q)_x} q^{x(x+1)}b^{2x}$$
$$\times \sum_{k=0}^{m-x} \frac{(q^{x-m},q^x b/a;q)_k}{(q,q^{x-\left[\frac{m}{2}\right]+\left[\frac{n+1}{2}\right]}b/c;q)_k} q^k \sum_{l=0}^{n-x} \frac{(q^{x-n},q^x ab;q)_l}{(q,q^{1+x+\left[\frac{m}{2}\right]-\left[\frac{n+1}{2}\right]}bc;q)_l} q^l.$$

Computing the inner sums using [8, Eq. (II.6)] and simplifying, we arrive at (45).

To deduce (44), we observe that the right-hand side of (45) has the form

$$\sum_{x=0}^{\min(m,n)} (q^{x-[m/2]-[n/2]}a/c)_{m+n-2x}(\cdots),$$

where the missing factors are regular at $c=a$. When $c\to a$, the summand vanishes for

$$x \leqslant \min\left(\left[\frac{m}{2}\right]+\left[\frac{n}{2}\right], \left[\frac{m+1}{2}\right]+\left[\frac{n+1}{2}\right]-1\right) = \left[\frac{m+n-1}{2}\right].$$

Thus, the range of summation can be reduced to

$$\left[\frac{m+n+1}{2}\right] \leqslant x \leqslant \min(m,n),$$

which is empty for $m\neq n$ and consists of the point $x=m=n$ otherwise. After simplification, this gives (44). □

We can now conclude that

$$\det_{0\leqslant m,n\leqslant N-1}\left(\frac{(1-\omega)(1-b\omega^2)}{3(1+b)}\mu_{\omega,b}\left(P_m^{(\omega,b;q)}P_m^{(\omega,b;q)}\right)\delta_{mn} + \frac{(1-\omega^2)(1-b\omega)}{3(1+b)}\mu_{\omega^2,b}\left(P_m^{(\omega,b;q)}P_n^{(\omega,b;q)}\right)\right) = \prod_{n=0}^{N-1} C_n^2 h_n^{(0)},$$

where $h_n^{(0)}$ is as in (40) and the remaining quantities are given in Lemma 11 and Proposition 13. To simplify, we pull out the factor $(1-\omega)(1-b\omega^2)/3(1+b)$ and multiply the $n$th row and column with $\omega^{2[n/2]}(b\omega;q)_n$, for each $n$. We also write

$$X_m = (-1)^m \frac{(b\omega,b\omega^2;q)_m}{(qb\omega;q)_{2[m/2]}(b\omega^2;q)_{2[(m+1)/2]}} = \begin{cases} \dfrac{1-b\omega}{1-b\omega q^m}, & m \text{ even}, \\ -\dfrac{1-b\omega}{1-b\omega^2 q^m}, & m \text{ odd}. \end{cases} \tag{46}$$

After simplification, we find the following $q$-analogue of Andrews's determinant.

**Theorem 14.** *Let* $\omega = e^{2\pi i/3}$ *and let* $X_m$ *be as in* (46)*. Then, the following determinant evaluation holds:*

$$\det_{0\leqslant m,n\leqslant N-1}\Bigg(X_m(q,b^2;q)_m\,\delta_{mn} - \omega^2 \frac{(b^2;q)_m(b^2;q)_n(q\omega;q)_{[\frac{m}{2}]+[\frac{n}{2}]}(\omega^2;q)_{[\frac{m+1}{2}]+[\frac{n+1}{2}]}}{(b\omega^2;q)_{[\frac{m}{2}]+[\frac{n}{2}]+1}(qb\omega;q)_{[\frac{m+1}{2}]+[\frac{n+1}{2}]-1}}$$

$$\times\ {}_4\phi_3\left(\begin{matrix} q^{-m}, q^{-n}, b\omega, b\omega^2 \\ b^2, q^{-[\frac{m}{2}]-[\frac{n}{2}]}\omega^2, q^{1-[\frac{m+1}{2}]-[\frac{n+1}{2}]}\omega \end{matrix}; q, q\right)\Bigg)$$

$$= \left(\frac{3(1+b)}{(1-\omega)(1-b\omega^2)}\right)^N \left(-\frac{(1-b)(1-bq^2)(1-b\omega^2)^2\omega^2}{(1-qb\omega)(1-qb\omega^2)}\right)^{[N/2]}$$

$$\times \prod_{n=0}^{[(N-1)/2]} \frac{\omega^n(b\omega;q)_{2n}^2(q^3,b^3;q^3)_n(b,b^2q;q)_{3n}}{(b^3,b^3q^3;q^3)_{2n}}$$

$$\times \prod_{n=0}^{[(N-2)/2]} \frac{\omega^{2n}(qb\omega^2;q)_{2n}^2(q^3,b^3q^3;q^3)_n(bq^3,b^2q;q)_{3n}}{(b^3q^3,b^3q^6;q^3)_{2n}}.$$

Replacing $b$ by $q^{b/2}$ and letting $q \to 1$, the ${}_4\phi_3$ reduces to a summable ${}_2F_1$ and we recover Andrews's determinant evaluation. Incidentally, replacing $b$ by $-q^{b/2}$ and letting $q \to 1$ gives Theorem 10.

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