A note on automorphisms of the infinite-dimensional hypercube graph

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Abstract

We define the infinite-dimensional hypercube graph $H_{\aleph_0}$ as the graph whose vertex set is formed by the so-called singular subsets of $\mathbb{Z}\setminus\{0\}$. This graph is not connected, but it has isomorphic connected components. We show that the restrictions of its automorphisms to the connected components are induced by permutations on $\mathbb{Z}\setminus\{0\}$ preserving the family of singular subsets. As an application, we describe the automorphism group of the connected components.

Keywords: infinite-dimensional hypercube graph; graph automorphism; weak wreath product of groups.

1 Introduction

By [3], typical graphs have no non-trivial automorphisms. On the other hand, the classical Frucht result [4] states that every abstract group can be realized as the automorphism group of some graph (we refer [2] for more information concerning graph automorphisms). In particular, the Coxeter group of type $\mathsf{B}_n = \mathsf{C}_n$ (the wreath product $S_2 \wr S_n$) is isomorphic to the automorphism group of the n-dimensional hypercube graph H_n.

In this note we consider the infinite-dimensional hypercube graph $H_{\aleph_0}$. This is the Cartesian product of infinitely many factors K_2, but it also can be defined as a graph whose vertex set is formed by the maximal singular subsets of $\mathbb{Z}\setminus\{0\}$ (Section 2). This graph is not connected, but it has isomorphic connected components. We show that the restrictions of its automorphisms to the connected components are induced by permutations on $\mathbb{Z}\setminus\{0\}$ preserving the family of singular subsets (Theorem 2). As a simple consequence, we establish that the automorphism group of each connected component is isomorphic to the so-called weak wreath product of S_2 and $S_{\aleph_0}$ (Corollary 5). Since $H_{\aleph_0}$ is the Cartesian product of infinitely many factors K_2, the latter statement can be drawn from some results concerning the automorphism group of Cartesian product of graphs [5, 6].

2 Infinite-dimensional hypercube graph

A subset $X \subset \mathbb{Z} \setminus \{0\}$ is said to be *singular* if

$$i \in X \implies -i \notin X.$$

For every natural i each maximal singular subset contains precisely one of the numbers i or $-i$; in other words, if X is a maximal singular subset then the same holds for its complement in $\mathbb{Z} \setminus \{0\}$. Two maximal singular subsets X, Y are called *adjacent* if

$$|X \setminus Y| = |Y \setminus X| = 1.$$

In this case, we have

$$X = (X \cap Y) \cup \{i\} \text{ and } Y = (X \cap Y) \cup \{-i\}$$

for some number $i \in \mathbb{Z} \setminus \{0\}$.

Following Example 2.6 in [7], we say that a permutation s on the set $\mathbb{Z} \setminus \{0\}$ is *symplectic* if

$$s(-i) = -s(i) \quad \forall\, i \in \mathbb{Z} \setminus \{0\}.$$

A permutation is symplectic if and only if it preserves the family of singular subsets. The group of symplectic permutations is isomorphic to the wreath product $S_2 \wr S_{\aleph_0}$ (we write S_α for the group of permutations on a set of cardinality α, see Section 5 for the definition of wreath product). The action of this group on the family of maximal singular subsets is transitive.

Denote by $H_{\aleph_0}$ the graph whose vertex set is formed by all maximal singular subsets and whose edges are adjacent pairs of such subsets. This graph is not connected. The connected component containing $X \in H_{\aleph_0}$ will be denoted by $H(X)$; it consists of all $Y \in H_{\aleph_0}$ such that

$$|X \setminus Y| = |Y \setminus X| < \infty.$$

Any two connected components $H(X)$ and $H(Y)$ are isomorphic. Indeed, every symplectic permutation s on the set $\mathbb{Z} \setminus \{0\}$ induces an automorphism of $H_{\aleph_0}$; this automorphism transfers $H(X)$ to $H(Y)$ if $s(X) = Y$.

It is clear that $H_{\aleph_0}$ can be identified with the graph whose vertices are sequences

$$\{a_n\}_{n\in\mathbb{N}} \text{ with } a_n \in \{0, 1\}$$

and $\{a_n\}_{n\in\mathbb{N}}$ is adjacent with $\{b_n\}_{n\in\mathbb{N}}$ (connected by an edge) if

$$\sum_{n\in\mathbb{N}} |a_n - b_n| = 1.$$

Then one of the connected components is formed by all sequences having a finite number of non-zero elements.

3 Automorphisms

Every automorphism of $H_{\aleph_0}$ induced by a symplectic permutation will be called *regular*. An easy verification shows that distinct symplectic permutations induce distinct regular automorphisms. Therefore, the group of regular automorphisms is isomorphic to $S_2 \wr S_{\aleph_0}$.

Non-regular automorphisms exist. The following example is a modification of examples given in [1, 8], see also Example 3.14 in [7].

Example 1. Let $A \in H_{\aleph_0}$ and B be a vertex of the connected component $H(A)$ distinct from A. We take any symplectic permutation s transferring A to B. This permutation preserves $H(A)$ and the mapping

$$f(X) := \begin{cases} s(X) & X \in H(A) \\ X & X \in H_{\aleph_0} \setminus H(A) \end{cases}$$

is well-defined. Clearly, f is a non-trivial automorphism of $H_{\aleph_0}$. Suppose that this automorphism is regular and t is the associated symplectic permutation. For every $i \in \mathbb{Z} \setminus \{0\}$ there exists a singular subset N such that

$$X = N \cup \{i\} \text{ and } Y = N \cup \{-i\}$$

are elements of $H_{\aleph_0} \setminus H(A)$. Then

$$t(N) = t(X \cap Y) = t(X) \cap t(Y) = f(X) \cap f(Y) = X \cap Y = N$$

and

$$N \cup \{i\} = X = f(X) = t(X) = t(N) \cup \{t(i)\} = N \cup \{t(i)\}$$

which implies that $t(i) = i$. Thus t is identity which is impossible. So, the automorphism f is non-regular.

Theorem 2. *The restriction of every automorphism of $H_{\aleph_0}$ to any connected component coincides with the restriction of some regular automorphism to this connected component.*

A similar result was obtained in [8] for infinite Johnson graphs. The proof of that result is based on the same idea.

4 Proof of Theorem 2

Let $A \in H_{\aleph_0}$ and f be the restriction of an automorphism of $H_{\aleph_0}$ to the connected component $H(A)$. For every $X \in H_{\aleph_0}$ we denote by $X^{\sim}$ the set which contains X and all vertices of $H_{\aleph_0}$ adjacent with X. It is clear that $X^{\sim}$ is contained in $H(A)$ if $X \in H(A)$.

Lemma 3. *For every $X \in H(A)$ there is a symplectic permutation s_X such that*

$$f(Y) = s_X(Y) \qquad \forall\, Y \in X^{\sim}. \tag{1}$$

Proof. We can assume that $f(X)$ coincides with X (if $f(X) \neq X$ then we take any symplectic permutation t sending $f(X)$ to X and consider tf). In this case, the restriction of f to $X^{\sim}$ is a bijective transformation of $X^{\sim}$.

For every $i \in \mathbb{Z} \setminus \{0\}$ one of the following possibilities is realized:

- $i \notin X$,
- $i \in X$.

Consider the first case. Then $-i \in X$ and there is unique element of $X^{\sim}$ containing i, this is

$$Y = \{i\} \cup (X \setminus \{-i\}). \tag{2}$$

Since $f|_{X^{\sim}}$ is a transformation of $X^{\sim}$, $f(Y)$ is adjacent with X and the set $f(Y) \setminus X$ contains only one element. We denote it by $s_X(i)$. It is clear that $s_X(i) \notin X$.

In the second case, $-i \notin X$ and we define $s_X(i)$ as $-s_X(-i)$. Since $s_X(-i)$ does not belong to X, we have $s_X(i) \in X$.

So, s_X is a symplectic permutation on $\mathbb{Z} \setminus \{0\}$ such that

$$s_X(X) = X.$$

Now, we check (1).

Let $Y \in X^{\sim}$. Then we have (2) for some i and

$$s_X(Y) = \{s_X(i)\} \cup (s_X(X) \setminus \{-s_X(i)\}) = \{s_X(i)\} \cup (X \setminus \{-s_X(i)\})$$

is the unique element of $X^{\sim}$ containing $s_X(i)$. On the other hand, $s_X(i)$ belongs to $f(Y)$ by the definition of s_X. Therefore, $f(Y)$ coincides with $s_X(Y)$. □

Lemma 4. *If $X, Y \in H(A)$ are adjacent then $s_X = s_Y$.*

Proof. Since X, Y are adjacent, we have

$$X = \{i\} \cup (X \cap Y) \text{ and } Y = \{-i\} \cup (X \cap Y)$$

for some $i \in X$. We can assume that

$$f(X) = X \text{ and } f(Y) = Y.$$

Indeed, in the general case

$$f(X) = \{j\} \cup (f(X) \cap f(Y)) \text{ and } f(Y) = \{-j\} \cup (f(X) \cap f(Y))$$

(since $f(X)$ and $f(Y)$ are adjacent); we take any symplectic permutation t sending j and $f(X) \cap f(Y)$ to i and $X \cap Y$ (respectively) and consider tf.

Then

$$s_X(X \cap Y) = s_X(X) \cap s_X(Y) = f(X) \cap f(Y) = X \cap Y;$$

similarly,

$$s_Y(X \cap Y) = X \cap Y.$$

We have

$$(X \cap Y) \cup \{i\} = X = f(X) = s_X(X) = s_X((X \cap Y) \cup \{i\}) = (X \cap Y) \cup \{s_X(i)\}$$

and the same arguments show that

$$(X \cap Y) \cup \{i\} = (X \cap Y) \cup \{s_Y(i)\}.$$

Therefore,

$$s_X(i) = s_Y(i) = i \text{ and } s_X(-i) = s_Y(-i) = -i.$$

Now, we show that the equality

$$s_X(j) = s_Y(j) \tag{3}$$

holds for every $j \neq \pm i$. Since s_X and s_Y are symplectic, it is sufficient to establish (3) only in the case when $j \notin X \cup Y$. Indeed, if $j \in X \cap Y$ then $-j$ does not belong to $X \cup Y$.

Let j be an element of $\mathbb{Z} \setminus \{0\}$ which does not belong to $X \cup Y$. Then $-j \in X \cap Y$ and

$$X' := \{j\} \cup (X \setminus \{-j\}) \in X^{\sim}, \quad Y' := \{j\} \cup (Y \setminus \{-j\}) \in Y^{\sim}$$

are adjacent. Hence

$$f(X') = s_X(X') = \{s_X(j)\} \cup (X \setminus \{-s_X(j)\})$$

and

$$f(Y') = s_Y(Y') = \{s_Y(j)\} \cup (Y \setminus \{-s_Y(j)\})$$

are adjacent. The latter is possible only in the case when $s_X(j) = s_Y(j)$. $\square$

Using the connectedness of $H(A)$ and Lemma 4, we establish that $s_X = s_Y$ for all $X, Y \in H(A)$.

5 Automorphisms of connected components

Let G_1 and G_2 be permutation groups on sets X_1 and X_2, respectively. Recall that the *wreath product* $G_1 \wr G_2$ is a permutation group on $X_1 \times X_2$ and its elements are compositions of the following two types of permutations:

(1) for each element $g \in G_2$, the permutation $(x_1, x_2) \to (x_1, g(x_2))$;

(2) for each function $i : X_2 \to G_1$, the permutation $(x_1, x_2) \to (i(x_2)x_1, x_2)$.

Consider the subgroup of $G_1 \wr G_2$ whose elements are compositions of permutations of type (1) and permutations of type (2) such that the set

$$\{\, x_2 \in X_2 \ :\ i(x_2) \neq \mathrm{id}_{X_1} \,\}$$

is finite. This is a proper subgroup only in the case when X_2 is infinite; it will be called the *weak wreath product* and denoted by $G_1 \wr_w G_2$.

Corollary 5. *The automorphism group of the connected component of $H_{\aleph_0}$ is isomorphic to the weak wreath product $S_2 \wr_w S_{\aleph_0}$.*

Proof. Let $A \in H_{\aleph_0}$ and f be an automorphism of the connected component $H(A)$. By the previous section, f is induced by a symplectic permutation s. Since $f(A) = s(A)$ belongs to $H(A)$, the set $s(A) \setminus A$ is finite. So, the automorphism group of $H(A)$ is isomorphic to the group of symplectic permutations s such that the set $s(A) \setminus A$ is finite. The latter group is isomorphic to the weak wreath product $S_2 \wr_w S_{\aleph_0}$ (indeed, we can identify the set $\mathbb{Z} \setminus \{0\}$ with the Cartesian product $\mathbb{Z}_2 \times A$ and the group $S_{\aleph_0}$ with the group of all permutation on A). $\square$

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