Distance Powers and Distance Matrices of Integral Cayley Graphs over Abelian Groups

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Abstract

It is shown that distance powers of an integral Cayley graph over an abelian group Γ are again integral Cayley graphs over Γ. Moreover, it is proved that distance matrices of integral Cayley graphs over abelian groups have integral spectrum.

1 Introduction

Eigenvalues of an undirected graph G are the eigenvalues of an arbitrary adjacency matrix of G. General facts about graph spectra can e.g. be found in [7] or [8]. Harary and Schwenk [10] defined G to be *integral* if all of its eigenvalues are integers. For a survey of integral graphs see [4]. In [2] the number of integral graphs on n vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes, see e.g. [1], [13], or [15]. Here we concentrate on integral Cayley graphs over abelian groups and their distance powers.

Let Γ be a finite, additive group, $S \subseteq \Gamma$, $-S = \{-s : s \in S\} = S$. The undirected *Cayley graph over* Γ *with shift set (or symbol)* S, $\mathrm{Cay}(\Gamma, S)$, has vertex set Γ. Vertices $a, b \in \Gamma$ are adjacent if and only if $a - b \in S$. For general properties of Cayley graphs we refer to Godsil and Royle [9] or Biggs [5]. Note that $0 \in S$ generates a loop at every vertex of $\mathrm{Cay}(\Gamma, S)$. Many definitions of Cayley graphs exclude this case, but its inclusion saves us from sacrificing clarity of presentation later on.

In our paper [12] we proved for an abelian group Γ that $\mathrm{Cay}(\Gamma, S)$ is integral if S belongs to the Boolean algebra $B(\Gamma)$ generated by the subgroups of Γ. Our conjecture that the converse is true for all integral Cayley graphs over abelian groups has recently been proved by Alperin and Peterson [3].

Proposition 1. *Let* Γ *be a finite abelian group,* $S \subseteq \Gamma, \ -S = S$*. Then* $G = \mathrm{Cay}(\Gamma, S)$ *is integral if and only if* $S \in B(\Gamma)$*.*

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, D a finite set of nonnegative integers. The *distance power* G^D of G is an undirected graph with vertex set V. Vertices x and y are adjacent in G^D, if their distance $d(x, y)$ in G belongs to D. We prove that if G is an integral Cayley graph over the abelian group Γ, then every distance power G^D is also an integral Cayley graph over Γ. Moreover, we show that in a very general sense distance matrices of integral Cayley graphs over abelian groups have integral spectrum. This extends an analogous result of Ilić [11] for integral circulant graphs, which are the integral Cayley graphs over cyclic groups. Finally, we show that the class of gcd-graphs, another subclass of integral Cayley graphs over abelian groups (see [13]), is also closed under distance power operations.

2 The Boolean Algebra $B(\Gamma)$

Let Γ be an arbitrary finite, additive group. We collect facts about the Boolean algebra $B(\Gamma)$ generated by the subgroups of Γ.

2.1 Atoms of $B(\Gamma)$

Let us determine the minimal elements of $B(\Gamma)$. To this end, we consider elements of Γ to be equivalent, if they generate the same cyclic subgroup. The equivalence classes of this relation partition Γ into nonempty disjoint subsets. We shall call these sets *atoms*. The atom represented by $a \in \Gamma$, $\mathrm{Atom}(a)$, consists of the generating elements of the cyclic group $\langle a \rangle$.

$$\begin{aligned}\mathrm{Atom}(a) &= \{b \in \Gamma : \ \langle a \rangle = \langle b \rangle\} \\ &= \{ka : \ k \in \mathbb{Z}, \ 1 \leqslant k \leqslant \mathrm{ord}_\Gamma(a), \ \ \gcd(k, \mathrm{ord}_\Gamma(a)) = 1\}.\end{aligned}$$

Here, $\mathbb{Z}$ stands for the set of all integers. For a positive integer k and $a \in \Gamma$ we denote as usual by ka the k-fold sum of terms a, $(-k)a = -(ka), \ 0a = 0$. By $\mathrm{ord}_\Gamma(a)$ we mean the order of a in Γ.

Each set $\mathrm{Atom}(a)$ can be obtained by removing from $\langle a \rangle$ all elements of its proper subgroups. We bear in mind that every set $S \in B(\Gamma)$ can be derived from the cyclic subgroups of Γ by means of repeated union, intersection and complement (with respect to Γ). Thus we easily arrive at the following proposition [3].

Proposition 2. *For an arbitrary finite group* Γ *the following statements are true:*

1. $\mathrm{Atom}(a) \in B(\Gamma)$ *for every* $a \in \Gamma$*.*

2. For no $a \in \Gamma$ there exists a nonempty proper subset of $\mathrm{Atom}(a)$ *that belongs to* $B(\Gamma)$.

3. Every nonempty set $S \in B(\Gamma)$ is the union of some sets $\mathrm{Atom}(a),\ a \in \Gamma$.

2.2 Sums of Sets in $B(\Gamma)$

In this subsection Γ denotes a finite, additive, abelian group. We define the sum of nonempty subsets $S,\ T$ of Γ:

$$S + T \;=\; \{s + t:\; s \in S,\; t \in T\}.$$

We are going to show that the sum of sets in $B(\Gamma)$ is again a set in $B(\Gamma)$.

Lemma 1. *If Γ is a finite abelian group and $a,\ b \in \Gamma$ then*

$$\mathrm{Atom}(a) + \mathrm{Atom}(b) \;\in\; B(\Gamma).$$

Proof. We know that Γ can be represented (see Cohn [6]) as a direct sum of cyclic groups of prime power order. This can be grouped as

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_r,$$

where Γ_i is a direct sum of cyclic groups, the order of which is a power of a prime p_i, $|\Gamma_i| = p_i^{\alpha_i}$, $\alpha_i \geqslant 1$ for $i = 1, \ldots, r$ and $p_i \neq p_j$ for $i \neq j$. Hence we can write each element $x \in \Gamma$ as an r-tuple (x_i) with $x_i \in \Gamma_i$ for $i = 1, \ldots, r$.

The order of $x_i \in \Gamma_i$, $\mathrm{ord}_{\Gamma_i}(x_i)$, is a divisor of $p_i^{\alpha_i}$. Therefore, integer factors in the i-th coordinate of x may be reduced modulo $p_i^{\alpha_i}$. The order of $x \in \Gamma$, $\mathrm{ord}_\Gamma(x)$, is the least common multiple of the orders of its coordinates:

$$\mathrm{ord}_\Gamma(x) \;=\; \mathrm{lcm}(\mathrm{ord}_{\Gamma_1}(x_1), \ldots, \mathrm{ord}_{\Gamma_r}(x_r)). \tag{1}$$

This implies that all prime divisors of $\mathrm{ord}_\Gamma(x)$ belong to $\{p_1, \ldots, p_r\}$.

Let $a = (a_i),\ b = (b_i)$ be elements of Γ. The statement of the lemma becomes trivial for $a = 0$ or $b = 0$. So we may assume $a \neq 0$ and $b \neq 0$. An arbitrary element $w \in \mathrm{Atom}(a) + \mathrm{Atom}(b)$ has the following form:

$$\begin{aligned} w \;&=\; \lambda a + \mu b, \\ 1 \leqslant \lambda &\leqslant \mathrm{ord}_\Gamma(a), \quad \gcd(\lambda, \mathrm{ord}_\Gamma(a)) = 1, \\ 1 \leqslant \mu &\leqslant \mathrm{ord}_\Gamma(b), \quad \gcd(\mu, \mathrm{ord}_\Gamma(b)) = 1. \end{aligned} \tag{2}$$

We have to show $\mathrm{Atom}(w) \subseteq \mathrm{Atom}(a) + \mathrm{Atom}(b)$. To this end, we choose the integer ν with $1 \leqslant \nu \leqslant \mathrm{ord}_\Gamma(w)$, $\gcd(\nu, \mathrm{ord}_\Gamma(w)) = 1$, and show $\nu w \in \mathrm{Atom}(a) + \mathrm{Atom}(b)$.

Case 1. $\quad (p_1 p_2 \cdots p_r) \mid \mathrm{ord}_\Gamma(w)$.

By $\gcd(\nu, \mathrm{ord}_\Gamma(w)) = 1$ we know that ν has no prime divisor in $\{p_1, \ldots, p_r\}$. On the other hand all prime divisors of $\mathrm{ord}_\Gamma(a)$ and of $\mathrm{ord}_\Gamma(b)$ are in $\{p_1, \ldots, p_r\}$. This implies $\gcd(\nu, \mathrm{ord}_\Gamma(a)) = 1$ and $\gcd(\nu, \mathrm{ord}_\Gamma(b)) = 1$. Setting $\lambda' = \nu\lambda$ and $\mu' = \nu\mu$ we achieve

$$\gcd(\lambda', \mathrm{ord}_\Gamma(a)) = 1,\ \lambda' a \in \mathrm{Atom}(a),$$
$$\gcd(\mu', \mathrm{ord}_\Gamma(b)) = 1,\ \mu' b \in \mathrm{Atom}(b).$$

Now we have by (2):

$$\nu w = \nu\lambda a + \nu\mu b = \lambda' a + \mu' b \in \mathrm{Atom}(a) + \mathrm{Atom}(b).$$

Case 2. $(p_1 p_2 \cdots p_r) \not| \,\mathrm{ord}_\Gamma(w)$.

Trivially, for $w = 0 \in \mathrm{Atom}(a)+\mathrm{Atom}(b)$ we have $\nu w \in \mathrm{Atom}(a)+\mathrm{Atom}(b)$. Therefore, we may assume $w \neq 0$. Without loss of generality let

$$(p_1 \cdots p_k) \mid \mathrm{ord}_\Gamma(w),\ \ \gcd(\mathrm{ord}_\Gamma(w), p_{k+1} \cdots p_r) = 1,\ 1 \leqslant k < r. \tag{3}$$

Now (1) and (3) imply

$$\begin{aligned} w \ = \ \lambda a + \mu b \ = \ (\lambda a_1 + \mu b_1, \ldots, \lambda a_k + \mu b_k, 0, \ldots, 0), \\ \lambda a_i + \mu b_i \neq 0 \text{ for } i = 1, \ldots, k. \end{aligned} \tag{4}$$

By $\gcd(\nu, ord_\Gamma(w)) = 1$ we know $\gcd(\nu, p_1 \cdots p_k) = 1$. If even more $\gcd(\nu, p_1 \cdots p_r) = 1$ then we deduce $\nu w \in \mathrm{Atom}(a) + \mathrm{Atom}(b)$ as in Case 1. So we may assume that ν has at least one prime divisor in $\{p_{k+1}, \ldots, p_r\}$. Without loss of generality let

$$\gcd(\nu, p_1 \cdots p_l) = 1,\ (p_{l+1} \cdots p_r) \mid \nu,\ k \leqslant l < r.$$

We define

$$\nu' \ = \ \nu \ + \ p_1^{\alpha_1} \cdots p_l^{\alpha_l}. \tag{5}$$

If we observe that integer factors in the i-th coordinate of w can be reduced modulo $p_i^{\alpha_i}$, then we see by (4): $\nu' w = \nu w$. Moreover, (5) and the properties of ν imply $\gcd(\nu', p_1 \cdots p_r) = 1$. As in Case 1 we now conclude $\nu w = \nu' w \in \mathrm{Atom}(a) + \mathrm{Atom}(b)$. □

Corollary 1. *If Γ is a finite abelian group with nonempty subsets $S, T \in B(\Gamma)$ then $S + T \in B(\Gamma)$.*

Proof. According to Proposition 2 the sets S and T are unions of atoms of $B(\Gamma)$.

$$S \ = \ \bigcup_{i=1}^{k} \mathrm{Atom}(a_i),\ \ T \ = \ \bigcup_{j=1}^{l} \mathrm{Atom}(b_j).$$

Then we have

$$S + T \ = \ \bigcup_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l} (\mathrm{Atom}(a_i) + \mathrm{Atom}(b_j)). \tag{6}$$

According to Lemma 1 the sum $\mathrm{Atom}(a_i) + \mathrm{Atom}(b_j)$ is an element of $B(\Gamma)$. Therefore, (6) implies $S + T \in B(\Gamma)$. □

3 Distance Powers and Distance Matrices

We repeat the definition of the distance power G^D of an undirected graph $G = (V, E)$ from the Introduction. Let D be a set of nonnegative integers. The distance power G^D has vertex set V. Vertices x, y are adjacent in G^D, if their distance in G is $d(x, y) \in D$. If G is not connected, it makes sense to allow $\infty \in D$. Clearly, $G^\emptyset$ is the graph without edges on V. The edge set of $G^{\{0\}}$ consists of a single loop at every vertex of G. If G has no loops then $G^{\{1\}} = G$.

Theorem 1. *If $G = \mathrm{Cay}(\Gamma, S)$ is an integral Cayley graph over the finite abelian group Γ and if D is a set of nonnegative integers (possibly including ∞), then the distance power G^D is also an integral Cayley graph over Γ.*

Proof. If $D = \emptyset$ then $G^D = \mathrm{Cay}(\Gamma, \emptyset)$ is an integral Cayley graph over Γ. We now consider the case, where D has only one element,

$$D = \{d\}, \quad d \in \{0, 1, \ldots, \infty\}.$$

In several steps we define $S^{(d)} \in B(\Gamma)$ such that $G^{\{d\}} = \mathrm{Cay}(\Gamma, S^{(d)})$ is an integral Cayley graph over Γ. If d is a distance not attained in G, then the assertion is confirmed by $G^{\{d\}} = \mathrm{Cay}(\Gamma, S^{(d)})$ with $S^{(d)} = \emptyset$. If $d = 0$ then we achieve our goal by $S^{(0)} = \{0\}$. Suppose now that $d = \infty$ and G is disconnected. If $U = \langle S \rangle$ is the subgroup generated by S in Γ, then G consists of disjoint subgraphs on the cosets of U, all of them isomorphic to $\mathrm{Cay}(U, S)$. Vertices x, y in $G^{\{\infty\}}$ are adjacent if and only if they belong to different cosets of U, and this is true if and only if $x - y \notin U$. Therefore, we have

$$G^{\{\infty\}} = \mathrm{Cay}(\Gamma, S^{(\infty)}) \text{ with } S^{(\infty)} = \overline{U} = \Gamma \backslash U \in B(\Gamma).$$

Assume now that $d \geqslant 1$ is a finite distance attained between vertices x, y in G. The sequence of vertices in a shortest path P between x and y in $G = \mathrm{Cay}(\Gamma, S)$ has the form

$$x, x + s_1, x + s_1 + s_2, \ldots, x + s_1 + \ldots + s_d = y, \; s_i \in S \text{ for } 1 \leqslant i \leqslant d.$$

This implies $y - x = s_1 + \ldots + s_d \in dS$, where dS denotes the d-fold sum of the set S. To guarantee that there is no shorter path from x to y than P we remove from dS all multiples kS for $0 \leqslant k < d$, $0S = \{0\}$. Setting

$$S^{(d)} \;=\; dS \setminus \bigcup_{0 \leqslant k < d} kS \tag{7}$$

we achieve $G^{\{d\}} = \mathrm{Cay}(\Gamma, S^{(d)})$. If $G = \mathrm{Cay}(\Gamma, S)$ is integral, then we have $S \in B(\Gamma)$ by Proposition 1, $kS \in B(\Gamma)$ for every $k \geqslant 2$ by Corollary 1, and trivially $0S = \{0\} \in B(\Gamma)$. By (7) this implies $S^{(d)} \in B(\Gamma)$, so $G^{\{d\}}$ is an integral Cayley graph over Γ.

To complete our proof, let

$$D = \{d_1, \ldots, d_r\} \subseteq \{0, 1, \ldots, \infty\} \text{ and } S^{(D)} = \bigcup_{i=1}^{r} S^{(d_i)}.$$

Then we have $S^{(D)} \in B(\Gamma)$ and $G^D = \mathrm{Cay}(\Gamma, S^{(D)})$ is an integral Cayley graph over Γ by Proposition 1. □

Let Γ be a finite additive group. A *character* ψ of Γ is a homomorphism from Γ into the multiplicative group of complex numbers. An abelian group Γ with n elements has exactly n distinct characters, which represent an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors for every Cayley graph over Γ. More precisely, we have (see e. g. [12] or [14])

Proposition 3. *Let $\psi_1, \ldots, \psi_n$ be the distinct characters of the additive abelian group $\Gamma = \{v_1, \ldots, v_n\}$, $S \subseteq \Gamma$, $-S = S$. Assume that $A = (a_{i,j})$ is the adjacency matrix of $G = \mathrm{Cay}(\Gamma, S)$ with respect to the given ordering of the vertex set $V(G) = \Gamma$. Then the vectors $(\psi_i(v_j))_{j=1,\ldots,n}$, $1 \leqslant i \leqslant n$, constitute an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors of A. To the eigenvector $(\psi_i(v_j))_{j=1,\ldots,n}$ belongs the eigenvalue $\psi_i(S) = \sum_{s\in S} \psi_i(s)$.*

Now we define a generalized distance matrix $\mathrm{DM}(k, G)$ of a given undirected graph G with vertex set $\{v_1, \ldots, v_n\}$ as follows. Let $d_0 = 0 < d_1 < \ldots < d_r$ be the sequence of possible distances between vertices in G, possibly $d_r = \infty$. If $k = (k_0, \ldots, k_r)$ is a vector with integral entries, then we define the entries of $\mathrm{DM}(k, G) = (d^{(k)}_{i,j})$ for $i, j \in \{1, \ldots, n\}$ by

$$d^{(k)}_{i,j} \;=\; k_t \text{ , if } \; d(v_i, v_j) = d_t.$$

The ordinary distance matrix $\mathrm{DM}(G)$ for a connected graph G is established for $k = (0, 1, ..., r)$, where r is the diameter of G.

Let $\Gamma = \{v_1, \ldots, v_n\}$ be an abelian group and consider some integral Cayley graph $G = \mathrm{Cay}(\Gamma, S)$. Any generalized distance matrix $\mathrm{DM}(k, G)$ is an integer weighted sum of the adjacency matrices of the graphs $G^{\{d\}}$ with $d \in \{d_0, d_1, \ldots, d_r\}$, assuming $v_1, \ldots, v_n$ as their common vertex order. To make it more precise, for $j = 0, \ldots, r$ we denote by $A^{(j)}$ the adjacency matrix of the distance power $G^{\{d_j\}}$, $A^{(0)} = I_n$ is the $n \times n$ unit matrix. Then we have

$$\mathrm{DM}(k, G) = k_0 A^{(0)} + k_1 A^{(1)} + \ldots + k_r A^{(r)}.$$

By Theorem 1, all matrices $A^{(j)}$, $0 \leqslant j \leqslant r$, are adjacency matrices of integral Cayley graphs over Γ. According to Proposition 3, all Cayley graphs over Γ have a universal common basis of complex eigenvectors. As a result, integrality extends to $\mathrm{DM}(k, G)$. This proves the following theorem.

Theorem 2. *Let $G = \mathrm{Cay}(\Gamma, S)$ be an integral Cayley graph over the abelian group Γ, $|\Gamma| = n$. Then every distance matrix $\mathrm{DM}(k, G)$ as defined above has integral spectrum. Moreover, the characters $\psi_1, \ldots, \psi_n$ of Γ represent an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors of $\mathrm{DM}(k, G)$.*

As we have seen in Theorem 1, the class of integral Cayley graphs over an abelian group is closed under distance power operations. We shall conclude this section by presenting a subclass which has the same closure property.

We introduce the class of *gcd-graphs* as in [13]. To this end, let the finite abelian group Γ be represented as the direct product of cyclic groups, $\Gamma = \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_r}$, $m_i \geqslant 1$ for $i = 1, \ldots, r$. Hence the elements $x \in \Gamma$ take the form of r-tuples.

$$x = (x_i) = (x_1, \ldots, x_r),\ x_i \in \mathbb{Z}_{m_i} = \{0, 1, \ldots, m_i - 1\},\ 1 \leqslant i \leqslant r.$$

Addition is coordinatewise modulo m_i. For $x=(x_1,\dots,x_r)\in\Gamma$ and $m=(m_1,\dots,m_r)$ we define

$$\gcd(x,m)=(\gcd(x_1,m_1),\dots,\gcd(x_r,m_r)).$$

Here we agree upon $gcd(0,m_i)=m_i$. For a divisor tuple $d=(d_1,\dots,d_r)$ of m, $d\mid m$, we require $d_i\geqslant 1$ and $d_i\mid m_i$ for every $i=1,\dots,r$. Every divisor tuple d of m defines an *elementary gcd-set* given by

$$S_\Gamma(d)=\{x\in\Gamma:\ \gcd(x,m)=d\}.$$

Clearly, the sets $S_\Gamma(d)$ with $d\mid m$ form a partition of the elements of Γ. We denote by $E_\Gamma(x)$ the unique elementary gcd-set that contains x, i.e. $E_\Gamma(x)=S_\Gamma(d)$ with $d=\gcd(x,m)$. A *gcd-set* is a union of elementary gcd-sets. By construction, the elementary gcd-sets are the atoms of the Boolean algebra $B_{\mathrm{gcd}}(\Gamma)$ consisting of all gcd-sets of Γ. According to Theorem 1 in [13], $B_{\mathrm{gcd}}(\Gamma)$ is a Boolean sub-algebra of $B(\Gamma)$. Hence by Proposition 1, all gcd-graphs $Cay(\Gamma,S)$, $S\in B_{gcd}(\Gamma)$, are integral.

Lemma 2. *If* $\Gamma=\mathbb{Z}_{m_1}\oplus\ldots\oplus\mathbb{Z}_{m_r}$ *and* $x=(x_1,\dots,x_r)\in\Gamma$ *then*

$$E_\Gamma(x)=E_{\mathbb{Z}_{m_1}}(x_1)\times\ldots\times E_{\mathbb{Z}_{m_r}}(x_r).$$

Proof. Let $m=(m_1,\dots,m_r)$ and $d=(d_1,\dots,d_r)=\gcd(x,m)$. Then we have $y=(y_1,\dots,y_r)\in E_\Gamma(x)$ if and only if $\gcd(y_i,m_i)=d_i$ for $i=1,\dots,r$. This is equivalent to $y\in S_{\mathbb{Z}_{m_1}}(d_1)\times\ldots\times S_{\mathbb{Z}_{m_r}}(d_r)$, which is the same as $y\in E_{\mathbb{Z}_{m_1}}(x_1)\times\ldots\times E_{\mathbb{Z}_{m_r}}(x_r)$. □

Lemma 3. *For every finite abelian group* Γ*, any sum of its gcd-sets is again a gcd-set.*

Proof. As in the proof of Corollary 1 it suffices to show that any sum of elementary gcd-sets is a gcd-set. If Γ is cyclic, then $B_{gcd}(\Gamma)=B(\Gamma)$ (see Theorem 3 in [13]) and the result follows from Lemma 1.

Now let $\Gamma=\mathbb{Z}_{m_1}\oplus\ldots\oplus\mathbb{Z}_{m_r}$, $m=(m_1,\dots,m_r)$, $r\geqslant 2$. Further let $x=(x_1,\dots,x_r)\in\Gamma$, $\gcd(x,m)=d=(d_1,\dots,d_r)$ and let $y=(y_1,\dots,y_r)\in\Gamma$, $\gcd(y,m)=\delta=(\delta_1,\dots,\delta_r)$. By Lemma 2 we have

$$E_\Gamma(x)+E_\Gamma(y)=(E_{\mathbb{Z}_{m_1}}(x_1)+E_{\mathbb{Z}_{m_1}}(y_1))\times\ldots\times(E_{\mathbb{Z}_{m_r}}(x_r)+E_{\mathbb{Z}_{m_r}}(y_r)).$$

Since the cyclic case is already solved, it follows that $E_{\mathbb{Z}_{m_i}}(x_i)+E_{\mathbb{Z}_{m_i}}(y_i)$ is a gcd-set of $\mathbb{Z}_{m_i}$ for $i=1,\dots,r$. Hence $E_{\mathbb{Z}_{m_i}}(x_i)+E_{\mathbb{Z}_{m_i}}(y_i)$ is a disjoint union of elementary gcd-sets $E_{\mathbb{Z}_{m_i}}(z_1^{(i)}),\dots,E_{\mathbb{Z}_{m_i}}(z_{\varrho_i}^{(i)})$, with $z_j^{(i)}\in\mathbb{Z}_{m_i}$ for $j=1,\dots,\varrho_i$. It follows that

$$E_\Gamma(x)+E_\Gamma(y)=\bigcup_{1\leqslant j_k\leqslant\varrho_k,\ k=1,\dots,r}\left(E_{\mathbb{Z}_{m_1}}(z_{j_1}^{(1)})\times\ldots\times E_{\mathbb{Z}_{m_r}}(z_{j_r}^{(r)})\right).$$

Writing $z^{(j_1,\dots,j_r)}=(z_{j_1}^{(1)},\dots,z_{j_r}^{(r)})$, we get by Lemma 2

$$E_\Gamma(x)+E_\Gamma(y)=\bigcup_{1\leqslant j_k\leqslant\varrho_k,\ k=1,\dots,r}E_\Gamma(z^{(j_1,\dots,j_r)})\ \in B_{gcd}(\Gamma).$$

□

The following theorem is readily deduced from Lemma 3 applying the same reasoning as in the proof of Theorem 1.

Theorem 3. *If* $G = \mathrm{Cay}(\Gamma, S)$ *is a gcd-graph over* $\Gamma = \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_r}$ *and if* D *is a set of nonnegative integers (possibly including* ∞*), then the distance power* G^D *is also a gcd-graph over* Γ*.*

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