Invariant Principal Order Ideals under Foata's Transformation

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Abstract

Let Φ denote Foata's second fundamental transformation on permutations. For a permutation σ in the symmetric group S_n, let $\widetilde{\Lambda}_\sigma = \{\pi \in S_n \colon \pi \leqslant_w \sigma\}$ be the principal order ideal generated by σ in the weak order $\leqslant_w$. Björner and Wachs have shown that $\widetilde{\Lambda}_\sigma$ is invariant under Φ if and only if σ is a 132-avoiding permutation. In this paper, we consider the invariance property of Φ on the principal order ideals $\Lambda_\sigma = \{\pi \in S_n \colon \pi \leqslant \sigma\}$ with respect to the Bruhat order $\leqslant$. We obtain a characterization of permutations σ such that Λ_σ are invariant under Φ. We also consider the invariant principal order ideals with respect to the Bruhat order under Han's bijection H. We find that Λ_σ is invariant under the bijection H if and only if it is invariant under the transformation Φ.

Keywords: Foata's second fundamental transformation; Han's bijection; Bruhat order; principal order ideal

1 Introduction

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$. Foata's second fundamental transformation Φ on permutations in S_n maps the major index of a permutation π to the inversion number of $\Phi(\pi)$, see Foata [5]. Björner and Wachs [2] have shown that Foata's second fundamental transformation can also be used in the study of subsets U of S_n over which the inversion number and the major index are equidistributed. In particular, they

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showed that if the subset U is an order ideal of permutations in S_n with respect to the weak order, then U is invariant under Φ if and only if the maximal elements of U are 132-avoiding permutations.

In this paper, we investigate principal order ideals $\Lambda_\sigma = \{\pi \in S_n \colon \pi \leqslant \sigma\}$ with respect to the Bruhat order $\leqslant$ that are invariant under Foata's transformation. We obtain a characterization of permutations σ for which Λ_σ is invariant under Φ.

We also consider the principal order ideals Λ_σ that are invariant under Han's bijection [6] while restricted to permutations. Recall that Han's bijection, denoted H, is a Foata-style bijection defined on words, which can be used to show that the Z-statistic introduced by Zeilberger and Bressoud [7] is Mahonian. We shall show that a principal order ideal Λ_σ with respect to the Bruhat order is invariant under H if and only if it is invariant under Φ.

Let us give a brief review of Foata's transformation Φ and Han's bijection H on permutations. To describe Φ, we need to define a factorization for permutations on a finite set of positive integers. Let A be a set of n positive integers, and let x be an integer not belonging to A. For any permutation $w = w_1 w_2 \cdots w_n$ on A, x induces a factorization

$$w = \gamma_1 \gamma_2 \cdots \gamma_j,$$

where each subword γ_i $(1 \leqslant i \leqslant j)$ is determined uniquely as follows:

(i) If $w_n < x$, then the last element of γ_i is smaller than x and all the remaining elements of γ_i are greater than x;

(ii) If $w_n > x$, then the last element of γ_i is greater than x and all the remaining elements of γ_i are smaller than x.

For example, let $w = 387125$. Then the factorization induced by $x = 4$ is $w = 38{\cdot}7{\cdot}125$, while the factorization induced by $x = 6$ is $w = 3 \cdot 871 \cdot 2 \cdot 5$, where we use dots to separate the factors.

For a factorization $w = \gamma_1 \gamma_2 \cdots \gamma_j$ induced by x, let

$$\delta_x(w) = \gamma_1' \gamma_2' \cdots \gamma_j',$$

where γ_i' $(1 \leqslant i \leqslant j)$ is obtained from γ_i by moving the last element to the beginning of γ_i. For example, for the permutation $w = 387125$, based on the above factorization, we have $\delta_4(w) = 837512$.

To define Φ, we still need the k-th $(1 \leqslant k \leqslant n)$ Foata bijection $\phi_k \colon S_n \longrightarrow S_n$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$. For $k = 1$, define $\phi_k(\sigma) = \sigma$. For $k > 1$, define

$$\phi_k(\sigma) = \delta_{\sigma_k}(\sigma_1 \sigma_2 \cdots \sigma_{k-1}) \cdot \sigma_k \sigma_{k+1} \cdots \sigma_n.$$

The transformation Φ is defined to be the composition $\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$, that is, for $\sigma \in S_n$,

$$\Phi(\sigma) = \phi_n(\cdots \phi_2((\phi_1(\sigma))) \cdots).$$

We next describe Han's bijection H on permutations, and we need two maps C_x and C^x ($x \in [n]$) from the set of permutations on $[n]\setminus\{x\}$ to S_{n-1}. Assume that $w = w_1w_2\cdots w_{n-1}$ is a permutation on $[n]\setminus\{x\}$. Let $\tau_i = w_i - x \pmod{n}$, namely,

$$\tau_i = \begin{cases} w_i - x + n, & \text{if } w_i < x; \\ w_i - x, & \text{if } w_i > x, \end{cases}$$

and let

$$\nu_i = \begin{cases} w_i, & \text{if } w_i < x; \\ w_i - 1, & \text{if } w_i > x. \end{cases}$$

The maps C_x and C^x are defined by

$$C^x(w) = \tau_1\tau_2\cdots\tau_{n-1} \quad \text{and} \quad C_x(w) = \nu_1\nu_2\cdots\nu_{n-1}.$$

It is easy to check that both C^x and C_x are one-to-one correspondences between the set of permutations on $[n]\setminus\{x\}$ and S_{n-1}. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. The bijection H can be defined recursively as follows

$$H(\sigma) = C_{\sigma_n}^{-1}(H(C^{\sigma_n}(\sigma')))\sigma_n,$$

where we set $H(1) = 1$ and $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. Note that the bijection H can also be described in terms of permutation codes, see Chen, Fan and Li [3].

We now recall the Bruhat order on permutations. To describe this partial order, we need to clarify the definition of the multiplication of permutations. First, we regard a permutation $\pi \in S_n$ as a bijection on $[n]$ by setting $\pi(i) = \pi_i$. The product $\pi\sigma$ of two permutations $\pi, \sigma \in S_n$ is defined as the composition of π and σ as functions, that is, $\pi\sigma(i) = \pi(\sigma(i))$ for $i \in [n]$. For $1 \leqslant i < j \leqslant n$, let (i, j) denote the transposition of S_n that interchanges the elements i and j. Thus, the multiplication on the right of a permutation π by a transposition (i, j) has the same effect as interchanging the elements π_i and π_j. Similarly, the multiplication on the left of a permutation π by a transposition (i, j) is equivalent to the exchange of the elements i and j. For example, for $\pi = 236514$, we have $\pi(2, 5) = 216534$ and $(2, 5)\pi = 536214$.

The Bruhat order of S_n is defined as follows. For two permutations $\pi, \sigma \in S_n$, we say that $\pi \leqslant \sigma$ if there exists a sequence of transpositions $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ such that

$$\sigma = \pi(i_1, j_1)(i_2, j_2)\cdots(i_k, j_k)$$

and

$$\mathrm{inv}(\pi(i_1, j_1)\cdots(i_{t-1}, j_{t-1})) < \mathrm{inv}(\pi(i_1, j_1)\cdots(i_t, j_t)), \quad \text{for} \quad t = 1, 2, \ldots, k,$$

where $\mathrm{inv}(\pi)$ is the inversion number of π, namely,

$$\mathrm{inv}(\pi) = |\{(\pi_i, \pi_j)\colon\ 1 \leqslant i < j \leqslant n \text{ and } \pi_i > \pi_j\}|.$$

If replacing transpositions by adjacent transpositions in the above definition, then the Bruhat order reduces to the weak order $\leqslant_w$. Denote by Λ_σ and $\widetilde{\Lambda}_\sigma$ the principal order ideals generated by σ in the Bruhat order and the weak order respectively.

The following theorem gives a characterization of the covering relation in the Bruhat order.

Theorem 1 (Björner and Brenti [1], Lemma 2.1.4)**.** *Let $\pi, \sigma \in S_n$. Then π is covered by σ in the Bruhat order if and only if $\sigma = \pi(i,j)$ for some $1 \leqslant i < j \leqslant n$ such that $\pi_i < \pi_j$ and there does not exist k such that $i < k < j$ and $\pi_i < \pi_k < \pi_j$.*

The following theorem is due to Ehresmann [4], see also Björner and Brenti [1, Theorem 2.1.5], which gives a criterion for the comparison of two permutations in the Bruhat order. This characterization will be employed in the proof of Theorem 3. For a permutation $\pi \in S_n$ and two integers $1 \leqslant i, j \leqslant n$, let $\pi[i,j]$ be the number of elements in $\{\pi_1, \pi_2, \ldots, \pi_i\}$ that are greater than or equal to j, that is,

$$\pi[i,j] = |\{\pi_k \colon 1 \leqslant k \leqslant i \quad \text{and} \quad \pi_k \geqslant j\}|.$$

Theorem 2 (Ehresmann [4])**.** *Let $\pi, \sigma \in S_n$. Then $\pi \leqslant \sigma$ if and only if*

$$\pi[i,j] \leqslant \sigma[i,j]$$

for any $1 \leqslant i, j \leqslant n$,

This paper is organized as follows. In Section 2, we present a characterization of permutations σ such that the principal order ideals Λ_σ are invariant under Φ. Section 3 is devoted to the proof of the fact that Λ_σ is invariant under Han's bijection H if and only if it is invariant under Foata's transformation Φ.

2 Invariant principal order ideals under Foata's map

The objective of this section is to give a characterization of invariant principal ideals Λ_σ under Foata's transformation Φ. To this end, we need a property on the maximal elements of the following subset $\Lambda_\sigma(k)$ of Λ_σ which is defined by

$$\Lambda_\sigma(k) = \{\tau \colon \tau \in \Lambda_\sigma \quad \text{and} \quad \tau_n = k\},$$

where $1 \leqslant k \leqslant n$. It should be noted that by Theorem 2, $\Lambda_\sigma(k)$ is nonempty unless $k \geqslant \sigma_n$. We need to consider a special element in $\Lambda_\sigma(k)$, denoted $M(\sigma,k)$. Define $M(\sigma,k) = \sigma$ if $k = \sigma_n$. The definition of $M(\sigma,k)$ for $k > \sigma_n$ can be described as follows.

Let i_1 be the largest element in σ such that i_1 is to the right of k and $\sigma_n \leqslant i_1 < k$. If $i_1 = \sigma_n$, then define $M(\sigma,k) = (k,i_1)\sigma$. Otherwise, we continue to consider the permutation $(k,i_1)\sigma$. Let i_2 be the largest element in $(k,i_1)\sigma$ such that i_2 is to the right of k and $\sigma_n \leqslant i_2 < k$. If $i_2 = \sigma_n$, then define $M(\sigma,k) = (k,i_2)(k,i_1)\sigma$. Otherwise, we consider the permutation $(k,i_2)(k,i_1)\sigma$, and let i_3 be the largest element in $(k,i_2)(k,i_1)\sigma$

such that i_3 is to the right of k and $\sigma_n \leqslant i_3 < k$. Repeating this procedure, we end up with an element i_s ($1 \leqslant s \leqslant n$) such that $i_s = \sigma_n$. Define

$$M(\sigma, k) = (k, i_s) \cdot \cdots (k, i_2)(k, i_1)\sigma.$$

For example, for $\sigma = 875169423$, $M(\sigma, 7)$ is constructed as follows. It is clear that $i_1 = 6$. So, we have $(7, i_1)\sigma = 865179423$. Now we see that $i_2 = 4$, and hence $(7, i_2)(7, i_1)\sigma = 865149723$. Since $i_3 = 3 = \sigma_n$, we find $M(\sigma, 7) = 865149327$.

The following theorem will be employed in the proof of Theorem 5.

Theorem 3. *Suppose that σ is a permutation in S_n and k is an integer such that $\sigma_n \leqslant k \leqslant n$. Then, $M(\sigma, k)$ is the unique maximal element of $\Lambda_\sigma(k)$, that is, $M(\sigma, k) \geqslant \tau$ for any $\tau \in \Lambda_\sigma(k)$.*

Proof. It is clear that the theorem holds when $k = \sigma_n$. Now we consider the case $k > \sigma_n$. Assume that

$$M(\sigma, k) = (k, i_s) \cdots (k, i_2)(k, i_1)\sigma.$$

Let $\tau \in \Lambda_\sigma(k)$, and denote $\sigma^{(1)} = (k, i_1)\sigma$. To prove $\tau \leqslant M(\sigma, k)$, we first show that

$$\tau \leqslant \sigma^{(1)}. \tag{1}$$

Let m and t be the indices such that $\sigma_m = k$ and $\sigma_t = i_1$. By the choice of i_1, we see that $1 \leqslant m < t \leqslant n$. It is easy to verify the following relation

$$\sigma^{(1)}[i, j] = \begin{cases} \sigma[i, j] - 1, & \text{if } m \leqslant i < t \text{ and } i_1 < j \leqslant k; \\ \sigma[i, j], & \text{otherwise.} \end{cases} \tag{2}$$

Since $\tau[i, j] \leqslant \sigma[i, j]$ for $1 \leqslant i, j \leqslant n$, we see that (1) can be deduced from the following relation

$$\tau[i, j] < \sigma[i, j], \qquad \text{for } m \leqslant i < t \text{ and } i_1 < j \leqslant k. \tag{3}$$

We now proceed to prove (3). We shall present detailed argument for the case $i = m$ and $j = i_1 + 1$ and the remaining cases can be dealt with in the same vein. Suppose to the contrary that $\tau[m, i_1 + 1] = \sigma[m, i_1 + 1]$. By the choice of i_1, we see that the elements $i_1 + 1, i_1 + 2, \ldots, k - 1$ in σ are all to the left of σ_m. Thus, we get

$$\sigma[m, i_1 + 2] = \sigma[m, i_1 + 1] - 1.$$

It follows that

$$\tau[m, i_1 + 2] \geqslant \tau[m, i_1 + 1] - 1 = \sigma[m, i_1 + 1] - 1 = \sigma[m, i_1 + 2]. \tag{4}$$

On the other hand, since $\tau \leqslant \sigma$, we see that $\tau[m, i_1 + 2] \leqslant \sigma[m, i_1 + 2]$. Hence,

$$\tau[m, i_1 + 2] = \sigma[m, i_1 + 2].$$

In a similar fashion, we find that

$$\tau[m, i_1 + 3] = \sigma[m, i_1 + 3], \quad \tau[m, i_1 + 4] = \sigma[m, i_1 + 4], \quad \ldots \quad , \tau[m, k] = \sigma[m, k].$$

From the relation $\tau[m, k] = \sigma[m, k]$, we assert that the number k belongs to the set $\{\tau_1, \tau_2, \ldots, \tau_m\}$. This can be seen as follows. Suppose to the contrary that $k \notin \{\tau_1, \tau_2, \ldots, \tau_m\}$. Then, we have $\tau[m, k+1] = \tau[m, k]$. But, since $\sigma_m = k$, we get $\sigma[m, k+1] = \sigma[m, k] - 1$. Thus we obtain that $\tau[m, k+1] > \sigma[m, k+1]$, a contradiction.

Obviously, the above assertion that $k \in \{\tau_1, \tau_2, \ldots, \tau_m\}$ is contrary to the assumption that $\tau_n = k$. Thus, relation (3) holds for $i = m$ and $j = i_1 + 1$. This completes the proof of the relation in (1).

Using the same argument, we can deduce that

$$\tau \leqslant (k, i_2)\sigma^{(1)}.$$

Repeating this procedure, we finally get

$$\tau \leqslant (k, i_s) \cdots (k, i_2)\sigma^{(1)} = M(\sigma, k).$$

This completes the proof. □

Before stating our main theorem, we need a result of Björner and Wachs [2] on the weak order. Let $\leqslant_w$ denote the weak order on permutations. For a given permutation $\sigma \in S_n$, they constructed a permutation $\gamma(\sigma) \in S_n$ such that $\gamma(\sigma) \leqslant_w \sigma$, $\mathrm{inv}(\sigma) \leqslant \mathrm{maj}(\gamma(\sigma))$, and the last element of $\gamma(\sigma)$ is equal to σ_n. Moreover, they showed that $\mathrm{inv}(\sigma) = \mathrm{maj}(\gamma(\sigma))$ if and only if σ is 132-avoiding. Recall that a permutation σ is 132-avoiding if there do not exist numbers $1 \leqslant a < b < c \leqslant n$ such that $\sigma_a < \sigma_c < \sigma_b$. We shall not give the precise definition of $\gamma(\sigma)$, since we only require the properties of $\gamma(\sigma)$ as mentioned above.

The following lemma will be used in the proof of Theorem 5.

Lemma 4. *Suppose that $\sigma \in S_n$ is a permutation such that Λ_σ is invariant under Foata's map Φ. Then σ is a 132-avoiding permutation.*

Proof. Since Λ_σ is invariant under Φ, it is easy to see that $\Phi(\gamma(\sigma)) \leqslant \sigma$. Thus we have

$$\mathrm{inv}(\Phi(\gamma(\sigma))) \leqslant \mathrm{inv}(\sigma). \tag{5}$$

Recall that $\mathrm{inv}(\Phi(\gamma(\sigma))) = \mathrm{maj}(\gamma(\sigma))$. Hence, by (5), we deduce that $\mathrm{maj}(\gamma(\sigma)) \leqslant \mathrm{inv}(\sigma)$. On the other hand, as we have mentioned above, $\gamma(\sigma)$ possesses the property that $\mathrm{maj}(\gamma(\sigma)) \geqslant \mathrm{inv}(\sigma)$. So we get $\mathrm{maj}(\gamma(\sigma)) = \mathrm{inv}(\sigma)$. In other words, σ is 132-avoiding. This completes the proof. □

We are now ready to state the main result in this paper.

Theorem 5. *Suppose that σ is a permutation in S_n. Then the following assertions hold:*

(1) If $\sigma_n = n$, then Λ_σ is invariant under Φ if and only if $\Lambda_{\sigma'}$ is invariant under Φ, where

$$\sigma' = \sigma_1\sigma_2 \cdots \sigma_{n-1}.$$

(2) If $\sigma_n = k < n$, then Λ_σ is invariant under Φ if and only if

$$\sigma = n\,(n-1)\,\cdots\,(k+1)\,(k-1)\,\cdots\,2\,1\,k, \tag{6}$$

where $\sigma = n\,(n-1)\,\cdots\,2\,1$ for $k = 1$.

Proof. We first prove (1). Assume that Λ_σ is invariant under Foata's map Φ. To prove that $\Lambda_{\sigma'}$ is invariant under Φ, let π' be a permutation in S_{n-1} such that $\pi' \leqslant \sigma'$. It is easy to see that $\pi'\,n \leqslant \sigma'\,n = \sigma$. Thus, $\Phi(\pi' n) \leqslant \sigma$. Since $\Phi(\pi' n) = \Phi(\pi')n$, we have

$$\Phi(\pi')n \leqslant \sigma.$$

By the fact that $\sigma_n = n$, we find $\Phi(\pi') \leqslant \sigma'$. Hence $\Lambda_{\sigma'}$ is invariant under Φ. Conversely, it can be easily checked that if $\Lambda_{\sigma'}$ is invariant under Φ, then Λ_σ is invariant under Φ.

Next, we proceed to prove (2). In this case, $\sigma_n = k < n$. Assume that σ is a permutation in (6). It follows from Theorem 2 that

$$\Lambda_\sigma = \{\pi \in S_n \colon \pi_n \geqslant k\}. \tag{7}$$

Let π be a permutation in Λ_σ. Since the last element of $\Phi(\pi)$ is equal to π_n, from (7) we see that $\Phi(\pi)$ belongs to Λ_σ. So we deduce that $\Phi(\Lambda_\sigma) \subseteq \Lambda_\sigma$. Since Φ is a bijection, we have $\Phi(\Lambda_\sigma) = \Lambda_\sigma$, that is, Λ_σ is invariant under Φ.

It remains to prove the reverse direction of (2), that is, if Λ_σ is invariant under Φ, then σ is a permutation of form (6). We have the following two cases.
Case 1: $\sigma_n = k = n-1$. We use induction on n. Assume that the assertion in (2) is true for permutations in S_{n-1}. Let σ be a permutation in S_n such that Λ_σ is invariant under Φ.

By Lemma 4, we see that σ is 132-avoiding. It is readily checked that $\sigma_1 = n$. Let $\tau = (n-1, n)\sigma$. Evidently,

$$\tau = M(\sigma, n) \quad \text{and} \quad \Lambda_\tau = \{\pi\colon\ \pi \leqslant \sigma, \pi_n = n\},$$

which implies that Λ_τ is invariant under Φ. Since $\tau_n = n$, by Part (1) of the theorem, we obtain that $\Lambda_{\tau'}$ is invariant under Φ, where $\tau' = \tau_1\tau_2\cdots\tau_{n-1}$. Therefore, by the induction hypothesis, we deduce that

$$\tau_1\tau_2\cdots\tau_{n-1} = \begin{cases} (n-1)\,(n-2)\,\cdots\,(i+1)\,(i-1)\,\cdots\,2\,1\,i, & \text{if } \tau_{n-1} = i > 1; \\ (n-1)\,(n-2)\,\cdots\,2\,1, & \text{if } \tau_{n-1} = 1. \end{cases}$$

Now, we claim that $\tau_{n-1} = 1$. Suppose to the contrary that $\tau_{n-1} = i > 1$. Then we have

$$\tau = (n-1)\,(n-2)\,\cdots\,(i+1)\,(i-1)\,\cdots\,2\,1\,i\,n,$$

and so

$$\sigma = n\,(n-2)\,\cdots\,(i+1)\,(i-1)\,\cdots\,2\,1\,i\,(n-1).$$

Consider the permutation

$$\pi = (n-2)\,\cdots\,(i+1)\,i\,(i-1)\,\cdots 2\,1\,n\,(n-1).$$

Clearly,

$$\begin{aligned}\pi &\leqslant (n-2)\,\cdots\,(i+1)\,n\,(i-1)\,\cdots\,2\,1\,i\,(n-1)\\ &\leqslant n\,(n-2)\,\cdots\,(i+1)\,(i-1)\,\cdots\,2\,1\,i\,(n-1)\\ &= \sigma.\end{aligned}$$

However,

$$\Phi(\pi) = n\,(n-2)\,\cdots\,(i+1)\,i\,(i-1)\,\cdots\,2\,1\,(n-1) \not\leqslant \sigma,$$

which contradicts the assumption that Λ_σ is invariant under Φ. This completes the proof of the above claim that $\tau_{n-1} = 1$.

We now arrive at the conclusion that $\tau = (n-1)\,(n-2)\,\cdots\,2\,1\,n$. Hence

$$\sigma = (n-1, n)\tau = n\,(n-2)\,\cdots\,2\,1\,(n-1),$$

as required.
Case 2: $\sigma_n = k < n-1$. The proof is by induction on σ_n. Assume that the assertion is true for $\sigma_n > k$. We now consider the case $\sigma_n = k$.

To apply the induction hypothesis, we need to consider the principal order ideal generated by

$$w = M(\sigma, k+1) = (k, k+1)\sigma.$$

We first show that the principal order ideal Λ_w has the following form

$$\Lambda_w = \{\pi \in \Lambda_\sigma \colon \pi_n \geqslant k+1\}. \tag{8}$$

It is clear that $\Lambda_w \subseteq \{\pi \in \Lambda_\sigma \colon \pi_n \geqslant k+1\}$. It remains to show that

$$\{\pi \in \Lambda_\sigma \colon \pi_n \geqslant k+1\} \subseteq \Lambda_w. \tag{9}$$

To prove (9), we need the permutation $\gamma(w)$. Keep in mind that the last element of $\gamma(w)$ is equal to w_n. This implies that $\gamma(w) \in \Lambda_\sigma(k+1)$. It follows that $\Phi(\gamma(w)) \in \Lambda_\sigma(k+1)$. By Theorem 3, we see that $\Phi(\gamma(w)) \leqslant M(\sigma, k+1) = w$. Using the arguments in the proof of Lemma 4, we deduce that w is 132-avoiding. Hence we conclude $\sigma^{-1}(i) < \sigma^{-1}(k+1)$ for $i > k+1$.

In view of the construction of $M(\sigma, i)$, for any $i > k+1$, there exist integers $i_1 > i_2 > \cdots > i_{s-2} > k+1$ such that

$$\begin{aligned}M(\sigma, i) &= (i,k)(i,k+1)(i,i_{s-2})\cdots(i,i_1)\sigma\\ &\leqslant (i,k+1)M(\sigma,i)\\ &= (k,k+1)(i,i_{s-2})\cdots(i,i_1)\sigma\\ &\leqslant (k,k+1)\sigma.\end{aligned}$$

Thus, for $i \geqslant k+1$ and $\pi \in \Lambda_\sigma(i)$, we have

$$\pi \leqslant M(\sigma, i) \leqslant (k, k+1)\sigma = w,$$

which implies the relation (9). Hence the proof of (8) is complete.

From (8), we see that Λ_w is invariant under Φ. By the induction hypothesis, we get

$$w = n\,(n-1)\,\cdots(k+2)\,k\,(k-1)\,\cdots\,2\,1\,(k+1),$$

which yields that

$$\sigma = n\,(n-1)\,\cdots\,(k+2)\,(k+1)\,(k-1)\,\cdots\,2\,1\,k.$$

This completes the proof. □

Theorem 5 has the following consequence.

Corollary 6. *There are $\binom{n}{2}+1$ permutations σ in S_n such that the principal order ideals Λ_σ are invariant under Foata's transformation Φ.*

Proof. Let a_n denote the number of principal order ideals in S_n that are invariant under Φ. By Theorem 5, it is easy to derive the following recurrence relation

$$a_n = a_{n-1} + n - 1.$$

Since $a_1 = 1$, the formula for a_n is easily verified. This completes the proof. □

For example, for $n = 3$, there are four permutations σ for which Λ_σ is invariant under Φ: 123, 213, 312, 321. The following two figures are the Hasse diagrams of $(S_3, \leqslant)$ and $(S_3, \leqslant_w)$. The permutations σ such that Λ_σ is Φ-invariant are written in boldface in Figure 1, and the permutations σ such that $\widetilde{\Lambda}_\sigma$ is Φ-invariant are written in boldface as well in Figure 2.

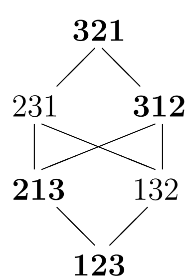

Figure 1: $(S_3, \leqslant)$

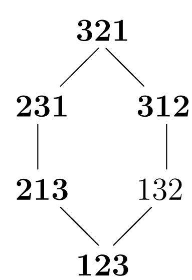

Figure 2: $(S_3, \leqslant_w)$

3 Invariant principal ideals under Han's map

In this section, we show that in the Bruhat order, Han's bijection H has the same invariant principal order ideals as Foata's transformation Φ.

Theorem 7. *Let σ be a permutation in S_n. Then Λ_σ is invariant under H if and only if it is invariant under Φ.*

Proof. We first show that if Λ_σ is invariant under Φ then it is invariant under H. We have the following two cases.
Case 1: $\sigma_n = n$. We use induction on n. Assume that the assertion is true for S_{n-1}. By Theorem 5, we see that $\Lambda_{\sigma'}$ is invariant under Φ, where $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. Thus, by the induction hypothesis, we obtain that $\Lambda_{\sigma'}$ is invariant under H. Notice that

$$\Lambda_\sigma = \{\tau\, n\colon\ \tau \in \Lambda_{\sigma'}\}.$$

Since $H(\pi\, n) = H(\pi)\, n$ for any permutation $\pi \in S_{n-1}$, we deduce that Λ_σ is invariant under H.
Case 2: $\sigma_n = k < n$. Again, by Theorem 5, we see that

$$\sigma = n\,(n-1)\,\cdots\,(k+1)\,(k-1)\,\cdots\,2\,1\,k,$$

which implies that $\Lambda_\sigma = \{\pi \in S_n \colon \pi_n \geqslant k\}$. Since the map H preserves the last element of a permutation in S_n, we see that Λ_σ is invariant under H.

Conversely, assume that Λ_σ is invariant under H. We wish to show that Λ_σ is invariant under Φ. We also have the following two cases.
Case 1: $\sigma_n = n$. We proceed to use induction on n. Assume that the assertion is true for S_{n-1}. It is easy to see that

$$\Lambda_\sigma = \{\tau\, n\ :\ \tau \in \Lambda_{\sigma'}\},$$

where $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. So we have

$$H(\Lambda_\sigma) = \{H(\tau\, n)\ :\ \tau \in \Lambda_{\sigma'}\} = \{H(\tau)\, n\ :\ \tau \in \Lambda_{\sigma'}\}.$$

By the assumption that Λ_σ is invariant under H, we see that $\Lambda_{\sigma'}$ is invariant under H. Thus, by the induction hypothesis, we find that $\Lambda_{\sigma'}$ is invariant under Φ. Therefore,

$$\begin{aligned}\Phi(\Lambda_\sigma) &= \{\Phi(\tau\, n)\ :\ \tau \in \Lambda_{\sigma'}\}\\ &= \{\Phi(\tau)\, n\ :\ \tau \in \Lambda_{\sigma'}\}\\ &= \{\tau\, n\ :\ \tau \in \Lambda_{\sigma'}\}\\ &= \Lambda_\sigma.\end{aligned}$$

So we arrive at the assertion that Λ_σ is invariant under Φ.
Case 2: $\sigma_n = k < n$. We claim that

$$\sigma = n\,(n-1)\,\cdots\,(k+1)\,(k-1)\,\cdots\,1\,k. \tag{10}$$

To prove (10), we use induction on k. We first verify that (10) is valid for $k = n-1$. In this case, by an argument similar to the proof of Lemma 4, we may deduce that σ is 132-avoiding. This implies that $\sigma_1 = n$. Assume that $\sigma_{n-1} = i$. It is easy to check that

$$H(\sigma)_{n-1} = i+1, \quad H^2(\sigma)_{n-1} = i+2, \quad \ldots, \quad H^{n-i-1}(\sigma)_{n-1} = n, \quad H^{n-i}(\sigma)_{n-1} = 1.$$

Since Λ_σ is invariant under H, we see that $H^{n-i}(\sigma) \leqslant \sigma$, which implies $i = 1$. Thus we have

$$\sigma = n\,\sigma_2 \cdots \sigma_{n-2}\,1\,(n-1).$$

To show that σ is a permutation of form (10), we notice that the last element of the permutation $\tau = (n, n-1)\sigma$ is n. Clearly, $\Lambda_\tau = \{\pi\colon \pi \in \Lambda_\sigma, \pi_n = n\}$. So we deduce that Λ_τ is invariant under H. Let $\tau' = \tau_1 \cdots \tau_{n-1}$. Since $\tau_n = n$, by the assertion in Case 1, we see that $\Lambda_{\tau'}$ is invariant under H. Therefore, by the induction hypothesis, we conclude that $\Lambda_{\tau'}$ is invariant under Φ. In view of Theorem 5, we have

$$\tau' = (n-1)\,(n-2)\,\cdots\,2\,1,$$

and hence

$$\tau = \tau' n = (n-1)\,(n-2)\,\cdots\,2\,1\,n,$$

which yields that $\sigma = n\,(n-2)\,\cdots 2\,1\,(n-1)$. This completes the proof of (10) in the case $k = n-1$.

We next consider the case $\pi_n = k < n-1$. Let

$$\tau = (k, k+1)\sigma.$$

It is clear that $\tau = M(\sigma, k+1)$. By an argument analogous to the proof of relation (8), we deduce that

$$\Lambda_\tau = \{\pi\colon \pi \leqslant \sigma, \pi_n \geqslant k+1\},$$

from which it is easily seen that Λ_τ is invariant under H. Since $\tau_n > k$, by the induction hypothesis, we get

$$\tau = n\,(n-1)\,\cdots\,(k+2)\,k\,(k-1)\,\cdots\,2\,1\,(k+1).$$

It follows that

$$\sigma = (k, k+1)\tau = n\,(n-1)\,\cdots\,(k+2)\,(k+1)\,(k-1)\,\cdots\,2\,1\,k.$$

Thus the proof of (10) is complete.

By Theorem 5, we see that Λ_σ is invariant under Φ. This completes the proof. □

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