Identifying Vertex Covers in Graphs

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Abstract

An identifying vertex cover in a graph G is a subset T of vertices in G that has a nonempty intersection with every edge of G such that T distinguishes the edges, that is, $e \cap T \neq \emptyset$ for every edge e in G and $e \cap T \neq f \cap T$ for every two distinct edges e and f in G. The identifying vertex cover number $\tau_D(G)$ of G is the minimum size of an identifying vertex cover in G. We observe that $\tau_D(G) + \rho(G) = |V(G)|$, where $\rho(G)$ denotes the packing number of G. We conjecture that if G is a graph of order n and size m with maximum degree Δ, then $\tau_D(G) \leqslant \left(\frac{\Delta(\Delta-1)}{\Delta^2+1}\right) n + \left(\frac{2}{\Delta^2+1}\right) m$. If the conjecture is true, then the bound is best possible for all $\Delta \geqslant 1$. We prove this conjecture when $\Delta \geqslant 1$ and G is a Δ-regular graph. The three known Moore graphs of diameter 2, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph, are examples of regular graphs that achieves equality in the upper bound. We also prove this conjecture when $\Delta \in \{2, 3\}$.

Keywords: Vertex cover, Identifying vertex cover, Transversal.

1 Introduction

For notation and graph theory terminology, we in general follow [2]. Specifically, let $G = (V, E)$ be a graph with vertex set $V = V(G)$, edge set $E = E(G)$, order $n(G) = |V|$ and size $m(G) = |E|$. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of v is $d_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write n, m, $N(v)$, $N[v]$ and $d(v)$ rather than $n(G)$, $m(G)$, $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively. The set of

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vertices at distance 2 from a vertex v in G we denote by $N_2(G;v)$, or simply by $N_2(v)$ is the graph G is clear from the context. A *subcubic graph* is a graph with maximum degree 3. A path and a cycle on n vertices are denoted by P_n and C_n, respectively.

For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A set S of vertices in G is a *packing* if the vertices of S are pairwise at distance at least 3 apart in G; that is, for distinct vertices u and v in S, we have $d_G(u,v) \geqslant 3$ or, equivalently, $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ of G is the maximum cardinality of a packing in G. A packing of cardinality $\rho(G)$ in G is called a $\rho(G)$-packing.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A (vertex) *cover* in G is a set of vertices that covers all the edges of G. We remark that a cover is also called a *transversal* or *hitting set* in the literature. Thus a cover T has a nonempty intersection with every edge of G. The (vertex) *covering number* $\tau(G)$ of G is the minimum cardinality of a cover in G. A cover of size $\tau(G)$ is called a $\tau(G)$-cover. We say that an edge e in G is *covered* by a set T if $e \cap T \neq \emptyset$. In particular, if T is a cover in G, then T covers every edge of G.

We define a subset T of vertices in G to be an *identifying vertex cover*, abbreviated *id-cover*, if T is a cover in G that distinguishes the edges, that is, $e \cap T \neq f \cap T$ for every two distinct edges e and f in G. We remark that every graph has an id-cover since $V(G)$ is such a set. The *identifying vertex covering number*, abbreviated *id-covering number*, $\tau_D(G)$ of G is the minimum size of an id-cover in G. An id-cover of size $\tau_D(G)$ is called a $\tau_D(G)$-cover.

Julien Moncel[1] in his PhD thesis (in French) was the first to observe that an id-cover is precisely the complement of a packing.

Observation 1. ([9]) *A set of vertices in a graph is an id-cover if and only if it is the complement of a packing.*

As an immediate consequence of Observation 1, we have the following relationship between the id-covering number and the packing number of a graph.

Corollary 2. ([9]) *For every graph G, we have $\tau_D(G) + \rho(G) = |V(G)|$.*

Covers in graphs and more generally, transversals in hypergraphs, are very well studied in the literature. Identifying vertex covers in graphs and hypergraphs have applications, for example, in identifying open codes (also called *open-locating-dominating sets* or a *strong identifying codes* in the literature (see, for example, [3, 7, 11, 12]). Identification of edges of a graph have been studied, for example, in [5, 6] and elsewhere. We remark that there is a strong link between id-covers in graphs and *test covers*, which are studied for example in [10]. An id-cover is both a set cover and a test cover of a 2-regular hypergraph. The dual case of test covers of graphs was studied in [1]. Recently the authors [3] obtained results on identifying open codes in cubic graphs using distinguishing transversal in hypergraphs. A bibliography of papers concerned with identifying codes, currently listing over 170 papers, is maintained by Lobstein [8].

[1]We remark that Moncel used slightly different terminology to the authors and viewed a packing as an error-correcting code.

2 Main Results

Our aim in this paper is to establish an upper bound on the identifying vertex covering number of a graph in terms of its order, size and maximum degree. In particular, we establish upper bounds on the id-covering number of a regular graph in terms of its order and size and of a subcubic graph in terms of its order and size. The results we present in this paper on id-covers in graphs give support for the following conjecture.

Conjecture 3. If G is a graph of order n and size m with maximum degree Δ, then

$$\tau_D(G) \leqslant \left(\frac{\Delta(\Delta-1)}{\Delta^2+1} \right) n + \left(\frac{2}{\Delta^2+1} \right) m.$$

When $\Delta = 1$, the bound in Conjecture 3 simplifies to $\tau_D(G) \leqslant m$, which is trivially true since in this case G consists of a disjoint union of copies of K_1 and K_2.

In this paper, we prove the following results. Proofs of Theorem 4, Theorem 5 and Theorem 6 are given in Section 3, Section 4 and Section 5, respectively.

Theorem 4. *Conjecture 3 is true when $\Delta \geqslant 1$ and G is a Δ-regular graph.*

Theorem 5. *Conjecture 3 is true when $\Delta = 2$.*

Theorem 6. *Conjecture 3 is true when $\Delta = 3$.*

Recall (or see [4, 13]) that the Moore graphs of diameter 2 are Δ-regular graphs (of girth 5) and order $n = \Delta^2 + 1$ and exist for $\Delta = 2, 3, 7$ and possibly 57, but for no other degrees. The Moore graphs for the first three values of Δ are unique, namely

- The 5-cycle (2-regular graph on $n = 5$ vertices)
- The Petersen graph (3-regular graph on $n = 10$ vertices)
- The Hoffman-Singleton graph (7-regular graph on $n = 50$ vertices).

Since these graphs G have diameter 2, their packing number $\rho(G) = 1$, implying by Corollary 2 that $\tau_D(G) = n - 1$. However in this case the conjectured bound in Conjecture 3 simplifies to precisely $n - 1$. Hence the bound in Conjecture 3 is achieved by the three known Moore graphs of diameter 2.

We remark that if Conjecture 3 is true, then the bound is best possible due to the star $G = K_{1,\Delta}$, where $\Delta \geqslant 1$, which has order $n = \Delta + 1$, size $m = \Delta$, maximum degree Δ and $\tau_D(G) = \Delta$, implying that

$$\tau_D(G) = \Delta = \frac{\Delta(\Delta-1)(\Delta+1)}{\Delta^2+1} + \frac{2\Delta}{\Delta^2+1} = \left(\frac{\Delta(\Delta-1)}{\Delta^2+1} \right) n + \left(\frac{2}{\Delta^2+1} \right) m.$$

Although we have yet to find a family of connected graphs with arbitrary large order relative to the maximum degree that achieve equality in the bound of Conjecture 3, there are such graphs with id-covering number arbitrarily close to the conjectured bound. For example, for $s < \Delta$ if we take s disjoint copies of a star $K_{1,\Delta}$ and add $s-1$ edges by joining a leaf from one of the stars to a leaf from each of the other stars, and denote the resulting connected graph by G and its order by n, then $\tau_D(G) = n - s$, while the upper bound in Conjecture 3 simplifies to $n-s+2(s-1)/(\Delta^2+1)$ which can be made arbitrarily close to $\tau_D(G)$ by letting $s \ll \Delta$.

3 Proof of Theorem 4

In this section, we give a proof of Theorem 4. We first present the following theorem.

Theorem 7. *If G is a graph of order n with maximum degree Δ, then*

$$\tau_D(G) \leqslant \left(\frac{\Delta^2}{\Delta^2+1}\right) n.$$

Proof. Let S be a $\rho(G)$-packing. Then, $N(v) \cup N_2(v) \subseteq V \setminus S$ for every vertex $v \in S$. Further if $u \notin S$, then, by the maximality of S, we must have that $d(u,v) \leqslant 2$ for some vertex $v \in S$, implying that

$$V \setminus S = \bigcup_{v \in S} (N(v) \cup N_2(v)).$$

Hence since $|N(v)| \leqslant \Delta$ and $|N_2(v)| \leqslant |N(v)|(\Delta - 1)$ for each $v \in S$, we have that

$$n - |S| = |V \setminus S| \leqslant \sum_{v \in S} (|N(v)| + |N_2(v)|) \leqslant \Delta^2 \cdot |S|,$$

and so, $\rho(G) = |S| \geqslant n/(\Delta^2+1)$. The desired result now follows from Corollary 2. □

As a consequence of Theorem 7, we can readily deduce Theorem 4. Recall its statement.

Theorem 4. *If G is a Δ-regular graph of order n and size m, then*

$$\tau_D(G) \leqslant \left(\frac{\Delta(\Delta-1)}{\Delta^2+1}\right) n + \left(\frac{2}{\Delta^2+1}\right) m.$$

Proof. Since G is a Δ-regular graph, $2m = \Delta n$. Hence by Theorem 7, we have

$$\begin{aligned}
\tau_D(G) &\leqslant \left(\frac{\Delta^2}{\Delta^2+1}\right) n \\
&= \left(\frac{\Delta^2-\Delta}{\Delta^2+1}\right) n + \left(\frac{\Delta}{\Delta^2+1}\right) n \\
&= \left(\frac{\Delta^2-\Delta}{\Delta^2+1}\right) n + \left(\frac{2}{\Delta^2+1}\right) m.
\end{aligned}$$

□

As observed earlier, the 5-cycle (2-regular graph on $n = 5$ vertices), the Petersen graph (3-regular graph on $n = 10$ vertices) and the Hoffman-Singleton graph (7-regular graph on $n = 50$ vertices) are examples of regular graphs that achieve equality in the bound of Theorem 4.

4 Proof of Theorem 5

The packing number of a path and a cycle is well-known. Hence using the relationship between the id-covering number and the packing number of a graph in Corollary 2, we have the following result.

Proposition 8. *The following holds.*
(a) *For $n \geqslant 1$, $\tau_D(P_n) = \lfloor 2n/3 \rfloor$.*
(b) *For $n \geqslant 3$, $\tau_D(C_n) = \lceil 2n/3 \rceil$.*

As a consequence of Proposition 8, we have that if G is a path on n vertices and m edges, then $\tau_D(G) \leqslant 2(2n-1)/5 = 2(n+m)/5$ with equality if and only if $G = P_3$. Further if G is a cycle on n vertices and m edges, then $\tau_D(G) \leqslant 4n/5 = 2(n+m)/5$ with equality if and only if $G = C_5$. Hence we have the following result, which proves Theorem 5.

Theorem 9. *If G is a graph of order n and size m with maximum degree 2, then $\tau_D(G) \leqslant 2(n+m)/5$ with equality if and only if every component of G is a path P_3 or a cycle C_5.*

5 Proof of Theorem 6

In order to prove Theorem 6, we prove a stronger result. For this purpose, we shall need the following notation. Let $G = (V, E)$ be a graph.

We call $\mathcal{P} = (E_2, F_2)$ a 2-edge-partition of E if $\mathcal{P}$ is a weak partition of E (that is, some of the subsets of the partition may be empty) such that $E_2 \cup F_2 = E$. Let T be a cover in G such that the edges in F_2 are distinguished, i.e., if $e, f \in F_2$ and $e \neq f$, then $e \cap T \neq f \cap T$. We call T a $\mathcal{P}$-cover of G. We call an edge in E_2 an E_2-edge and an edge in F_2 an F_2-edge. We define the $\mathcal{P}$-covering number of G, denoted $\tau_{\mathcal{P}}(G)$, to be the minimum cardinality of a $\mathcal{P}$-cover in G.

Let X be a subset of vertices in G (possibly, $X = \emptyset$). For a given 2-edge-partition $\mathcal{P}$ of E, let T be chosen to be a $\mathcal{P}$-cover of G such that $X \subseteq T$. We call such a $\mathcal{P}$-cover T a $(\mathcal{P}, X)$-cover of G and we define the $(\mathcal{P}, X)$-covering number of G, denoted $\tau(G; \mathcal{P}, X)$ to be the minimum size of a $(\mathcal{P}, X)$-cover in G. If $X = \emptyset$, a $(\mathcal{P}, X)$-cover of G is a $\mathcal{P}$-cover of G and we write $\tau(G; \mathcal{P})$ rather than $\tau(G; \mathcal{P}, X)$.

In order to prove Theorem 6, we need to prove the following stronger result.

Theorem 10. *Let $G = (V, E)$ be a graph with $\Delta(G) \leqslant 3$ and let $\mathcal{P} = (E_2, F_2)$ be a 2-edge-partition of E and let $X \subseteq V$. Then,*

$$10\tau(G; \mathcal{P}, X) \leqslant 6n(G) + 2|F_2| + |E_2| + 4|X|.$$

Proof. Define $\xi(G; \mathcal{P}, X) = 6n(G) + 2|F_2| + |E_2| + 4|X|$ for all graphs G with associated 2-edge-partition $\mathcal{P}$ and subset $X \subseteq V$. We wish to prove that $10\tau(G; \mathcal{P}, X) \leqslant \xi(G; \mathcal{P}, X)$ when $\Delta(G) \leqslant 3$. Assume that the theorem is false. Among all counterexamples, let $G = (V, E)$ with associated 2-edge-partition $\mathcal{P}$ and subset $X \subseteq V$ be chosen so that

(1) G has minimum order $n(G)$.

(2) Subject to (1), $|F_2|$ is a minimum.

Since G is a counterexample to the theorem, $10\tau(G; \mathcal{P}, X) > \xi(G; \mathcal{P}, X)$. Clearly, $|E| \geqslant 1$, for otherwise $10\tau(G; \mathcal{P}, X) = 10|X| \leqslant 6n(G) + 4|X| = \xi(G; \mathcal{P}, X)$, a contradiction. Further, $n(G) \geqslant 3$, for otherwise $10\tau(G; \mathcal{P}, X) = 10 < \xi(G; \mathcal{P}, X)$ if $|X| < 2$ and $10\tau(G; \mathcal{P}, X) = 20 < \xi(G; \mathcal{P}, X)$ if $|X| = 2$, a contradiction. If G is not connected, then by the minimality of the order of G the theorem holds for all components of G and therefore for G. This is a contradiction to G being a counterexample. Hence, G is connected.

We will now prove a number of claims. In these claims we shall adopt the following notation. Let $G' = (E', V')$ be a graph with $\Delta(G') \leqslant 3$ and let $\mathcal{P}' = (E'_2, F'_2)$ be a 2-edge-partition of E' and let $X' \subseteq V'$. We now define

$$\begin{aligned} \xi^{\Delta}(G'; \mathcal{P}', X') &= \xi(G; \mathcal{P}, X) - \xi(G'; \mathcal{P}', X') \\ \tau^{\Delta}(G'; \mathcal{P}', X') &= \tau(G; \mathcal{P}, X) - \tau(G'; \mathcal{P}', X'). \end{aligned}$$

The usefulness of these definitions will become clear in Lemma 11 and the following claims. We shall invoke Lemma 11 throughout the proof of Theorem 10. The essential idea when applying Lemma 11 is to prove properties on the structure of $G = (V, E)$ with associated 2-edge-partition $\mathcal{P}$ and subset $X \subseteq V$ by extending a cover of a modified instance $G' = (V', E')$ with associated 2-edge-partition $\mathcal{P}'$ and subset $X' \subseteq V'$ that is smaller than G in terms of the order defined by conditions (a) and (b) in Lemma 11 and deriving a contradiction.

Lemma 11. *If $G' = (E', V')$ is a graph, $X' \subseteq V'$, and $\mathcal{P}' = (E'_2, F'_2)$ a 2-edge-partition of E' such that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, then the following hold.*

(a) $n(G') \geqslant n(G)$.

(b) *If equality holds in* (a)*, then $|F'_2| \geqslant |F_2|$.*

Proof. Suppose to the contrary that such a graph G', subset X', and associated 2-edge-partition $\mathcal{P}'$ exists such that (a) or (b) do not hold. By the minimality of G we have $10\tau(G'; \mathcal{P}', X') \leqslant \xi(G'; \mathcal{P}', X')$. By assumption, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$. Hence, $\xi(G; \mathcal{P}, X) - \xi(G'; \mathcal{P}', X') = \xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X') = 10\tau(G; \mathcal{P}, X) - 10\tau(G'; \mathcal{P}', X') \geqslant 10\tau(G; \mathcal{P}, X) - \xi(G'; \mathcal{P}', X')$, and so, $\xi(G; \mathcal{P}, X) \geqslant 10\tau(G; \mathcal{P}, X)$, a contradiction. $\square$

Claim A: $X = \emptyset$.

Proof. Suppose that $X \neq \emptyset$. Suppose that there exist distinct edges $e_1, e_2 \in F_2$ that intersect X such that $e_1 \cap X = e_2 \cap X$. Then, $|e_1 \cap X| = 1$. Let $e_1 = \{v, v_1\}$ and $e_2 = \{v, v_2\}$. Let G' be obtained from $G - \{e_1, e_2\}$ by adding the edge $e' = \{v_1, v_2\}$ if this edge does not already exist. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \cup \{e'\}$ and $F_2' = F_2 \setminus \{e_1, e_2\}$. Let $X' = X$. Then, $n(G') = n(G)$, $|E_2'| \leqslant |E_2| + 1$, $|F_2'| = |F_2| - 2$ and $|X'| = |X|$, implying that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 4 - 1 = 3$. Every $(\mathcal{P}', X')$-cover in G' is a $(\mathcal{P}, X)$-cover in G, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X')$ and therefore $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 0$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 3 > 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. Hence if $e_1, e_2 \in F_2$ are distinct edges that intersect X, then $e_1 \cap X \neq e_2 \cap X$.

If $X = V$, then $10\tau(G; \mathcal{P}, X) = 10n(G) < \xi(G; \mathcal{P}, X)$, a contradiction. Hence, $X \neq V$. Let $G' = G - X$ (and so, the vertices in X and the edges incident with X are deleted). Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \cap E(G')$ and $F_2' = F_2 \cap E(G')$. Let $X' = \emptyset$. Since distinct edges in F_2 have distinct intersections with X, every $\tau(G'; \mathcal{P}', X')$-cover can be extended to a $\tau(G; \mathcal{P}, X)$-cover by adding to it the set X, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X') + |X|$, and so $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant |X|$. We note that $n(G') = n(G) - |X|$, and so $\xi^{\Delta}(G'; \mathcal{P}', X') > 6|X| + 4|X| = 10|X| \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. □

Claim B: *Every degree-3 vertex is incident with three E_2-edges or three F_2-edges.*

Proof. Suppose that some vertex v of degree 3 is not incident with three E_2-edges or three F_2-edges. Then either v is incident with one E_2-edge and two F_2-edges or with one F_2-edge and two E_2-edges. Suppose that v is incident with exactly two F_2-edges, say $e_1 = \{v, v_1\}$ and $e_2 = \{v, v_2\}$, and with one E_2-edge, say e_3. Let G' be obtained from $G - \{e_1, e_2, e_3\}$ by deleting the isolated vertex v and adding the edge $e' = \{v_1, v_2\}$ if this edge does not already exist. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = (E_2 \setminus \{e_3\}) \cup \{e'\}$ and $F_2' = F_2 \setminus \{e_1, e_2\}$. Let $X' = X$. Recall that by Claim A, $X = \emptyset$. Then, $n(G') = n(G) - 1$, $|E_2'| \leqslant |E_2|$ and $|F_2'| = |F_2| - 2$, implying that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 6 + 4 = 10$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the vertex v, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X') + 1$ and therefore $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 1$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 10 \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11.

Hence, v is incident with exactly two E_2-edges, say f_1 and f_2, and with one F_2-edge, say f_3. Let G' be obtained from $G - \{f_1, f_2, f_3\}$ by deleting the resulting isolated vertex v. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{f_1, f_2\}$ and $F_2' = F_2 \setminus \{f_3\}$. Let $X' = X$. Then, $n(G') = n(G) - 1$, $|E_2'| = |E_2| - 2$ and $|F_2'| = |F_2| - 1$, implying that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 6 + 2 + 2 = 10$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the vertex v, implying that $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 1$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 10 \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. □

Claim C: $\delta(G) \geqslant 2$.

Proof. Suppose that $\delta(G) < 2$. As observed earlier, $n(G) \geqslant 3$. The connectivity of G implies that $\delta(G) = 1$. Let u be a vertex of degree 1 in G and let $e_1 = \{u, v\}$ be the edge incident with u.

Suppose that $e_1 \in E_2$. Let $G' = G - u$ and let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{e_1\}$ and $F_2' = F_2$. Let $X' = X \cup \{v\} = \{v\}$ (recall that by Claim C, $X = \emptyset$). Then, $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 6+1-4 = 3$. Every $(\mathcal{P}', X')$-cover in G' is a $(\mathcal{P}, X)$-cover in G, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X')$ and therefore $\tau^\Delta(G'; \mathcal{P}', X') \leqslant 0$. Hence, $\xi^\Delta(G'; \mathcal{P}', X') = 3 > \tau^\Delta(G'; \mathcal{P}', X')$, contradicting Lemma 11. Therefore, $e_1 \in F_2$.

Now let $G^* = G - \{u, v\}$ and let $\mathcal{P}^* = (E_2^*, F_2^*)$, where $E_2^* = E_2 \cap E(G^*)$ and $F_2^* = F_2 \cap E(G^*)$. Let $X^* = \{w \mid \{v, w\} \in F_2 \setminus \{e_1\}\}$. In other words, X^* contains all vertices different from u that are joined to v by an edge in F_2. Then, $\xi^\Delta(G^*; \mathcal{P}^*, X^*) \geqslant 12 + 2 + 2|X^*| - 4|X^*| = 14 - 2|X^*| \geqslant 10$, since $|X^*| \leqslant d(v) - 1 \leqslant 2$. Every $(\mathcal{P}^*, X^*)$-cover in G^* can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the vertex v, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G^*; \mathcal{P}^*, X^*) + 1$ and therefore $\tau^\Delta(G^*; \mathcal{P}^*, X^*) \leqslant 1$. Hence, $\xi^\Delta(G^*; \mathcal{P}^*, X^*) \geqslant 10 \geqslant 10\tau^\Delta(G^*; \mathcal{P}^*, X^*)$, contradicting Lemma 11. This completes the proof of Claim C. □

Claim D: *Every vertex is incident with at least one F_2-edge.*

Proof. Suppose that some vertex v is incident with no F_2-edge.

Claim D.1: $d_G(v) = 3$.

Proof. Suppose that $d_G(v) = 2$. Let $e_1 = \{v, v_1\}$ and $e_2 = \{v, v_2\}$ be the two edges incident with v. Let $G' = G - v$. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{e_1, e_2\}$ and $F_2' = F_2$. Let $X' = X \cup \{v_1, v_2\}$. Then, $n(G') = n(G) - 1$, $|E_2'| = |E_2| - 2$, $|F_2'| = |F_2|$ and $|X'| = |X| + 2$, implying that $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 6+2-8 = 0$. Every $(\mathcal{P}', X')$-cover in G' is a $(\mathcal{P}, X)$-cover in G, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X')$ and therefore $\tau^\Delta(G'; \mathcal{P}', X') \leqslant 0$. Hence, $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 0 \geqslant 10\tau^\Delta(G'; \mathcal{P}', X')$, contradicting Lemma 11. □

By Claim D.1, $d_G(v) = 3$. Let $e_1 = \{v, v_1\}$, $e_2 = \{v, v_2\}$ and $e_3 = \{v, v_3\}$ be the three edges incident with v. Let $G' = G - \{v, v_1, v_2, v_3\}$. Let $\mathcal{P}' = (E_2', F_2')$, where E_2' is obtained from E_2 by deleting all E_2-edges incident with v_1, v_2 or v_3 and where F_2' is obtained from F_2 by deleting all F_2-edges incident with with v_1, v_2 or v_3. Let $X' = X$. Then, $n(G') = n(G) - 4$. If a neighbor of v is incident with no F_2-edge, then as shown in the proof of Claim D.1 such a vertex is incident with three E_2-edges. If a neighbor of v is incident with an F_2-edge, then by Claim B, such a vertex is incident with exactly one F_2-edge (and one E_2-edge, namely the edge joining it to v). In particular, if no neighbor of v is incident with an F_2-edge, then we note that at least six E_2-edges are deleted when constructing G', implying that $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 24+6 = 30$. If only one F_2-edge is deleted when constructing G', we note that at least five E_2-edges were deleted, implying that $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 24 + 2 + 5 = 31$. If at least two F_2-edges are deleted when constructing G', we note that at least three E_2-edges were deleted, implying that $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 24 + 4 + 3 = 31$. In all three cases, we have $\xi^\Delta(G'; \mathcal{P}', X') \geqslant 30$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the three vertices v_1, v_2 and v_3, implying that $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X') + 3$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 3$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 30 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. This completes the proof of Claim D. □

Claim E: *For all F_2-edges $\{u, v\}$, we have $d(u) = 2$ or $d(v) = 2$ (or both)*

Proof. Suppose that there is an F_2-edge $e = \{u, v\}$ with $d(u) = d(v) = 3$. Let $e_1 = \{u, u_1\}$, $e_2 = \{u, u_2\}$, $f_1 = \{v, v_1\}$ and $f_2 = \{v, v_2\}$ be the edges in G adjacent with e. By Claim B, all these edges are F_2-edges. Let G' be obtained from $G - \{u, v\}$ by adding the two edges $\{u_1, u_2\}$ and $\{v_1, v_2\}$, and let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \cup \{\{u_1, u_2\}, \{v_1, v_2\}\}$ and $F_2' = F_2 \setminus \{e, e_1, e_2, f_1, f_2\}$. Let $X' = X = \emptyset$. Then, $n(G') = n(G) - 2$, $|E_2'| \leqslant |E_2| + 2$, $|F_2'| = |F_2| - 5$ and $|X'| = |X| = 0$, implying that $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 12+10-2 = 20$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the two vertices u and v, implying that $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X') + 2$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 2$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 20 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. □

Claim F: *Every edge is an F_2-edge.*

Proof. Suppose that there is an E_2-edge $e = \{v_1, v_2\}$. By Claims D, E, and F, every vertex that is incident with an E_2-edge has degree exactly 2 and is incident with an F_2-edge. In particular, $d(v_1) = d(v_2) = 2$. Let e_1 and e_2 be the F_2-edges incident with v_1 and v_2, respectively.

Claim F.1: *The vertices v_1 and v_2 do not have a common neighbor.*

Proof. Suppose that v_1 and v_2 have a common neighbor, v_3 say. Hence, $e_1 = \{v_1, v_3\}$ and $e_2 = \{v_2, v_3\}$. Since $\Delta(G) = 3$ and G is connected, we have that $d(v_3) = 3$. Let $e_3 = v_3v_4$ be the third edge incident with v_3 that is distinct from e_1 and e_2. By Claim B, e_3 is an F_2-edge. We now consider the graph $G' = G - \{v_1, v_2, v_3\}$. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{e\}$ and $F_2' = F_2 \setminus \{e_1, e_2, e_3\}$. Let $X' = X \cup \{v_4\} = \{v_4\}$. Then, $n(G') = n(G) - 3$, $|E_2'| = |E_2| - 1$, $|F_2'| = |F_2| - 3$ and $|X'| = |X| + 1$, implying that $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 18 + 6 + 1 - 4 = 21$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the vertices v_1 and v_3, implying that $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X') + 2$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 2$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 21 > 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. □

Let $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$. By Claim F.1, $w_1 \neq w_2$.

Claim F.2: $d(w_1) = d(w_2) = 3$.

Proof. Suppose that $d(w_1) = 2$. Let $e_3 = \{w_1, w_3\}$ be the edge incident with w_1 that is distinct from e_1. Suppose that $e_3 \in E_2$. Let G' be obtained from G by deleting the four edges e, e_1, e_2, e_3 and the resulting isolated vertices v_1, v_2 and w_1. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{e, e_3\}$ and $F_2' = F_2 \setminus \{e_1, e_2\}$. Let $X' = X \cup \{w_2\} = \{w_2\}$. Then, $n(G') =$

$n(G)-3$, $|E_2'| = |E_2|-2$, $|F_2'| = |F_2|-2$ and $|X'| = |X|+1$, implying that $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 18+4+2-4 = 20$. Every $(\mathcal{P}',X')$-cover in G' can be extended to a $(\mathcal{P},X)$-cover in G by adding to it the vertices v_2 and w_1, implying that $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X') + 2$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 2$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 20 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. Therefore, $e_3 \in F_2$.

We now let $G' = G - \{v_1, v_2, w_1\}$. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 \setminus \{e\}$ and $F_2' = F_2 \setminus \{e_1, e_2, e_3\}$. Let $X' = X \cup \{w_3\} = \{w_3\}$. Then, $n(G') = n(G)-3$, $|E_2'| = |E_2|-1$, $|F_2'| = |F_2|-3$ and $|X'| = |X|+1$, implying that $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 18+6+1-4 = 21$. Every $(\mathcal{P}',X')$-cover in G' can be extended to a $(\mathcal{P},X)$-cover in G by adding to it the vertices v_2 and w_1, implying that $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X')+2$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 2$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') > 20 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. Therefore, $d(w_1) = 3$. Analogously, $d(w_2) = 3$. □

By Claim F.2, $d(w_1) = d(w_2) = 3$. By Claim B, all three edges incident with w_1 (respectively, w_2) are F_2-edges. Let $N(w_1) = \{v_1, x_1, y_1\}$ and let $f_1 = \{w_1, x_1\}$ and $g_1 = \{w_1, y_1\}$ be the two edges incident with w_1 that are distinct from e_1 (note that $w_2 \in \{x_1, y_1\}$ is possible). Renaming x_1 and y_1, if necessary, we may assume that $x_1 \neq w_2$.

Let $G^* = G - \{v_1, v_2, w_1, x_1, y_1\}$. Let $\mathcal{P}^* = (E_2^*, F_2^*)$, where $E_2^* = E_2 \cap E(G^*)$ and $F_2^* = F_2 \cap E(G^*)$. Let $X^* = X = \emptyset$. Then, $n(G^*) = n(G) - 5$ and $|X^*| = |X|$. Since $x_1 \neq w_2$, we note that apart from the edges e, e_1, e_2, f_1, g_1 we also delete at least one further edge which is incident to x_1. On the one hand, if such an edge is an E_2-edge, then $|E_2^*| \leqslant |E_2| - 2$ and $|F_2^*| \leqslant |F_2| - 4$, implying that $\xi^{\Delta}(G^*;\mathcal{P}^*,X^*) \geqslant 30+8+2 = 40$. On the other hand, if such an edge is an F_2-edge, then $|E_2^*| \leqslant |E_2| - 1$ and $|F_2^*| \leqslant |F_2| - 5$, implying that $\xi^{\Delta}(G^*;\mathcal{P}^*,X^*) \geqslant 30+10+1 = 41$. In both cases, $\xi^{\Delta}(G^*;\mathcal{P}^*,X^*) \geqslant 40$. By Claim C and Claim E, we note that $d(x_1) = d(y_1) = 2$. Hence every $(\mathcal{P}^*,X^*)$-cover in G^* can be extended to a $(\mathcal{P},X)$-cover in G by adding to it the vertices v_2, w_1, x_1 and y_1, and so $\tau(G;\mathcal{P},X) \leqslant \tau(G';\mathcal{P}',X') + 4$ and therefore $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 4$. Hence, $\xi^{\Delta}(G';\mathcal{P}',X') \geqslant 40 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. □

Claim G: *G is triangle-free.*

Proof. Suppose that there is a triangle $T\colon v_1v_2v_3v_1$ in G. By Claim C and Claim E, at least two of the vertices in T have degree 2 in G. Renaming vertices, if necessary, we may assume that $d(v_1) = d(v_2) = 2$. Since $\Delta(G) = 3$ and G is connected, we have that $d(v_3) = 3$. Let w be the neighbor of v_3 not in T. Let $G' = G - \{v_1, v_2, v_3\}$ and let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 = \emptyset$ and where F_2' is obtained from F_2 by deleting the four edges incident with vertices in $\{v_1, v_2, v_3\}$. Let $X' = X \cup \{w\}$. Then, $\xi^{\Delta}(G';\mathcal{P}',X') = 18+8-4 = 22$. Every $(\mathcal{P}',X')$-cover in G' can be extended to a $(\mathcal{P},X)$-cover in G by adding to it the set $\{v_2, v_3\}$, implying that $\tau^{\Delta}(G';\mathcal{P}',X') \leqslant 2$ and $\xi^{\Delta}(G';\mathcal{P}',X') > 20 \geqslant 10\tau^{\Delta}(G';\mathcal{P}',X')$, contradicting Lemma 11. □

Claim H: *G contains no 5-cycle.*

Proof. Suppose that there is a 5-cycle $C\colon v_1v_2v_3v_4v_5v_1$ in G. By Claim G, G is triangle-free, and so C is an induced cycle in G. Since $\Delta(G) = 3$ and G is connected, we may assume, renaming vertices of C if necessary, that $d(v_1) = 3$. Let v_6 be the neighbor of v_1 not on C. By Claim C and Claim E, $d(v_2) = d(v_5) = d(v_6) = 2$. By Claim E, at least one of v_3 and v_4 has degree 2 in G. Renaming vertices if necessary, we may assume that $d(v_3) = 2$. Let $G' = G - V(C)$ and let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = E_2 = \emptyset$ and where F_2' is obtained from F_2 by deleting all edges incident with vertices in $V(C)$. On the one hand, if $d(v_4) = 2$, let $X' = X = \emptyset$. On the other hand, if $d(v_4) = 3$, let w be the neighbor of v_4 not in C and let $X' = X \cup \{w\}$. Therefore if $d(v_4) = 2$, we have that $\xi^{\Delta}(G'; \mathcal{P}', X') = 30 + 12 = 42$, while if $d(v_4) = 3$, we have that $\xi^{\Delta}(G'; \mathcal{P}', X') = 30 + 14 - 4 = 40$. In both cases, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 40$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the set $\{v_1, v_2, v_4, v_5\}$, implying that $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 4$ and $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 40 \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. □

Claim I: *G contains no 4-cycle.*

Proof. Suppose that there is a 4-cycle $C\colon w_1w_2w_3w_4w_1$ in G. By Claim G, G is triangle-free, and so C is an induced cycle in G. Since $\Delta(G) = 3$ and G is connected we may assume, renaming vertices of C if necessary, that $d(w_1) = 3$. Let w_5 be the neighbor of w_1 not in C. By Claim C and Claim E, $d(w_2) = d(w_4) = d(w_5) = 2$. Let w_6 be the neighbor of w_5 different from w_1. If $w_3 = w_6$, then $G = K_{2,3}$ and $10\tau(G; \mathcal{P}, X) = 40 < 30 + 12 = \xi(G; \mathcal{P}, X)$, a contradiction. Hence, $w_3 \neq w_6$. Let $W = \{w_1, w_2, \ldots, w_6\}$. By Claim H, w_3 is not adjacent to w_6, and so the vertex w_5 is the only neighbor of w_6 that belongs to the set W.

On the one hand, if $d(w_6) = 2$, let $G' = G - W$. On the other hand if $d(w_6) = 3$, then let G' be obtained from $G - W$ by adding an E_2-edge e' joining the two neighbors of w_6 not in W. Let $\mathcal{P}' = (E_2', F_2')$, where $E_2' = \emptyset$ if $d(w_6) = 2$ and $E_2' = \{e'\}$ if $d(w_6) = 3$, and where F_2' is obtained from F_2 by deleting all edges incident with vertices in W. Let $X' = X = \emptyset$. Then, $n(G') = n(G) - 6$ and $|X'| = |X| = 0$.

If $d(w_6) = 2$, then $|E_2'| = |E_2| = 0$ and $|F_2'| = |F_2| - 5 - d(w_3) \leqslant |F_2| - 7$, implying that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 36 + 14 = 50$. If $d(w_6) = 3$, then $|E_2'| = |E_2| + 1 = 1$ and $|F_2'| = |F_2| - 6 - d(w_3) \leqslant |F_2| - 8$, implying that $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 36 + 16 - 1 = 51$. In both cases, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 50$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the set $W \setminus \{w_1\}$, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X') + 5$ and therefore $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 5$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 50 \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. □

We now return to the proof of Theorem 10. Let u be any vertex of degree 3 in G and let $N(u) = \{u_1, u_2, u_3\}$. Further let $e_1 = \{u, u_1\}$, $e_2 = \{u, u_2\}$ and $e_3 = \{u, u_3\}$ be the three edges incident with u in G. By Claim C and Claim E, we note that $d(u_1) = d(u_2) = d(u_3) = 2$. For $i \in \{1, 2, 3\}$, let $f_i = \{u_i, v_i\}$ be the edge incident with u_i that is different from e_i. By Claim G, the set $N(u)$ is an independent set. By Claim I, no two vertices in $N(u)$ have a common neighbor other than the vertex u; that is, $v_i \neq v_j$ for $1 \leqslant i < j \leqslant 3$.

By Claim H, the set $\{v_1, v_2, v_3\}$ is an independent set. Let W be the set of all vertices within distance 2 from u, and so $W = \{u, u_1, u_2, u_3, v_1, v_2, v_3\}$. As observed earlier, for $i \in \{1, 2, 3\}$, the vertex u_i is the only neighbor of v_i that belongs to the set W.

Let G' be obtained from $G - W$ as follows: For each vertex v_i, $1 \leqslant i \leqslant 3$, of degree 3 in G, add an E_2-edge containing the two neighbors of v_i not in W. Hence if $d(v_i) = 2$, then we delete two F_2-edges incident with v_i when constructing G', while if $d(v_i) = 3$, we delete three F_2-edges incident with v_i when construction G' but add an E_2-edge. Let $\mathcal{P}' = (E_2', F_2')$, where E_2' consists of the added E_2-edges, if any, and where F_2' is obtained from F_2 by deleting all edges incident with vertices in W. Hence if ℓ denotes the number of vertices in $\{v_1, v_2, v_3\}$ of degree 3 in G, then $|E_2'| = |E_2| + \ell = \ell$ and $|F_2'| = |F_2| - 9 - \ell$. Let $X' = X = \emptyset$. Then, $n(G') = n(G) - 7$ and $|X'| = |X| = 0$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 42 + 18 + 2\ell - \ell = 60 + \ell \geqslant 60$. Every $(\mathcal{P}', X')$-cover in G' can be extended to a $(\mathcal{P}, X)$-cover in G by adding to it the set $W \setminus \{u\}$, implying that $\tau(G; \mathcal{P}, X) \leqslant \tau(G'; \mathcal{P}', X') + 6$ and therefore $\tau^{\Delta}(G'; \mathcal{P}', X') \leqslant 6$. Hence, $\xi^{\Delta}(G'; \mathcal{P}', X') \geqslant 60 \geqslant 10\tau^{\Delta}(G'; \mathcal{P}', X')$, contradicting Lemma 11. This completes the proof of Theorem 10. $\square$

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