The (signless Laplacian) spectral radii of connected graphs with prescribed degree sequences

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Abstract

In this paper, some new properties are presented to the extremal graphs with largest (signless Laplacian) spectral radii in the set of all the connected graphs with prescribed degree sequences, via which we determine all the extremal tricyclic graphs in the class of connected tricyclic graphs with prescribed degree sequences, and we also prove some majorization theorems of tricyclic graphs with special restrictions.

Keywords: Spectral radius, signless Laplacian spectral radius, degree sequence, majorization

1 Introduction

Throughout the paper, G denotes a connected undirected simple graph with n vertices and m edges, unless specified otherwise. If $m = n+c-1$, then G is called a c-cyclic graph. In particular, when $c = 0$, 1, 2 or 3, then G is called a tree, unicyclic graph, bicyclic graph or tricyclic graph, respectively. As usual, denote $N_G(v)$ the neighbor set of vertex v in

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G, and let $d_G(v)$ be the degree of v. When there is no confusion, we simplify $N_G(v)$ and $d_G(v)$ as $N(v)$ and $d(v)$, respectively. If $d(v) = 1$, v is called a pendant vertex.

Let $A(G)$ be the adjacency matrix of G, and let $D(G)$ be the diagonal matrix whose (i,i)-entry is $d(v_i)$. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. We use the notations $\rho(G)$ and $\mu(G)$ to denote the spectral radius and signless Laplacian spectral radius of G, respectively, namely, $\rho(G)$ and $\mu(G)$ are, respectively, equal to the largest eigenvalues of $A(G)$ and $Q(G)$.

When G is connected, by the Perron-Frobenius Theorem of non-negative matrices (see e. g. [4]), $\rho(G)$ and $\mu(G)$ have multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$, and there also exists a unique positive unit eigenvector corresponding to $\mu(G)$. In this paper, we use $f = (f(v_1), \ldots, f(v_n))^T$ to indicate the unique positive unit eigenvector corresponding to $\rho(G)$ or $\mu(G)$, and call f the Perron vector of G.

If $d_i = d(v_i)$ for $i = 1, 2, \ldots, n$, then we call the sequence $\pi = (d_1, d_2, \ldots, d_n)$ the *degree sequence* of G. Throughout this paper, we enumerate the degrees in non-increasing order, i.e., $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Let $\Gamma(\pi)$ define the class of connected graphs with a prescribed degree sequence π, and let $\mathbf{S}(\pi)$ be the class of connected tricyclic graphs with a prescribed tricyclic degree sequence π. In the coming discussion, we call G an extremal graph if G has largest spectral radius or signless Laplacian spectral radius of $\Gamma(\pi)$.

Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two non-increasing integer sequences, we write $\pi \lhd \pi'$ if and only if $\pi \neq \pi'$, $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$, and $\sum_{i=1}^{j} d_i \leqslant \sum_{i=1}^{j} d'_i$ for all $j = 1, 2, \ldots, n$. Such an ordering is sometimes called majorization. Suppose that G and G' are the extremal graphs of $\Gamma(\pi)$ and $\Gamma(\pi')$, respectively. We say that the spectral radii (respectively, signless Laplacian spectral radii) of G and G' satisfy the majorization theorem if $\pi \lhd \pi'$ implies that $\rho(G) < \rho(G')$ (respectively, $\mu(G) < \mu(G')$).

The work on determining the graph which has largest spectral radius among some class of graphs, can be traced back to 1985 when Brualdi and Hoffman investigated the maximum spectral radius of the adjacency matrix of a (not necessarily connected) graph in the set of all graphs with a given number of vertices and edges. Their work was followed by other people, in the connected graph case as well as in the general case.

In this line, the unique extremal graph of $\Gamma(\pi)$ was characterized when $\Gamma(\pi)$ are restricted on trees, unicyclic graphs and/or bicyclic graphs, respectively [1, 2, 5, 11, 16, 17], and the (signless Laplacian) spectral radii of extremal graphs were proved to satisfy the majorization theorem when $\Gamma(\pi)$ are restricted on trees, unicyclic graphs and/or bicyclic graphs, respectively [2, 5, 6, 8, 16, 17]. Furthermore, Liu et al. [9] found that the majorization theorem is a good tool to deal with Cvetković's problem, asked how to classify and order graphs according to their spectral radii [3]. Unfortunately, this method (namely, the tool of majorization theorem) cannot be applied to deal with Cvetković's problem for the spectral radii of tricyclic graphs, since a counterexample to the majorization theorem of tricyclic graphs was discovered by Liu et al. [10].

In this paper, some new properties are presented to the extremal graphs of $\Gamma(\pi)$, and all the extremal tricyclic graphs of $\mathbf{S}(\pi)$ will be determined. Furthermore, we also verify some majorization theorems of tricyclic graphs with special restrictions.

The rest of this paper is organized as follows: We first give some new properties to the extremal graphs of $\Gamma(\pi)$ in Section 2, via which we characterize all the extremal tricyclic graphs of $\mathbf{S}(\pi)$ in Sections 3. Finally, some majorization theorems of tricyclic graphs with special restrictions are given in Section 4.

2 Extremal graphs of $\Gamma(\pi)$

Let $G-uv$ denote the graph obtained from G by deleting the edge $uv \in E(G)$. Similarly, denote by $G+uv$ the graph obtained from G by adding an edge $uv \notin E(G)$.

Lemma 1. [15, 16] *Let u, v be two vertices of the connected graph G, and $w_1, w_2, \ldots, w_k$ $(1 \leqslant k \leqslant d(v))$ be some vertices of $N(v) \setminus (N(u) \cup \{u\})$. Let $G' = G + w_1u + \cdots + w_ku - w_1v - \cdots - w_kv$. Suppose f is the Perron vector of G. If $f(u) \geqslant f(v)$, then $\rho(G') > \rho(G)$ and $\mu(G') > \mu(G)$.*

Corollary 2. *Suppose G is an extremal graph of $\Gamma(\pi)$ and f is the Perron vector of G. If $d(v) > d(u)$, then $f(v) > f(u)$. Moreover, if $f(v) = f(u)$, then $d(v) = d(u)$.*

Proof. Suppose that there exist vertices v and u such that $d(v) > d(u)$, but $f(v) \leqslant f(u)$. Since G is connected, we may suppose that P_{uv} is a shortest path from u to v. Note that $d(v) - d(u) = k > 0$. Then, there exist vertices $\{w_1, w_2, \ldots, w_k\} \subseteq N(v) \setminus (N(u) \cup \{u\})$ such that w_1, w_2, $\ldots$, $w_k \notin V(P_{uv})$. Let $G' = G + uw_1 + \cdots + uw_k - vw_1 - \cdots - vw_k$. Then, $G' \in \Gamma(\pi)$. Since $f(v) \leqslant f(u)$, $\rho(G') > \rho(G)$ and $\mu(G') > \mu(G)$ by Lemma 1, contradicting the choice of G. □

Lemma 3. ([4], P. 492–493) *Suppose $M = M_{n\times n}$ is a symmetric, nonnegative matrix, y is an n-tuple positive vector, α and β are two nonnegative real numbers. If $\alpha y \leqslant My \leqslant \beta y$, then $\alpha \leqslant \lambda \leqslant \beta$, where λ is the largest eigenvalue of M. Furthermore, $\alpha y < My$ implies that $\alpha < \lambda$, and $My < \beta y$ implies that $\lambda < \beta$.*

Proposition 4. *Let $G = (V, E)$ be a connected graph such that $ux \in E$, $vy \in E$, $uv \notin E$, $xy \notin E$, and let f be the Perron vector of G. Let $G' = G + uv + xy - ux - vy$ (not necessary simple). Suppose G' is not connected and G^* is a connected component of G' so that $uv \in E(G^*)$ and $xy \notin E(G^*)$. If $f(u) \geqslant f(y)$ and $f(v) \geqslant f(x)$, then $\rho(G^*) \geqslant \rho(G)$ and $\mu(G^*) \geqslant \mu(G)$. Moreover, $\rho(G^*) = \rho(G)$ (respectively, $\mu(G^*) = \mu(G)$) if and only if $f(u) = f(y)$ and $f(v) = f(x)$.*

Proof. Let f_1 be a vector which is the restriction of f on $V(G^*)$. Since $uv \in E(G^*)$ and $xy \notin E(G^*)$, $\rho(G)f_1 \leqslant A(G^*)f_1$ and $\mu(G)f_1 \leqslant Q(G^*)f_1$. By Lemma 3 we can conclude that $\rho(G^*) \geqslant \rho(G)$ and $\mu(G^*) \geqslant \mu(G)$, where both equalities hold if and only if $f(u) = f(y)$ and $f(v) = f(x)$. □

By Lemma 3, we can restate Lemma 3 of [2] and Lemma 3.3 of [16] as follows.

Lemma 5. [2, 16] *Let* $G = (V, E)$ *be a connected graph such that* $ux \in E$, $vy \in E$, $uv \notin E$, $xy \notin E$. *Let* $G' = G + uv + xy - ux - vy$. *Suppose* f *is the Perron vector of* G. *If* $f(u) \geqslant f(y)$ *and* $f(v) \geqslant f(x)$, *then* $\rho(G') \geqslant \rho(G)$ *and* $\mu(G') \geqslant \mu(G)$. *Moreover,* $\rho(G') = \rho(G)$ *(respectively,* $\mu(G') = \mu(G)$*) if and only if* $f(u) = f(y)$ *and* $f(v) = f(x)$.

Corollary 6. *Let* G *be an extremal graph of* $\Gamma(\pi)$ *and let* f *be the Perron vector of* G. *Suppose* $ux \in E$, $vy \in E$, $uv \notin E$, $xy \notin E$. *Let* $G' = G + uv + xy - ux - vy$. *If* $G' \in \Gamma(\pi)$, *then*

(1) $f(u) > f(y)$ *if and only if* $f(v) < f(x)$*;*

(2) $f(u) = f(y)$ *if and only if* $f(v) = f(x)$, *and* $f(u) = f(y)$ *(respectively,* $f(v) = f(x)$*) if and only if* G' *is also an extremal graph of* $\Gamma(\pi)$.

Definition 7. Let G be a connected graph and f be the Perron vector of G. A well-ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of $V(G)$ is called a BFS-ordering if the following hold for all vertices $u, v \in V(G)$:

(*i*) $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_n)$, $f(v_1) \geqslant f(v_2) \geqslant \cdots \geqslant f(v_n)$ and $h(v_1) \leqslant h(v_2) \leqslant \cdots \leqslant h(v_n)$, where $h(v_i)$ is the distance between v_i and v_1.

(*ii*) If $v \in N(u) \backslash N(x)$, $y \in N(x) \backslash N(u)$ such that $h(u) = h(x) = h(v) - 1 = h(y) - 1$, then $f(u) > f(x)$ if and only if $f(v) > f(y)$, and $f(u) = f(x)$ if and only if $f(v) = f(y)$.

Furthermore, if $V(G)$ has a BFS-ordering, then we call G a BFS-graph.

Suppose $v_1 \prec v_2 \prec \cdots \prec v_n$ is a BFS-ordering of $V(G)$. Denote by $dist(u, v)$ the distance between u and v in G, and let $h(v) = dist(v_1, v)$. Set $A_i = \{v : dist(v_1, v) = i\}$. In some literatures (for instance, [2, 5]), A_i is also called the i-th layer vertices of G. Clearly, $A_0 = \{v_1\}$ and $A_1 = N(v_1)$. We write $u \equiv v$ if and only if we can interchange the positions of u and v in $\prec$ to obtain another BFS-ordering of $V(G)$.

Lemma 8. [2, 17] *Suppose* G *is an extremal graph of* $\Gamma(\pi)$, *and* f *is the Perron vector of* G. *Then,* $V(G)$ *has a well-ordering* $v_1 \prec v_2 \prec \cdots \prec v_n$ *such that* $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_n)$, $f(v_1) \geqslant f(v_2) \geqslant \cdots \geqslant f(v_n)$ *and* $h(v_1) \leqslant h(v_2) \leqslant \cdots \leqslant h(v_n)$.

Lemma 9. *Suppose* G *is an extremal graph of* $\Gamma(\pi)$, *and* f *is the Perron vector of* G. *Then,* $V(G)$ *has a* BFS*-ordering such that*

(1) *if* $h(u) = h(v)$, *then* $u \equiv v$ *if and only if* $f(u) = f(v)$.

(2) *if* $h(u) = h(v) = h(w)$, $u \equiv v$ *and* $v \equiv w$, *then* $u \equiv w$.

Proof. We first show that $V(G)$ has a BFS-ordering. By Lemma 8, $V(G)$ has a well-ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ so that Definition 7 (*i*) holds. Thus, it suffices to deduce that Definition 7 (*ii*) also holds. Let $G' = G + uy + xv - uv - xy$. Then, $G' \in \Gamma(\pi)$, and hence Definition 7 (*ii*) follows from Corollary 6.

We secondly prove (1). Without loss of generality, suppose that $u \prec v$ in the ordering $\prec$. Clearly, $u \equiv v$ implies that $f(u) = f(v)$, since $f(u) \geqslant f(v)$ by Definition 7 (i). Thus, it suffices to show that $f(u) = f(v)$ also implies that $u \equiv v$.

Now, we suppose that $f(u) = f(v)$. We interchange the positions of u and v in the ordering $\prec$ to obtain a new ordering $\prec'$. Since $f(u) = f(v)$, $d(u) = d(v)$ by Corollary 2. So, $\prec'$ satisfies (i) of Definition 7 because $h(u) = h(v)$. Furthermore, $\prec'$ clearly satisfies (ii) of Definition 7, since $\prec$ satisfies (ii). So, (1) holds.

Finally, we turn to prove (2). Since $u \equiv v$ and $v \equiv w$, $f(u) = f(v) = f(w)$ by (1). Now, (1) implies that $u \equiv w$. Thus, (2) holds. □

Theorem 10. *Let G be an extremal graph of $\Gamma(\pi)$ and f be the Perron vector of G. Then, $V(G)$ has a BFS-ordering $\prec$ such that*

(1) *if $h(u) = h(v) = h(w)$, $u \prec v \prec w$, $uw \in E(G)$ and $uv \notin E(G)$, then $x \in N(w)$ holds for any $x \in N(v) \setminus \{w\}$ with $u \prec x$, and there must exist some $y \prec u$ such that $y \in N(v) \setminus N(w)$. Furthermore, if $h(v_2) = h(v) = h(w)$, $v_2 \prec v \prec w$, then $v_2w \in E(G)$ implies that $v_2v \in E(G)$.*

(2) *if $f(v_1) > f(v_2)$, then $h(u) < h(v)$ implies that $f(u) > f(v)$.*

Proof. We first prove (1). By Lemma 9, $V(G)$ has a BFS-ordering $\prec$. Assume, to the contrary, that the result is not true. Let u and w be the least vertices and v be the last vertex in the ordering $\prec$ of $V(G)$ such that $h(u) = h(v) = h(w)$, $u \prec v \prec w$, $uw \in E(G)$, $uv \notin E(G)$ and there exists some vertex $x \in N(v) \setminus \{w\}$ with $u \prec x$, but $x \notin N(w)$. We may suppose that $v \not\equiv w$ (Otherwise, we will consider the new BFS-ordering $\prec'$ of $V(G)$ obtained from $\prec$ by interchanging the positions of v and w). So, $f(v) > f(w)$ by Lemma 9 (1) and Definition 7 (i). Let $G' = G + uv + wx - uw - vx$. By Corollary 6 (1), it follows that $f(u) < f(x)$, contradicting (i) of Definition 7.

If $y \in N(w)$ holds for every $y \in N(v)$ with $y \prec u$, since $u \in N(w) \setminus N(v)$ and $x \in N(w)$ holds for any $x \in N(v) \setminus \{w\}$ with $u \prec x$, we have $d(v) < d(w)$, contradicting Definition 7 (i).

Thus, $V(G)$ has a BFS-ordering $\prec$ such that (1) holds. Now, we turn to show (2). It suffices to show that $f(u) > f(v)$ whenever $u \in A_j$ and $v \in A_{j+1}$ holds for $j \geqslant 0$ by induction. The result clearly follows for $j = 0$ by the condition $f(v_1) > f(v_2)$. Now, we assume that the result already holds for $0 \leqslant j \leqslant k-1$, and we will prove that the result also follows for $j = k$.

Suppose that there exist two vertices, say u and v, such that $u \in A_k$, $v \in A_{k+1}$ and $f(u) \leqslant f(v)$. Since $h(u) < h(v)$, we have $f(u) = f(v)$ by Definition 7 (i), and hence $d(u) = d(v)$ by Corollary 2. Let P_{uv_1} be a shortest path from v_1 to u, and let P_{vv_1} be a shortest path from v_1 to v. Choose $x \in V(P_{uv_1})$ such that $x \in N(u)$. Then, $xv \notin E(G)$ and $h(x) = h(u) - 1$.

Suppose $u \in V(P_{vv_1})$. Then $d(v) = d(u) \geqslant 2$ and $uv \in E(P_{vv_1})$. Since $d(v) = d(u) \geqslant 2$ and $x \in N(u) \setminus N(v)$, there exists some $y \in N(v) \setminus \{u\}$ such that $y \notin N(u)$. Since $u \in N(v) \cap V(P_{vv_1})$, $h(y) \geqslant h(u) = h(x) + 1$. If $h(y) > h(u)$, by Definition 7 (i) we have $f(u) \geqslant f(y)$. By the induction hypothesis and $h(x) = k - 1 < k = h(u)$, $f(x) > f(u)$.

Thus, $f(x) > f(y)$. If $h(y) = h(u)$, since $h(x) = k - 1 < k = h(u) = h(y)$, by the induction hypothesis we have $f(x) > f(y)$. So, we get $f(x) > f(y)$ whenever $h(y) > h(u)$ or $h(y) = h(u)$. Let $G' = G + vx + uy - ux - vy$. Since $ux \in E(G)$, $uv \in E(G)$ and $vy \in E(G)$, we can conclude that $G' \in \Gamma(\pi)$. Now, Corollary 6 (1) implies that $f(v) < f(u)$, a contradiction.

Suppose $u \notin V(P_{vv_1})$. Choose $y \in N(v) \cap V(P_{vv_1})$ such that $h(y) = h(u)$. Thus, $f(x) > f(y)$ by the induction hypothesis and $h(x) = k - 1 < k = h(u) = h(y)$. If $yu \notin E(G)$, let $G' = G + vx + uy - ux - vy$. Then, $f(v) < f(u)$ by Corollary 6 (1), a contradiction. If $yu \in E(G)$, since $x \in N(u) \setminus N(v)$ and $d(u) = d(v)$, there exists some $z \in N(v) \setminus \{u\}$ such that $z \notin N(u)$. Since $h(z) \geqslant h(y) = h(u)$, $f(x) > f(z)$ by the induction hypothesis and Definition 7 (i). Let $G' = G + xv + uz - ux - vz$. Now, Corollary 6 (1) implies that $f(v) < f(u)$, a contradiction. □

In the following, if G is an extremal graph of $\Gamma(\pi)$, we always suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ has a BFS-ordering $\prec$ such that $\prec$ satisfies the conclusion of Theorem 10.

Lemma 11. *Let G be an extremal graph of $\Gamma(\pi)$, and uv be an edge on a cycle of G. Suppose $P = w_1 \cdots w_{s+1}$ is a path of G, and $f(w_{s+1}) < \min\{f(u), f(v)\}$, where f is the Perron vector of G. If there exists some $j \in \{1, 2, \ldots, s\}$ such that $uw_j \notin E(G)$, $uw_t \notin E(G)$ and $vw_t \notin E(G)$ hold for every $t \in \{j+1, \ldots, s+1\}$, then $f(v) > f(w_j)$.*

Proof. To the contrary, suppose that $f(v) \leqslant f(w_j)$. Let $G' = G + uw_j + vw_{j+1} - uv - w_jw_{j+1}$. Then, $G' \in \Gamma(\pi)$. By Lemma 5, $f(u) \leqslant f(w_{j+1})$. Let $G'' = G + uw_{j+2} + vw_{j+1} - uv - w_{j+1}w_{j+2}$. Then, $G'' \in \Gamma(\pi)$. Since $f(u) \leqslant f(w_{j+1})$, by Lemma 5 it follows that $f(v) \leqslant f(w_{j+2})$. Let $G''' = G + uw_{j+2} + vw_{j+3} - uv - w_{j+2}w_{j+3}$. Then, $G''' \in \Gamma(\pi)$. Since $f(v) \leqslant f(w_{j+2})$, $f(u) \leqslant f(w_{j+3})$ by Lemma 5. By repeating the similar arguments, we can conclude that $f(w_{s+1}) \geqslant \min\{f(u), f(v)\}$, contradicting the condition. □

Denote by $\mathcal{R}(G)$ the reduced graph obtained from G by recursively deleting pendant vertices of the resultant graph until no pendant vertices remain. If G is a connected c-cyclic graph, it is easy to see that $\mathcal{R}(G)$ is unique and $\mathcal{R}(G)$ is also a connected c-cyclic graph. Thus, we have

$$\sum_{w \in V(\mathcal{R}(G))} d_{\mathcal{R}(G)}(w) = 2|V(\mathcal{R}(G))| + 2c - 2. \quad (1)$$

Proposition 12. *Let G be an extremal graph of $\Gamma(\pi)$, and f be the Perron vector of G. Suppose $u \in V(\mathcal{R}(G))$ and $v \in V(G) \setminus V(\mathcal{R}(G))$. Then, $f(u) > f(v)$.*

Proof. If $d(v) = 1$, then $f(u) > f(v)$ by Corollary 2 and $d(u) \geqslant 2$. Now, we suppose that $d(v) \geqslant 2$. Then, there exists some pendant path, say $P = vx_1 \cdots x_s$, where $d(x_s) = 1$.

Suppose u lies on some cycle C. Then, there exists some vertex $w \in V(C) \cap N(u)$ such that $wv \notin E(G)$, $wx_i \notin E(G)$ and $ux_i \notin E(G)$ hold for $1 \leqslant i \leqslant s$. Thus, $f(u) > f(v)$ by Lemma 11.

Suppose u does not lie on any cycle. Since $u \in \mathcal{R}(G)$, u lies on a path, say P, where P is the unique path of $\mathcal{R}(G)$ connecting two cycles, say C and C'. Suppose $\{x\} = V(P) \cap V(C)$ and $\{y\} = V(P) \cap V(C')$. Let P_{v_1x} be a shortest path connecting v_1 and x, and let P_{v_1y} be a shortest path connecting v_1 and y. If $u \in V(P_{v_1x})$, then $f(u) \geqslant f(x) > f(v)$ by the former arguments and Theorem 10. Similarly, if $u \in V(P_{v_1y})$, then $f(u) \geqslant f(y) > f(v)$. Now, we consider the case that $u \notin V(P_{v_1x})$ and $u \notin V(P_{v_1y})$.

If either $y \in V(P_{v_1x})$ or $x \in V(P_{v_1y})$, since $u \notin V(P_{v_1x})$ and $u \notin V(P_{v_1y})$, it is easy to see that u lies on a cycle, a contradiction. If $y \notin V(P_{v_1x})$ and $x \notin V(P_{v_1y})$, since $u \notin V(P_{v_1x})$ and $u \notin V(P_{v_1y})$, u also lies on a cycle, a contradiction. □

Proposition 13. *Let G be an extremal graph of $\Gamma(\pi)$ and u and v be two vertices of $\mathcal{R}(G)$. Suppose f is the Perron vector of G. If $d_{\mathcal{R}(G)}(u) > d_{\mathcal{R}(G)}(v)$, then $f(u) > f(v)$. Furthermore, $f(u) = f(v)$ implies that $d_{\mathcal{R}(G)}(u) = d_{\mathcal{R}(G)}(v)$.*

Proof. By Corollary 2, we may suppose that $d_G(u) \leqslant d_G(v)$. Let P_{uv} be a shortest path from u to v in $\mathcal{R}(G)$. Since $d_{\mathcal{R}(G)}(u) > d_{\mathcal{R}(G)}(v)$, there exists some vertex $w \in N_{\mathcal{R}(G)}(u) \setminus N_{\mathcal{R}(G)}(v)$ so that $w \notin P_{uv}$. Moreover, since $d(u) \leqslant d(v)$ and $d_{\mathcal{R}(G)}(u) > d_{\mathcal{R}(G)}(v)$, there exists some vertex z such that $z \in N_G(v) \setminus N_G(u)$ and $z \in V(G) \setminus V(\mathcal{R}(G))$. By Proposition 12, $f(w) > f(z)$. Let $G' = G + uz + vw - uw - vz$. By Corollary 6 (1), it follows that $f(v) < f(u)$. □

Corollary 14. *Suppose G is an extremal graph of $\Gamma(\pi)$. If $d_{\mathcal{R}(G)}(v) \geqslant 2$ holds for some $v \in A_j \cap \mathcal{R}(G)$ and $j \geqslant 2$, then*

(1) *$d_{\mathcal{R}(G)}(u) \geqslant d_{\mathcal{R}(G)}(v)$ holds for each $u \in A_i$, where $0 \leqslant i \leqslant j-1$;*

(2) *$d_{\mathcal{R}(G)}(w) = d_G(w)$ holds for each $w \in A_k$, where $0 \leqslant k \leqslant j-2$.*

Proof. (1) clearly follows from Proposition 13 and Theorem 10. Thus, we only need to show (2). Suppose that there exists some vertex x such that $x \in N_G(w) \setminus N_{\mathcal{R}(G)}(w)$. Then, $x \notin V(\mathcal{R}(G))$. So, $f(v) > f(x)$ by Proposition 12. On the other hand, since $x \in N_G(w)$, $dist(v_1, x) < dist(v_1, v)$. By Theorem 10, $f(v) \leqslant f(x)$, a contradiction. □

Let G be a connected graph and T be a tree such that T is attached to a vertex v of G. Then, v is called the root of T. In the coming discussion, we use the notation T_v to denote a root tree with root v, and we agree that T_v includes the root v.

An internal path, say $P = v_1v_2 \cdots v_{s+1}$ ($s \geqslant 1$), is a path joining v_1 and v_{s+1} (which need not be distinct) such that v_1 and v_{s+1} have degree greater than 2, while all other vertices $v_2, \ldots, v_s$ are of degree 2. Suppose P is an internal path. Denote $l(P)$ the length of P, i.e., $l(P) = s$.

Proposition 15. *Let G be an extremal graph of $\Gamma(\pi)$, where $d_n = 1$. Suppose $P = w_1 \cdots w_{s+1}$ is an internal path of $\mathcal{R}(G)$.*

(1) *If $w_1 \neq w_{s+1}$, then $l(P) \leqslant 2$. Furthermore, if $l(P) = 2$, then either $w_1w_3 \in E(G)$ or all the pendant vertices of G are on T_{w_2}.*

(2) *If $w_1 = w_{s+1}$, then $l(P) = 3$.*

Proof. Here we only prove (1), since (2) can be demonstrated analogously. Let f be the Perron vector of G, and let $f(w_k) = \min\{f(w_i)$, where $1 \leqslant i \leqslant s+1\}$. Suppose $s \geqslant 3$. By Proposition 13, we have $2 \leqslant k \leqslant s$.

If there exists at least one pendant vertex pertaining to $V(G)\backslash V(T_{w_k})$, let $G' = G + w_{k-1}w_{k+1} + w_kw_k - w_{k-1}w_k - w_kw_{k+1}$ (not simple), and let G^* be the component of G' containing the edge $w_{k-1}w_{k+1}$. Now, Proposition 4 implies that $\rho(G^*) \geqslant \rho(G)$ and $\mu(G^*) \geqslant \mu(G)$. Suppose u is a pendant vertex of G^*, and $uv \in E(G^*)$. Let G'' be the graph obtained from G^* by subdividing the edge uv, i.e., adding a new vertex w and edges wu, wv in $G - uv$. Then, we can construct a new graph G''' obtained from G'' (via replacing T_{w_k} by T_w) such that $G''' \in \Gamma(\pi)$. Since $G^* \subset G'''$, $\rho(G''') > \rho(G^*) \geqslant \rho(G)$ and $\mu(G''') > \mu(G^*) \geqslant \mu(G)$, a contradiction.

Thus, all the pendant vertices of G are on T_{w_k}. Since $s \geqslant 3$, there exists some vertex, say x, such that $x \in V(P)\backslash\{w_1, w_k, w_{s+1}\}$, $d_G(w_k) > 2 = d_G(x)$ and hence $f(w_k) > f(x)$ by Corollary 2, contradicting the choice of w_k. This contradiction implies that $l(P) \leqslant 2$.

Now, we assume that $w_1w_3 \notin E(G)$ and there exists at least one pendant vertex pertaining to $V(G)\backslash V(T_{w_2})$. Let $G' = G + w_1w_3 + w_2w_2 - w_1w_2 - w_2w_3$ (not simple). Similarly, we will reach a contradiction, since $f(w_2) < \min\{f(w_1), f(w_3)\}$ by Proposition 13. So, all the pendant vertices of G are on T_{w_2}. □

By the definition of internal path and Corollary 2, with the similar method as applied in the proof of Proposition 15, we have

Proposition 16. *Let G be an extremal graph of $\Gamma(\pi)$, where $d_n = 1$. Suppose P is an internal path of G from u to v. (1) If $u \neq v$, then $l(P) \leqslant 2$ and $uv \in E(G)$. (2) If $u = v$, then $l(P) = 3$.*

3 Extremal graphs of $\mathbf{S}(\pi)$

Denote P_n and K_n, respectively, a path and a complete graph on n vertices. Suppose u is a vertex of G, and $P_{s+1} = u_1u_2 \cdots u_{s+1}$, where $u_i \notin V(G)$ for $1 \leqslant i \leqslant s+1$. If we obtain G' by adding two edges between u and the two pendant vertices of P_{s+1}, i.e., by adding the edges uu_1 and uu_{s+1}, then we say that G' is obtained from G by appending the path P_{s+1} to u of G. If we obtain G' by adding the edge uu_1, then we say that G' is obtained from G by attaching the path P_{s+1} to u of G.

Suppose $\pi = (d_1, d_2, \ldots, d_n)$ is a tricyclic degree sequence. Then, $\sum_{i=1}^{n} d_i = 2n+4$, which implies that $d_n \leqslant 2$ holds for $n \geqslant 5$. It easily follows that

Proposition 17. *Suppose $\pi = (d_1, d_2, \ldots, d_n)$ is a tricyclic degree sequence. If $d_n = 2$, then $\pi \in \{(6,2,\ldots,2), (5,3,2,\ldots,2), (4,4,2,\ldots,2), (4,3,3,2,\ldots,2), (3,3,3,3,2,\ldots,2)\}$. If $d_n = 1$, then either (1) $d_1 \geqslant 4$ and $d_4 \geqslant 3$ or (2) $d_1 = 3$ and $d_4 = 3$ or (3) $d_2 = d_3 = d_4 = 2$ or (4) $d_2 = 3$ and $d_3 = d_4 = 2$ or (5) $d_2 = d_3 = 3$ and $d_4 = 2$ or (6) $d_2 \geqslant 4$ and $d_4 = 2$.*

In the following, we shall determined all the extremal tricyclic graphs of $\mathbf{S}(\pi)$ for any prescribed tricyclic degree sequence π according to Proposition 17. To do this, we need to introduce more notations as follows.

Let F_1 be the tricyclic graph obtained by appending two paths of lengths one and a path of length $n-6$, respectively, to a common vertex. Let $D=(V,E)$ be the bicyclic graph such that $V(D)=\{u_1,u_2,u_3,u_4\}$ and $E(D)=\{u_1u_2,u_1u_3,u_1u_4,u_2u_3,u_2u_4\}$. In other words, $D=K_4-e$. Let F_2 be the tricyclic graph obtained from D by appending a path of length $n-5$ to u_1 of D.

Suppose $P_{n-4}=w_1w_2\cdots w_{n-4}$. Let F_3 be the tricyclic graph obtained from D and P_{n-4} by adding two edges u_1w_1 and u_2w_{n-4}. Let F_4 be the tricyclic graph obtained from D and P_{n-4} by adding two edges u_1w_1 and u_3w_{n-4}. Let F_5 be the tricyclic graph obtained from D and P_{n-4} by adding two edges u_3w_1 and u_4w_{n-4}.

Theorem 18. *Suppose G is an extremal of $\mathbf{S}(\pi)$, where $\pi=(d_1,d_2,\ldots,d_n)$ and $d_n=2$.*

(1) *If $d_1=6$ and $d_2=\cdots=d_n=2$, then $G\cong F_1$;*

(2) *If $d_1=5$, $d_2=3$ and $d_3=\cdots=d_n=2$, then $G\cong F_2$;*

(3) *If $d_1=d_2=4$ and $d_3=\cdots=d_n=2$, then $G\cong F_3$;*

(4) *If $d_1=4$, $d_2=d_3=3$ and $d_4=\cdots=d_n=2$, then $G\cong F_4$;*

(5) *If $d_1=d_2=d_3=d_4=3$ and $d_5=\cdots=d_n=2$, then $G\cong F_5$.*

Proof of Theorem 18 (1). Since $d_1=6$ and $d_2=\cdots=d_n=2$, G is obtained by appending three paths, say $P_{l_i}=w_{i1}w_{i2}\cdots w_{il_i}$ $(i=1,2,3)$, respectively, to a common vertex u. Without loss of generality, suppose that $l_1\geqslant l_2\geqslant l_3$.

If $l_2\geqslant 3$, by Corollary 2, we have $u=v_1$ and $f(v_1)>f(v_2)$. Thus, $f(w_{21})>f(w_{12})$ and $f(w_{11})>f(w_{22})$ by Theorem 10 (2). Let $G'=G+w_{11}w_{21}+w_{12}w_{22}-w_{11}w_{12}-w_{21}w_{22}$. By Corollary 6 (1), $f(w_{11})<f(w_{22})$, a contradiction.

Therefore, $l_2=l_3=2$, and hence $G\cong F_1$. □

Proof of Theorem 18 (2). By Theorem 10, we can conclude that $v_1v_2\in E(G)$. If G contains a cut edge, say uv, then we may suppose that $u=v_1$ and $v=v_2$ by Corollary 2 and $d_3=2$. Suppose $x\in N(v_1)\setminus\{v_2\}$ and $y\in N(v_2)\setminus\{v_1\}$. By Corollary 2, $f(v_1)>f(v_2)$, and hence $f(x)>f(y)$ by Theorem 10 (2). Choose $z\in N(x)\setminus\{v_1\}$. Let $G'=G+v_2x+yz-v_2y-xz$. By Corollary 6 (1), $f(v_2)<f(z)$, a contradiction. Therefore, G contains no cut edge.

Since $d_3=2$, there are two paths, say $P_{l_i}=v_1w_{i1}w_{i2}\cdots w_{il_i-2}v_2$ $(i=1,2)$, respectively, connecting v_1 and v_2 such that $v_1v_2\notin E(P_{l_1})$ and $v_1v_2\notin E(P_{l_2})$. Without loss of generality, suppose that $l_1\geqslant l_2$. If $l_1\geqslant 4$, choose $x\in N(v_1)\backslash\{v_2\}$ such that $x\notin V(P_{l_1})$ and $x\notin V(P_{l_2})$, and let $y\in N(x)\setminus\{v_1\}$. By Corollary 2, $f(v_1)>f(v_2)$, and hence $f(x)>f(w_{1l_1-2})$ by Theorem 10 (2). Let $G'=G+v_2x+yw_{1l_1-2}-v_2w_{1l_1-2}-xy$. Now, Corollary 6 (1) implies that $f(v_2)<f(y)$, a contradiction. Therefore, $l_1=l_2=3$, and hence $G\cong F_2$. □

Proof of Theorem 18 (3). By Theorem 10, we can conclude that $v_1v_2 \in E(G)$.

Suppose v_2 is a cut vertex of G. Since $d_3 = 2$, G is obtained from a cycle $C = v_2v_1w_{11}\cdots w_{1l_1}v_2$ by appending the path $P_{l_2} = w_{21}w_{22}\cdots w_{2l_2}$ to v_1 and appending the path $P_{l_3} = w_{31}w_{32}\cdots w_{3l_3}$ to v_2. By Corollary 2, $f(v_2) > f(w_{22})$, and hence $f(w_{21}) < f(w_{31})$ by Corollary 6 (1), since $G' = G + v_2w_{21} + w_{22}w_{31} - w_{21}w_{22} - v_2w_{31}$ is connected. On the other hand, since $h(w_{31}) > h(w_{21})$, we have $f(w_{21}) \geqslant f(w_{31})$ by Theorem 10, a contradiction.

Thus, v_2 is not a cut vertex of G. Since $d_3 = 2$, there are three paths, say $P_{l_i} = v_1w_{i1}w_{i2}\cdots w_{il_i-2}v_2$ $(i = 1, 2, 3)$, respectively, connecting v_1 and v_2 such that $v_1v_2 \notin E(P_{l_i})$ holds for $i \in \{1, 2, 3\}$. Without loss of generality, suppose that $l_1 \geqslant l_2 \geqslant l_3$. Assume that $l_2 \geqslant 4$. Let $G'' = G + w_{11}w_{21} + w_{12}w_{22} - w_{11}w_{12} - w_{21}w_{22}$. Then, G'' is connected.

If $f(w_{11}) = f(w_{22})$ and $f(w_{21}) = f(w_{12})$, G'' is also an extremal graph of $\mathbf{S}(\pi)$ by Corollary 6 (2). But v_2 is a cut vertex of G'', a contradiction. Thus, either $f(w_{11}) > f(w_{22})$ or $f(w_{21}) > f(w_{12})$ holds by Theorem 10. By Corollary 6 (1), $f(w_{21}) > f(w_{12})$ implies that $f(w_{11}) < f(w_{22})$ and $f(w_{11}) > f(w_{22})$ implies that $f(w_{21}) < f(w_{12})$, a contradiction.

Thus, $l_2 = l_3 = 3$, and hence $G \cong F_3$. □

Proof of Theorem 18 (4). By Theorem 10, $v_1v_2 \in E(G)$ and $v_1v_3 \in E(G)$.

Case 1. $v_2v_3 \notin E(G)$.

Then, $N(v_1) \cap N(v_2) = \emptyset$ by Theorem 10 (1).

If v_1v_2 is a cut edge of G, choose $x \in N(v_2) \setminus \{v_1\}$, $y \in N(v_1) \setminus \{v_2\}$ and $z \in N(y) \setminus \{v_1\}$. Since $f(v_1) > f(v_2)$ by Corollary 2, $f(y) > f(x)$ by Theorem 10 (2). Let $G' = G + v_2y + xz - v_2x - yz$. By Corollary 6 (1), $f(v_2) < f(z)$, a contradiction.

If v_1v_2 is not a cut edge of G, choose $x \in N(v_2) \setminus \{v_1\}$ such that x is in a shortest path, say P, from v_2 to v_1 in $G - v_1v_2$. Choose $y \in N(v_3) \setminus \{v_1\}$ such that y is not in P (By $d_4 = 2$ and the choice of P, such y must exist). Let $G' = G + v_2v_3 + xy - v_2x - v_3y$. Since $f(v_2) > f(y)$ by Corollary 2, by Corollary 6 (1), we have $f(v_3) < f(x)$, a contradiction.

Case 2. $v_2v_3 \in E(G)$.

If v_1 is a cut vertex of G, choose $x \in N(v_1) \setminus \{v_2, v_3\}$ and $y \in N(v_2) \setminus \{v_1, v_3\}$. Let z be a vertex of $N(x) \setminus \{v_1\}$. Since $f(v_1) > f(v_2)$, $f(x) > f(y)$ by Theorem 10 (2). Let $G' = G + v_2x + yz - xz - v_2y$. By Corollary 6 (1), we have $f(v_2) < f(z)$, a contradiction.

Thus, v_1 is not a cut vertex of G. Since $d_4 = 2$, there is a path $P_{l_1} = v_1w_{11}w_{12}\cdots w_{1l_1-2}v_2$ connecting v_1 and v_2 in $G - v_3$, and there is a path $P_{l_2} = v_1w_{21}w_{22}\cdots w_{2l_2-2}v_3$ connecting v_1 and v_3 in $G - v_2$.

If $l_1 \geqslant 4$, by Theorem 10 (2), $f(w_{21}) > f(w_{1l_1-2})$. Let $G' = G + w_{21}v_2 + w_{22}w_{1l_1-2} - w_{21}w_{22} - v_2w_{1l_1-2}$ (if $l_2 = 3$, then replace w_{22} by v_3). By Corollary 6 (1), $f(v_2) < f(w_{22})$, a contradiction. Thus, $l_1 = 3$, and hence $G \cong F_4$. □

Proof of Theorem 18 (5). By Theorem 10, $v_1v_2 \in E(G)$, $v_1v_3 \in E(G)$ and $v_1v_4 \in E(G)$.

Suppose G contains a cut vertex u, where $u \in \{v_2, v_3, v_4\}$. Without loss of generality, assume that $u = v_2$. Choose $x \in N(v_2) \setminus \{v_1\}$ and $y \in N(v_3)$ such that $d(y) = 2$. Then, $f(v_2) > f(y)$ by Corollary 2. Let $G' = G + v_2v_3 + xy - v_2x - v_3y$. By Corollary 6 (1), $f(v_3) < f(x)$, which contradicts $d(v_3) > d(x) = 2$ and Corollary 2.

Thus, G is obtained from a cycle $C = uw_{11}\cdots w_{1l_1}vw_{21}\cdots w_{2l_2}ww_{31}\cdots w_{3l_3}u$ and an isolated vertex z by adding three edges zu, zv, and zw. Without loss of generality, suppose that $l_1 \geqslant l_2 \geqslant l_3 \geqslant 0$. If $l_2 \geqslant 1$, then $f(u) > f(w_{21})$ and $f(v) > f(w_{11})$ by Corollary 2. Let $G' = G + uv + w_{11}w_{21} - uw_{11} - vw_{21}$. By Corollary 6 (1), $f(v) < f(w_{11})$, a contradiction.

Therefore, $l_2 = l_3 = 0$, and hence $G \cong F_5$. □

Lemma 19. *Suppose G is an extremal of $\mathbf{S}(\pi)$, and $v \in \mathcal{R}(G)$ such that $dist(v_1, v)$ is as large as possible. If $v \notin A_1$ and $d_n = 1$, then $v \in A_2$, $d_{\mathcal{R}(G)}(v) = 2$, and the two neighbors of v of $\mathcal{R}(G)$ pertain to A_1.*

Proof. If $d_{\mathcal{R}(G)}(v) \geqslant 3$ holds for some $v \in A_j \cap \mathcal{R}(G)$ and $j \geqslant 2$, by Corollary 14 (1), $d_{\mathcal{R}(G)}(u) \geqslant 3$ holds for every $u \in A_i$, where $0 \leqslant i \leqslant j-1$. Thus, $\sum_{w\in V(\mathcal{R}(G))} d_{\mathcal{R}(G)}(w) \geqslant 5\times 3 + 2(|V(\mathcal{R}(G))| - 5) > 2|V(\mathcal{R}(G))| + 4$, contradicting equation (1). So, $d_{\mathcal{R}(G)}(v') = 2$ holds for each $v' \in A_j \cap \mathcal{R}(G)$, where $j \geqslant 2$.

Suppose that there exists some $v \in A_j \cap \mathcal{R}(G)$, where $j \geqslant 3$. Since $d_{\mathcal{R}(G)}(v') = 2$ holds for each $v' \in A_2 \cap \mathcal{R}(G)$, by Corollary 14, v lies on an internal path P of $\mathcal{R}(G)$ such that $l(P) \geqslant 4$, contradicting Proposition 15. Therefore, $v \in A_2$ and $d_{\mathcal{R}(G)}(v) = 2$.

To complete the proof, it suffices to show the following claim.

Claim. If $v \in A_2 \cap V(\mathcal{R}(G))$, then the two neighbors of v of $\mathcal{R}(G)$ pertain to A_1.

Assume the claim is not true, then at least one of the two neighbors of v of $\mathcal{R}(G)$ does not belong to A_1. We may assume that w is such a neighbor of v. Then, $d_{\mathcal{R}(G)}(w) = 2$ and $w \in A_2 \cap V(\mathcal{R}(G))$ by the former arguments. We consider the following two cases.

Case 1. v and w do not lie on a triangle.

Then, v lies on an internal path P from x to y, where $\{x, y\} \subseteq V(\mathcal{R}(G))$ by Corollary 14. Furthermore, $x = y$ implies that $l(P) \geqslant 4$ and $x \neq y$ implies that $l(P) \geqslant 3$, which is a contradiction to Proposition 15.

Case 2. v and w lie on a triangle, say C, where $V(C) = \{u, v, w\}$.

By Corollary 14, $A_1 \cap N_{\mathcal{R}(G)}(v) = \{u\} = A_1 \cap N_{\mathcal{R}(G)}(w)$.

Subcase 2.1. There exists some vertex x such that $x \in A_2 \cap (V(\mathcal{R}(G))\backslash\{v, w\})$.

By Case 1, either there exist vertices $y \in A_1 \cap V(\mathcal{R}(G))$ and $z \in A_2 \cap V(\mathcal{R}(G))$ such that x, y, z form a triangle, or there exist vertices $y, z \in A_1 \cap V(\mathcal{R}(G))$ such that $N_{\mathcal{R}(G)}(x) = \{y, z\}$.

We first suppose that there exist vertices $y \in A_1 \cap V(\mathcal{R}(G))$ and $z \in A_2 \cap V(\mathcal{R}(G))$ such that x, y, z form a triangle. If $u = y$, then $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(u) \geqslant 5$ by Proposition 13, and hence $\sum_{w\in V(\mathcal{R}(G))} d_{\mathcal{R}(G)}(w) \geqslant 5\times 2 + 2(|V(\mathcal{R}(G))| - 2) > 2|V(\mathcal{R}(G))| + 4$, contradicting equation (1). If $uy \in E(G)$, then $d_{\mathcal{R}(G)}(v_1) = 2 < d_G(v_1)$, contradicting Corollary 14 (2).

Thus, $u \neq y$ and $uy \notin E(G)$. By Proposition 13, $f(u) > f(x)$ and $f(y) > f(v)$. Let $G' = G + uy + vx - uv - yx$. Corollary 6 (1) implies that $f(y) < f(v)$, a contradiction.

Now, we suppose that there exist vertices $y, z \in A_1 \cap V(\mathcal{R}(G))$ such that $N_{\mathcal{R}(G)}(x) = \{y, z\}$.

If $u = y$, by Proposition 13, $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(u) \geqslant 4$. We claim that $d_{\mathcal{R}(G)}(z) \geqslant 3$. Otherwise, $P = yxzv_1$ is an internal path of $\mathcal{R}(G)$ of length three, contradicting Proposition 15 (1). So, $d_{\mathcal{R}(G)}(z) \geqslant 3$. On the other hand, recall that $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(u) \geqslant 4$, it is a contradiction to equation (1). Thus, $u \neq y$. Similarly, $u \neq z$.

Since G is a tricyclic graph, either $uy \notin E(G)$ or $uz \notin E(G)$. We may suppose that $uz \notin E(G)$. By Proposition 13 and Theorem 10, $f(u) > f(x)$ and $f(z) \geqslant f(v)$. Let $G' = G + uz + vx - uv - zx$. Now, Corollary 6 (1) implies that $f(z) < f(v)$, a contradiction.

Subcase 2.2. $A_2 \cap V(\mathcal{R}(G)) = \{v, w\}$.

If $ux \in E(G)$ holds for every $x \in N(v_1) \setminus \{u\}$, by Proposition 13, we have $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(u) \geqslant 5$, contradicting equation (1). Thus, there exists at least one vertex, say y, of $N(v_1) \setminus \{u\}$ such that $uy \notin E(G)$, and there exists a vertex z in $N(v_1) \setminus \{u, y\}$ such that $yz \in E(G)$.

If $uz \in E(G)$, by Proposition 13, we have $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(u) \geqslant 4$ and $d_{\mathcal{R}(G)}(z) \geqslant 3$, contradicting equation (1). Thus, $uz \notin E(G)$ and $uy \notin E(G)$.

Since G is a tricyclic graph, either $d_{\mathcal{R}(G)}(y) = 2$ or $d_{\mathcal{R}(G)}(z) = 2$. We may suppose that $d_{\mathcal{R}(G)}(z) = 2$. By Proposition 13 and Theorem 10, $f(u) > f(z)$ and $f(y) \geqslant f(v)$. Let $G' = G + uy + vz - uv - yz$. Now, Corollary 6 (1) implies that $f(y) < f(v)$, a contradiction. □

Lemma 20. *Suppose G is an extremal of $\mathbf{S}(\pi)$, and $v \in \mathcal{R}(G)$ such that $dist(v_1, v)$ is as large as possible. If $d_1 \geqslant 4$ and $d_n = 1$, then $v \in A_1$.*

Proof. Suppose $v \notin A_1$. By Lemma 19, $v \in A_2$ with $d_{\mathcal{R}(G)}(v) = 2$, and we may suppose that x and y are the two neighbors of v in $\mathcal{R}(G) \cap A_1$. Furthermore, Corollary 14 (1) implies that $w \in \mathcal{R}(G)$ holds for each $w \in A_1$.

If $xy \notin E(G)$, by Proposition 15 (1), all the pendant vertices of G lie on T_v. So, $d_G(v) \geqslant 3$. By Corollary 2, $d_{\mathcal{R}(G)}(w) = d_G(w) \geqslant d_G(v) \geqslant 3$ holds for each $w \in A_1 \cup \{v_1\}$, since $f(w) \geqslant f(v)$ by Theorem 10. Thus, $\sum_{w \in V(\mathcal{R}(G))} d_{\mathcal{R}(G)}(w) \geqslant 4+4\times3+2(|V(\mathcal{R}(G))|-5) > 2|V(\mathcal{R}(G))| + 4$, which contradicts equation (1).

If $xy \in E(G)$, then $d_1 = 4$ and $d_{\mathcal{R}(G)}(v_4) = d_{\mathcal{R}(G)}(v_5) = 2$ by equation (1) and Corollary 14 (2). By the virtue of the former arguments and equation (1), we may suppose that $A_2 \cap V(\mathcal{R}(G)) = \{v\}$ and hence $v_4v_5 \in E(G)$. By Proposition 13 and Theorem 10, we have $f(v_2) > f(v_5)$ and $f(v_4) \geqslant f(v)$. Let $G' = G + v_2v_4 + v_5v - v_4v_5 - v_2v$. By Corollary 6 (1), $f(v_4) < f(v)$, a contradiction. □

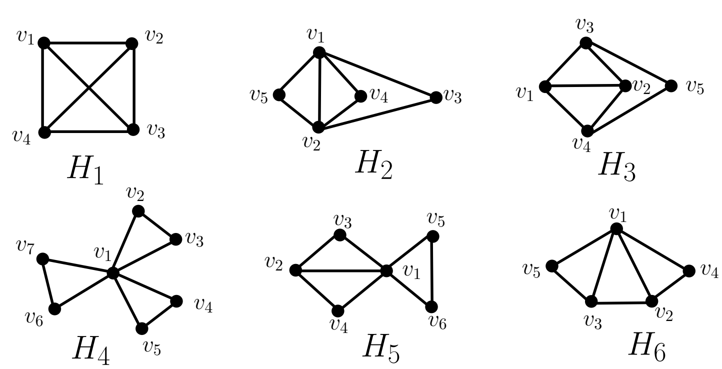

Figure 1: The tricyclic graphs H_1, H_2, …, H_6.

In the following, let H_1, H_2, ..., H_6 be the tricyclic graphs as shown in Figure 1.

Lemma 21. *Suppose G is an extremal of $\mathbf{S}(\pi)$, and $v \in \mathcal{R}(G)$ such that $dist(v_1, v)$ is as large as possible. If $v \in A_1$, then $\mathcal{R}(G) \in \{H_1, H_2, H_4, H_5, H_6\}$.*

Proof. Since $\mathcal{R}(G)$ is also a tricyclic graph, $1 \leqslant \max\{|N_{\mathcal{R}(G)}(u) \cap N_{\mathcal{R}(G)}(v_1)| : u \in V(\mathcal{R}(G)) \setminus \{v_1\}\} \leqslant 3$. If $\max\{|N_{\mathcal{R}(G)}(u) \cap N_{\mathcal{R}(G)}(v_1)| : u \in V(\mathcal{R}(G)) \setminus \{v_1\}\} = 1$, then $\mathcal{R}(G) \cong H_4$ by Proposition 12–13 and Lemma 9. If $\max\{|N_{\mathcal{R}(G)}(u) \cap N_{\mathcal{R}(G)}(v_1)| : u \in V(\mathcal{R}(G)) \setminus \{v_1\}\} = 2$, then $\mathcal{R}(G) \cong H_1$ or $\mathcal{R}(G) \cong H_5$ or $\mathcal{R}(G) \cong H_6$ by Theorem 10 (1) and Propositions 12–13. Similarly, if $\max\{|N_{\mathcal{R}(G)}(u) \cap N_{\mathcal{R}(G)}(v_1)| : u \in V(\mathcal{R}(G)) \setminus \{v_1\}\} = 3$, then $\mathcal{R}(G) \cong H_2$. □

Lemma 22. *Suppose G is an extremal of $\mathbf{S}(\pi)$, where $\pi = (d_1, d_2, \ldots, d_n)$ and $d_n = 1$.*

(1) *If $d_2 \geqslant 3$, then $\mathcal{R}(G) \ncong H_4$;.*

(2) *If $d_2 \geqslant 4$ or $d_3 \geqslant 3$, then $\mathcal{R}(G) \ncong H_5$;*

(3) *If $d_2 \geqslant 4$ or $d_4 \geqslant 3$, then $\mathcal{R}(G) \ncong H_6$.*

Proof. (1) Assume that $\mathcal{R}(G) \cong H_4$. Since $d_2 \geqslant 3$, there exists some neighbor, say x, of v_2 such that $x \in V(G) \setminus V(\mathcal{R}(G))$. By Lemma 11, we have $f(v_6) > f(v_2)$, a contradiction.

(2) Assume that $\mathcal{R}(G) \cong H_5$. If $d_3 \geqslant 3$, there exists some neighbor, say x, of v_3 such that $x \in V(G) \setminus V(\mathcal{R}(G))$. By Lemma 11, we have $f(v_6) > f(v_3)$, a contradiction. If $d_2 \geqslant 4$, we will yield a similar contradiction.

(3) Assume that $\mathcal{R}(G) \cong H_6$. If $d_4 \geqslant 3$, by Lemma 11 we have $f(v_5) > f(v_4)$, a contradiction. If $d_2 \geqslant 4$, by Lemma 11, we have $f(v_3) > f(v_2)$, a contradiction. □

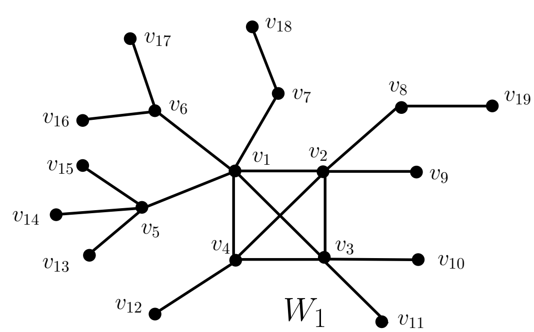

Figure 2: The tricyclic graph W_1.

Let W_1 be the unique tricyclic graph with $\mathcal{R}(W_1) = H_1$, and the remaining vertices appear in BFS-ordering (also called spiral like dispositions in [1, 11]) with respect to H_1 starting from v_5 that is adjacent to v_1. It means that, W_1 can be constructed by the breadth-first-search method as follows: Select a vertex v_1 as a root and begin with v_1 of the zeroth layer. Select the vertices $v_2, \ldots, v_{d_1+1}$ as the first layer such that $v_2, \ldots, v_{d_1+1}$ are adjacent with v_1. Let

$$\begin{aligned}
N(v_2) &= \{v_1, v_3, v_4, v_{d_1+2}, v_{d_1+3}, \ldots, v_{d_1+d_2-2}\},\\
N(v_3) &= \{v_1, v_2, v_4, v_{d_1+d_2-1}, \ldots, v_{d_1+d_2+d_3-5}\},\\
N(v_4) &= \{v_1, v_2, v_3, v_{d_1+d_2+d_3-4}, \ldots, v_{d_1+d_2+d_3+d_4-8}\}, \text{ and}\\
N(v_5) &= \{v_1, v_{d_1+d_2+d_3+d_4-7}, \ldots, v_{d_1+d_2+d_3+d_4+d_5-9}\}, \text{ etc.}
\end{aligned}$$

Informally, for a given tricyclic degree sequence $\pi = (6, 5^{(2)}, 4^{(2)}, 3, 2^{(2)}, 1^{(11)})$, W_1 is the tricyclic graph of order 19 as shown in Figure 2.

Let W_2 (respectively, W_5, W_6) be the unique tricyclic graph with $\mathcal{R}(W_2) = H_2$ (respectively, $\mathcal{R}(W_5) = H_5$, $\mathcal{R}(W_6) = H_6$), and the remaining vertices appear in BFS-ordering with respect to H_2 (respectively, H_5, H_6) starting from v_6 (respectively, v_7, v_6) that is adjacent to v_1. Denote W_3 the unique tricyclic graph with $\mathcal{R}(W_3) = H_3$ so that the remaining vertices appear in BFS-ordering with respect to H_3 starting from v_6 that is adjacent to v_5.

Paths $P_{l_1}, \ldots, P_{l_k}$ are said to have almost equal lengths if $l_1, \ldots, l_k$ satisfy $|l_i - l_j| \leqslant 1$ for $1 \leqslant i \leqslant j \leqslant k$. Let W_4 be the unique tricyclic graph obtained from H_4 by attaching k paths of almost equal lengths to v_1 of H_4, where $d_1 = k + 6$.

Theorem 23. *Suppose G is an extremal of $\mathbf{S}(\pi)$, where $\pi = (d_1, d_2, \ldots, d_n)$ and $d_n = 1$.*

(1) *If $d_1 \geqslant 4$, $d_4 \geqslant 3$, then $G \cong W_1$ or $G \cong W_2$;*

(2) *If $d_1 = 3$ and $d_4 = 3$, then $G \cong W_3$;*

(3) *If $d_2 = d_3 = d_4 = 2$, then $G \cong W_4$;*

(4) *If $d_2 = 3$ and $d_3 = d_4 = 2$, then $G \cong W_5$;*

(5) *If $d_2 = d_3 = 3$ and $d_4 = 2$, then $G \cong W_6$;*

(6) *If $d_2 \geqslant 4$ and $d_4 = 2$, then $G \cong W_2$.*

Proof. Choose $v \in \mathcal{R}(G)$ such that $dist(v_1, v)$ is as large as possible.

(1) By Lemma 20, $vv_1 \in E(G)$. By Lemmas 21–22, we can conclude that $\mathcal{R}(G) \cong H_1$ or $\mathcal{R}(G) \cong H_2$. Thus, either $G \cong W_1$ or $G \cong W_2$ by Theorem 10.

(3) Since G is a tricyclic graph, $d_1 \geqslant 7$. Otherwise, $\sum_{i=1}^{n} d_i \leqslant 6 + 2(n-2) + 1 = 2n+3$, a contradiction. By Lemmas 20–21 and Theorem 10, we have $G \cong W_4$.

(4) Since G is a tricyclic graph, $d_1 \geqslant 6$. Otherwise, $\sum_{i=1}^{n} d_i \leqslant 5 + 3 + 2(n-3) + 1 = 2n + 3$, a contradiction. By Lemmas 20–22, $\mathcal{R}(G) \cong H_5$, and hence $G \cong W_5$ by Theorem 10.

(5) Since G is a tricyclic graph, $d_1 \geqslant 4$. Otherwise, $\sum_{i=1}^{n} d_i \leqslant 3 \times 3 + 2(n-4) + 1 = 2n + 2$, a contradiction. By Lemmas 20–22 and Theorem 10, we have $G \cong W_6$.

(6) By Lemma 20, $vv_1 \in E(G)$, and hence $\mathcal{R}(G) \cong H_2$ by Lemmas 21–22. Now, Theorem 10 implies that $G \cong W_2$.

(2) Since G is a tricyclic graph, $d_1 = d_2 = \cdots = d_5 = 3$ and $d_n = 1$. By Lemma 19, either $v \in A_2$ with $d_{\mathcal{R}(G)}(v) = 2$ or $vv_1 \in E(G)$. If $vv_1 \in E(G)$, then $\mathcal{R}(G) \cong H_1$ by Lemmas 21–22 and hence G is not connected, a contradiction. Thus, $v \in A_2$ with $d_{\mathcal{R}(G)}(v) = 2$, and hence $d_{\mathcal{R}(G)}(v_1) = 3$ and $d_{\mathcal{R}(G)}(v_2) \geqslant d_{\mathcal{R}(G)}(v_3) \geqslant d_{\mathcal{R}(G)}(v_4) \geqslant 2$ by Corollary 14.

By Lemma 19, let x and y be the two neighbors of v of $\mathcal{R}(G)$ in A_1.

If $xy \in E(G)$, then $d_{\mathcal{R}(G)}(z) = 3$, where $z \in \{v_2, v_3, v_4\} \setminus \{x, y\}$ and there exist two vertices, say u and w, such that $u, w \in A_2$ and z, u, w form a triangle, which contradicts Lemma 19.

If $xy \notin E(G)$, by Proposition 15 (1), all the pendant vertices of G are on T_v. So, $d_G(v) = 3$. By Corollary 2, we can conclude that $d_{\mathcal{R}(G)}(v_2) = d_{\mathcal{R}(G)}(v_3) = d_{\mathcal{R}(G)}(v_4) = 3$. Thus, $G \cong W_3$ by Theorem 10, Lemma 19 and Proposition 15 (1). □

4 Further discussion

In view of Theorem 23, it is natural to consider the following question: Whether the construction of G of Theorem 23 (1) is unique? Unfortunately, as the following example shown, the answer is negative.

Example 24. Suppose $p \geqslant q \geqslant 0$ are two integers. Let S_1 and S_2 be the tricyclic graphs as shown in Figure 3. Let $S_1(p, q)$ (respectively, $S_2(p, q)$) be the tricyclic graph obtained from S_1 (respectively, S_2) by attaching p pendant vertices to v_1, and attaching q pendant vertices to v_2. Let G be the extremal graph of $\mathbf{S}(\pi)$, where $\pi = (p+4, q+4, 4^{(2)}, 3, 1^{(p+q+5)})$. Theorem 23 *(1)* implies that either $G \cong S_1(p, q)$ or $G \cong S_2(p, q)$. Using "Matlab", it easily follows that $\rho(S_1(4, 2)) > 3.7363 > 3.6888 > \rho(S_2(4, 2))$, $\rho(S_1(15, 10)) < 4.9168 < 4.9238 < \rho(S_2(15, 10))$, $\mu(S_1(4, 2)) < 9.7373 < 9.7374 < \mu(S_2(4, 2))$, and $\mu(S_1(1, 1)) > 7.8243 > 7.7439 > \mu(S_2(1, 1))$.

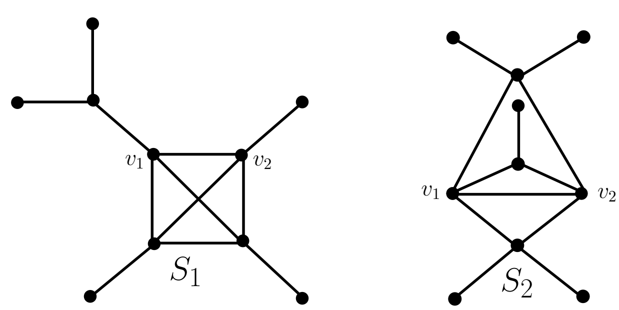

Figure 3: The tricyclic graphs S_1 and S_2.

Now, we present our main result of this section as follows.

Proposition 25. *Let $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ be two different tricyclic degree sequences, and $\pi \lhd \pi'$. Suppose G and G' are the extremal graphs of $\mathbf{S}(\pi)$ and $\mathbf{S}(\pi')$, respectively.*

(1) *If* $d_n \geqslant 2$*, then* $\rho(G) < \rho(G')$ *and* $\mu(G) < \mu(G')$*;*

(2) *If* $d_i = d'_i$ *holds for* $1 \leqslant i \leqslant 4$*, then* $\rho(G) < \rho(G')$ *and* $\mu(G) < \mu(G')$*;*

(3) *Suppose there exists some* t *such that* $d'_t \geqslant 3$ *and* $d_i = d'_i$ *holds for all* $1+t \leqslant i \leqslant n$*. If* $d_1 = d'_1$*, then* $\rho(G) < \rho(G')$ *and* $\mu(G) < \mu(G')$*;*

(4) *Suppose there exists some* t *such that* $d'_t \geqslant 2$ *and* $d_i = d'_i$ *holds for all* $1+t \leqslant i \leqslant n$*. If* $d_1 = d'_1$*,* $d_2 = d'_2$ *and* $d_3 = d'_3$*, then* $\rho(G) < \rho(G')$ *and* $\mu(G) < \mu(G')$*.*

To prove Proposition 25, we need to introduce more lemmas as follows.

Denote $\Phi(G,x)$ the characteristic polynomial of the adjacency matrix of G. The following result is often used to calculate $\Phi(G,x)$ of a graph G.

Lemma 26. [14] (*Schwenk's formulas*) *Let* G *be a graph. Denote by* C_v *the set of all cycles in* G *containing a vertex* v*. Then,*

$$\Phi(G,x) = x\Phi(G-v,x) - \sum_{w\sim v}\Phi(G-v-w,x) - 2\sum_{C\in C_v}\Phi(G-V(C),x).$$

Lemma 27. *If* $n \geqslant 7$*, then* $\rho(F_1) > \rho(F_2)$ *and* $\mu(F_1) > \mu(F_2)$*.*

Proof. We first show that $\rho(F_1) > \rho(F_2)$. Applying Lemma 26 to v_1 of F_1 and F_2, respectively, it follows that

$$\Phi(F_1,x) = (x-1)(x^2-x-4)(x+1)^2\Phi(P_{n-5},x) - 2(x^2-1)^2\Phi(P_{n-6},x) - 2(x^2-1)^2, \tag{2}$$

$$\Phi(F_2,x) = x(x+1)(x^2-x-4)\Phi(P_{n-4},x) - 2x(x^2-2)\Phi(P_{n-5},x) - 2x(x^2-2), \tag{3}$$

Furthermore, Lemma 26 implies that $\Phi(P_n,x) = x\Phi(P_{n-1},x) - \Phi(P_{n-2},x)$. Thus, by equations (2) and (3), we have

$$\Phi(F_2,x) - \Phi(F_1,x) = 2\Phi(P_{n-6},x) + (x^3+x+4)\Phi(P_{n-7},x) + 2(x^4-x^3-2x^2+2x+1).$$

Note that $x^4 - x^3 - 2x^2 + 2x + 1 > 0$ when $x \geqslant \rho(F_1) > \sqrt{d_{F_1}(v_1)} = \sqrt{6}$. Thus, when $x \geqslant \rho(F_1) > 2 > \rho(P_{n-6})$, $\Phi(F_2,x) > \Phi(F_1,x)$, which implies that $\rho(F_1) > \rho(F_2)$.

Now we turn to verify that $\mu(F_1) > \mu(F_2)$. Since F_1 is not bipartite, then $\mu(F_1) > d_{F_1}(v_1) + 1 = 7$ (see e. g. [13]). By the upper bound of [7] for $\mu(G)$, we have

$$\mu(F_2) \leqslant \max\left\{\frac{d(u)(d(u)+m(u)) + d(v)(d(v)+m(v))}{d(u)+d(v)}, uv \in E(F_2)\right\} \leqslant \frac{48}{7} < \mu(F_1),$$

where $m(v) = \sum_{w\in N(v)} d(w)/d(v)$. So, the result follows. □

If $d = (d_1,\ldots,d_n)$ is a non-increasing integer sequence and $d_i \geqslant d_j + 2$, then the following operation is called a unit transformation from i to j on d: subtract 1 from d_i and add 1 to d_j. The following famous lemma on majorization of integer sequences, is due to Muirhead (see [12]).

Lemma 28. (*Muirhead's Lemma*) *If d and d' are two non-increasing integer sequences and $d \lhd d'$, then d can be obtained from d' by a finite sequence of unit transformations.*

Suppose $\pi \lhd \pi'$, G and G' are the extremal graphs of $\Gamma(\pi)$ and $\Gamma(\pi')$, respectively. In the following, by Lemma 28, we may always suppose that π and π' differ only in two positions where the difference is 1, that is, $d_i = d'_i$, $i \neq p, q$, $1 \leqslant p < q \leqslant n$, and $d'_p = d_p + 1$, $d'_q = d_q - 1$. Let f be the Perron vector of G, and let $P_{v_p v_q}$ be a shortest path from v_p to v_q. By the choice of G and $p < q$, $f(v_p) \geqslant f(v_q)$ follows from Theorem 10. In the following, if w is a vertex of G such that $w \in N(v_q) \setminus (N(v_p) \cup \{v_p\})$ and $w \notin V(P_{v_p v_q})$, then we call w a surprising vertex of G. If G contains some surprising vertex, say w, let $G^* = G + v_p w - v_q w$. Then, $G^* \in \Gamma(\pi')$. Since $f(v_p) \geqslant f(v_q)$, Lemma 1 implies that $\rho(G) < \rho(G^*) \leqslant \rho(G')$ and $\mu(G) < \mu(G^*) \leqslant \mu(G')$. Therefore, if G contains a surprising vertex, then $\rho(G) < \rho(G')$ and $\mu(G) < \mu(G')$.

Proof of Proposition 25. It is easy to check that the result follows for $n \leqslant 6$ with the aid of computer. Thus, we may suppose that $n \geqslant 7$ in the following.

(1) If $\pi = (5, 3, 2, \ldots, 2)$ and $\pi' = (6, 2, 2, \ldots, 2)$, then $G = F_2$ and $G' = F_1$ by Theorem 18. Now, the result follows from Lemmas 27. If $\pi = (3, 3, 3, 3, 2, \ldots, 2)$, then $\pi' = (4, 3, 3, 2, \ldots, 2)$. By Theorem 18, $G = F_5$ and $G' = F_4$. Without loss of generality, suppose $f(u_3) \geqslant f(u_4)$. Choose $x \in N(u_4) \setminus \{u_1, u_2\}$. Then, x is a surprising vertex of F_5, and hence $\rho(F_5) < \rho(F_4)$ and $\mu(F_5) < \mu(F_4)$. We can also employ the similar method to deal with the other cases by Theorems 10 and 18.

(2) Since $d_i = d'_i$ holds for $1 \leqslant i \leqslant 4$, $q > p \geqslant 5$ and $d_n = 1$ by Theorem 18.

If $q \geqslant 8$ or $d_7 \geqslant 3$, by Theorem 23 G contains some surprising vertex.

If $q = 7$ and $d_7 = 2$, then $p = 6$ or $p = 5$. When $p = 5$, G contains some surprising vertex according to Theorem 23. When $p = 6$, then $d_5 = d'_5 \geqslant d'_6 = d_6 + 1 \geqslant 3$. By Theorem 23, G contains some surprising vertex.

If $q = 6$, then $p = 5$, and hence $d_4 = d'_4 \geqslant d'_5 = d_5 + 1 \geqslant 3$. By Theorem 23, G contains some surprising vertex.

(3) Note that $d'_t \geqslant 3$ and $d_i = d'_i$ holds for all $1 + t \leqslant i \leqslant n$. Then, $d_q = d'_q + 1 \geqslant d'_t + 1 \geqslant 4$. Since $d_1 = d'_1$, G contains some surprising vertex according to Theorem 18 and Theorem 23.

(4) Note that $d'_t \geqslant 2$ and $d_i = d'_i$ holds for all $1 + t \leqslant i \leqslant n$. Then, $d_q = d'_q + 1 \geqslant d'_t + 1 \geqslant 3$ and $d'_p = d_p + 1 \geqslant d_q + 1 \geqslant 4$. Since $d_1 = d'_1$, $d_2 = d'_2$ and $d_3 = d'_3 \geqslant d'_p \geqslant 4$, G contains some surprising vertex according to Theorem 18 and Theorem 23. □

Finally, we will verify the following majorization theorem to the c-cyclic graphs for $c \geqslant 4$.

Proposition 29. *Let $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ be two different c-cyclic degree sequences, and let G and G' be the extremal c-cyclic graphs of $\Gamma(\pi)$ and $\Gamma(\pi')$, respectively. Suppose $\pi \lhd \pi'$, $d_1 = d'_1$ and $c \geqslant 4$. If there exists some t such that $d'_t \geqslant c - 1$ and $d_i = d'_i$ holds for all $1 + t \leqslant i \leqslant n$, then $\rho(G) < \rho(G')$ and $\mu(G) < \mu(G')$.*

Proof. Note that $d'_t \geqslant c-1$ and $d_i = d'_i$ holds for all $1+t \leqslant i \leqslant n$. Then, $d_q = d'_q + 1 \geqslant d'_t + 1 \geqslant c \geqslant 4$. If $d_n = 2$, then $d'_1 \geqslant d'_p = d_p + 1 \geqslant d_q + 1 \geqslant c+1$. Recall that $d'_q \geqslant c-1 \geqslant 3$. Thus, $2(n+c-1) = \sum_{i=1}^n d'_i \geqslant 2(c+1) + 3 + 2(n-3) = 2n + 2c - 1$, a contradiction. Thus, we may suppose that $d_n = 1$.

If $v_q \in V(G) \setminus V(\mathcal{R}(G))$, then G contains some surprising vertex (since $d_q \geqslant 4$). Thus, we may suppose that $v_q \in V(\mathcal{R}(G))$ in the following. By Proposition 12 and $f(v_p) \geqslant f(v_q)$, it follows that $v_p \in V(\mathcal{R}(G))$. If there exists some $x \in N(v_q)$ such that $x \in V(G) \setminus V(\mathcal{R}(G))$, it is easy to see that x is a surprising vertex of G. If $N(v_q) \subseteq V(\mathcal{R}(G))$, then $d_{\mathcal{R}(G)}(v_1) \geqslant d_{\mathcal{R}(G)}(v_p) \geqslant d_{\mathcal{R}(G)}(v_q) = d_G(v_q) \geqslant c$ by Theorem 10 and Proposition 13. Thus,

$$2(|V(\mathcal{R}(G))| + c - 1) = \sum_{w \in V(\mathcal{R}(G))} d_{\mathcal{R}(G)}(w) \geqslant 3c + 2(|V(\mathcal{R}(G))| - 3), \tag{4}$$

which implies that $c = 4$ by $c \geqslant 4$.

By inequality (4) and $d_1 = d'_1$, we can conclude that $p = 2$, $q = 3$, $d_{\mathcal{R}(G)}(v_1) = d_{\mathcal{R}(G)}(v_2) = d_{\mathcal{R}(G)}(v_3) = 4$ and $d_{\mathcal{R}(G)}(w) = 2$ holds for $w \in V(\mathcal{R}(G)) \setminus \{v_1, v_2, v_3\}$. If there exists some $x \in N_{\mathcal{R}(G)}(v_3) \setminus \{v_1, v_2\}$ such that $x \notin N_{\mathcal{R}(G)}(v_2)$, then x is a surprising vertex of G. Now, we assume that $N_{\mathcal{R}(G)}(v_3) \setminus \{v_1, v_2\} = N_{\mathcal{R}(G)}(v_2) \setminus \{v_1, v_3\}$.

Since $c = 4$ and $d_{\mathcal{R}(G)}(w) = 2$ holds for $w \in V(\mathcal{R}(G)) \setminus \{v_1, v_2, v_3\}$, by Proposition 15, we have $v_2v_3 \in E(G)$ and $v_4v_5 \in E(G)$. Choose $y \in N_{\mathcal{R}(G)}(v_3) \setminus \{v_1, v_2\}$. By Theorem 10 and Proposition 13, $f(v_3) > f(v_5)$ and $f(v_4) \geqslant f(y)$. Let $G^* = G + v_3v_4 + v_5y - v_3y - v_4v_5$. By Corollary 6 (1), $f(v_4) < f(y)$, a contradiction. □

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