On cross-intersecting families of set partitions

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Abstract

Let $\mathcal{B}(n)$ denote the collection of all set partitions of $[n]$. Suppose $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B}(n)$ are cross-intersecting i.e. for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, we have $A_1 \cap A_2 \neq \varnothing$. It is proved that for sufficiently large n,

$$|\mathcal{A}_1||\mathcal{A}_2| \leqslant B_{n-1}^2$$

where B_n is the n-th Bell number. Moreover, equality holds if and only if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{A}_1$ consists of all set partitions with a fixed singleton.

Keywords: cross-intersecting family, Erdős-Ko-Rado, set partitions

1 Introduction

1.1 Finite sets

Let $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A fundamental result in extremal combinatorial set theory is the Erdős-Ko-Rado theorem ([6], [7], [22]) which asserts that if a family $\mathcal{A} \subseteq \binom{[n]}{k}$ is *t-intersecting* (i.e. $|A \cap B| \geqslant t$ for any $A, B \in \mathcal{A}$), then $|\mathcal{A}| \leqslant \binom{n-t}{k-t}$ for $n \geqslant (k-t+1)(t+1)$. Recently, there are several Erdős-Ko-Rado type results (see [2, 4, 5, 9, 11, 13, 15, 17, 20, 21]), most notably is the result of Ellis, Friedgut and Pilpel [5], which states that for sufficiently large n depending on t, a t-intersecting family $\mathcal{A}$ of permutations has size at most $(n-t)!$, with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of t points, thus settling an old conjecture of Deza and Frankl [3].

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Let $\mathcal{A}_i \subseteq \binom{[n]}{k_i}$ for $i = 1, 2, \ldots, r$. We say that the families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ are *r-cross t-intersecting* if $|A_1 \cap A_2 \cap \cdots \cap A_r| \geqslant t$ holds for all $A_i \in \mathcal{A}_i$. When $t = 1$, we will just say *r-cross intersecting* instead of r-cross 1-intersecting. Furthermore when $r = 2$ and $t = 1$, we will just say *cross-intersecting* instead of 2-cross intersecting. It has been shown by Frankl and Tokushige [8] that if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ are r-cross intersecting, then for $n \geqslant rk/(r-1)$,

$$\prod_{i=1}^{r} |\mathcal{A}_i| \leqslant \binom{n-1}{k-1}^r.$$

For differing values of k's, we have the following result.

Theorem 1.1 (Bey [1], Matsumoto and Tokushige [18], Pyber [19])**.** *Let $\mathcal{A}_1 \subseteq \binom{[n]}{k_1}$ and $\mathcal{A}_2 \subseteq \binom{[n]}{k_2}$ be cross-intersecting. If $k_1, k_2 \leqslant n/2$, then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leqslant \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}.$$

Equality holds for $k_1 + k_2 < n$ if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ consist of all k_1-element resp. k_2-element sets containing a fixed element.

1.2 Set partitions

A set partition of $[n]$ is a collection of pairwise disjoint nonempty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let $\mathcal{B}(n)$ denote the family of all set partitions of $[n]$. It is well-known that the size of $\mathcal{B}(n)$ is the n-th Bell number, denoted by B_n. A block of size one is also known as a *singleton*. We denote the number of all set partitions of $[n]$ which are singleton-free (i.e. without any singleton) by $\tilde{B}_n$.

A family $\mathcal{A} \subseteq \mathcal{B}(n)$ is said to be *t-intersecting* if any two of its members have at least t blocks in common. Ku and Renshaw [14, Theorem 1.7 and Theorem 1.8] proved the following analogue of the Erdős-Ko-Rado theorem for set partitions.

Theorem 1.2 (Ku-Renshaw)**.** *Suppose $\mathcal{A} \subseteq \mathcal{B}(n)$ is a t-intersecting family. Then, for $n \geqslant n_0(t)$,*

$$|\mathcal{A}| \leqslant B_{n-t},$$

with equality if and only if $\mathcal{A}$ consists of all set partitions with t fixed singletons.

Recently, Ku and Wong [16, Theorem 1.4] proved a generalization of Theorem 1.2, which is an analogue of the Hilton-Milner Theorem [10] for set partitions.

In this paper, we will prove the following analogue of Theorem 1.1 for set partitions.

Theorem 1.3. *Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B}(n)$ be cross-intersecting. Then, for $n \geqslant n_0$,*

$$|\mathcal{A}_1||\mathcal{A}_2| \leqslant B_{n-1}^2.$$

Moreover, equality holds if and only if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{A}_1$ consists of all set partitions with a fixed singleton.

2 Splitting operation

In this section, we will prove some important results regarding the splitting operation for r-cross t-intersecting families of set partitions. These results are the 'cross' version of [14, Proposition 3.1, 3.2, 3.3, 3.4].

Let $i, j \in [n]$, $i \neq j$, and $P \in \mathcal{B}(n)$. Denote by $P_{[i]}$ the block of P which contains i. We define the (i,j)-*split* of P to be the following set partition:

$$s_{ij}(P) = \begin{cases} P \setminus \{P_{[i]}\} \cup \{\{i\}, P_{[i]} \setminus \{i\}\} & \text{if } j \in P_{[i]}, \\ P & \text{otherwise.} \end{cases}$$

For a family $\mathcal{A} \subseteq \mathcal{B}(n)$, let $s_{ij}(\mathcal{A}) = \{s_{ij}(P) : P \in \mathcal{A}\}$. Any family $\mathcal{A}$ of set partitions can be decomposed with respect to given i, $j \in [n]$ as follows:

$$\mathcal{A} \;=\; (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},$$

where $\mathcal{A}_{ij} = \{P \in \mathcal{A} : s_{ij}(P) \notin \mathcal{A}\}$. Define the (i,j)-*splitting* of $\mathcal{A}$ to be the family

$$S_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup s_{ij}(\mathcal{A}_{ij}).$$

It is not hard to see that $|S_{ij}(\mathcal{A})| = |\mathcal{A}|$.

Let $I(n,r,t)$ denote the set of all r-cross t-intersecting families of set partitions of $[n]$. Let $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\} \in I(n,r,t)$. We set

$$S_{ij}(\mathbf{A}) = \{S_{ij}(\mathcal{A}_1), S_{ij}(\mathcal{A}_2), \ldots, S_{ij}(\mathcal{A}_r)\},$$

and write $S_{ij}(\mathbf{A}) = \mathbf{A}$ if $S_{ij}(\mathcal{A}_l) = \mathcal{A}_l$ for $l = 1, 2, \ldots, r$.

We define $|\mathbf{A}| = \prod_{l=1}^{r} |\mathcal{A}_l|$. It is not hard to see that

$$|\mathbf{A}| = \prod_{i=1}^{r} |S_{ij}(\mathcal{A}_l)|.$$

An element $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\} \in I(n,r,t)$ is said to be *trivial*, if $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_r$ and $\mathcal{A}_1$ consists of all set partitions containing t fixed singletons.

Proposition 2.1. *Let $i, j \in [n]$, $i \neq j$. If $\mathbf{A} \in I(n,r,t)$, then $S_{ij}(\mathbf{A}) \in I(n,r,t)$.*

Proof. Let $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\}$. For each $l = 1, 2, \ldots, r$, choose an $A_l \in S_{ij}(\mathcal{A}_l)$. If $A_l \in \mathcal{A}_l$ for all l, then $|A_1 \cap A_2 \cap \cdots \cap A_r| \geqslant t$. Without loss of generality, suppose $A_l \in s_{ij}((\mathcal{A}_l)_{ij})$ for $l = 1, \ldots, q$, and $A_l \in \mathcal{A}_l \setminus (\mathcal{A}_l)_{ij}$ for $l = q+1, \ldots, r$. Then $A_l = s_{ij}(P_l)$ for $l = 1, \ldots, q$, where $P_l \in (\mathcal{A}_l)_{ij} \subseteq A_l$.

Now there are at least t blocks, say $M_1, M_2, \ldots, M_t$, that are all contained in $P_1 \cap \cdots \cap P_q \cap A_{q+1} \cap \cdots \cap A_r$. If $\{i,j\} \nsubseteq M_y$ for $y = 1, \ldots, t$, then $M_1, M_2, \ldots, M_t \in A_l$ for $l = 1, \ldots, q$. This implies that $M_1, M_2, \ldots, M_t$ are contained in $A_1 \cap \cdots \cap A_q \cap A_{q+1} \cap \cdots \cap A_r$, and thus $|A_1 \cap \cdots \cap A_r| \geqslant t$.

Suppose one of the M_y contains $\{i,j\}$. We may assume that $\{i,j\} \subseteq M_1$. If $q = r$, then $\{i\}$, $M_2, \ldots, M_t$ are contained in $A_1 \cap \cdots \cap A_r$, and thus $|A_1 \cap \cdots \cap A_r| \geqslant t$. Suppose

$1 \leqslant q < r$. Since $A_l \in \mathcal{A}_l \setminus (\mathcal{A}_l)_{ij}$ for $l \geqslant q+1$, we must have $s_{ij}(A_l) \in \mathcal{A}_l$. Note that $M_2, \ldots, M_t$ are contained in $P_1 \cap \cdots \cap P_q \cap s_{ij}(A_{q+1}) \cap \cdots \cap s_{ij}(A_r)$. Since $|P_1 \cap \cdots \cap P_q \cap s_{ij}(A_{q+1}) \cap \cdots \cap s_{ij}(A_r)| \geqslant t$, there is a block M_{t+1} disjoint from $M_1, M_2, \ldots, M_t$, that is contained in $P_1 \cap \cdots \cap P_q \cap s_{ij}(A_{q+1}) \cap \cdots \cap s_{ij}(A_r)$. Now M_{t+1} is a block in $A_1 \cap \cdots \cap A_q \cap A_{q+1} \cap \cdots \cap A_r$, for $\{i, j\} \nsubseteq M_{t+1}$. Hence $|A_1 \cap \cdots \cap A_r| \geqslant t$. □

Proposition 2.2. *Let $n \geqslant t+1$. Suppose $\mathbf{A} \in I(n, r, t)$ and $|\mathbf{A}| > 1$. Let $i, j \in [n]$, $i \neq j$. If $S_{ij}(\mathbf{A})$ is trivial, then $\mathbf{A}$ is trivial.*

Proof. Let $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\}$. Then $S_{ij}(\mathbf{A}) = \{S_{ij}(\mathcal{A}_1), S_{ij}(\mathcal{A}_2), \ldots, S_{ij}(\mathcal{A}_r)\}$ and by Proposition 2.1, $S_{ij}(\mathbf{A}) \in I(n, r, t)$. Since $S_{ij}(\mathbf{A})$ is trivial, $S_{ij}(\mathcal{A}_1) = S_{ij}(\mathcal{A}_2) = \cdots = S_{ij}(\mathcal{A}_r)$ and $S_{ij}(\mathcal{A}_1)$ consists of all set partitions containing t fixed singletons, say $\{x_1\}$, $\{x_2\}$, …, $\{x_t\}$. Note that $T = \{\{x_1\}, \{x_2\}, \ldots, \{x_t\}, [n] \setminus \{x_1, \ldots, x_t\}\} \in S_{ij}(\mathcal{A}_1)$. If $T \in s_{ij}((\mathcal{A}_1)_{ij})$, then $T = s_{ij}(P)$ for a $P \in (\mathcal{A}_1)_{ij} \subseteq \mathcal{A}_1$. Note that P will have exactly t blocks. Now, if $Q_l \in \mathcal{A}_l$ for $l = 2, \ldots, r$, then $P = Q_2 = \cdots = Q_r$, for $|P \cap Q_2 \cap \cdots \cap Q_r| \geqslant t$. Therefore $\mathcal{A}_2 = \mathcal{A}_3 = \cdots = \mathcal{A}_r = \{P\}$, and this implies that $\mathcal{A}_1 = \{P\}$. So $|\mathbf{A}| = \prod_{l=1}^{r} |\mathcal{A}_l| = 1$, a contradiction. So we may assume that $T \in \mathcal{A}_1 \setminus (\mathcal{A}_1)_{ij} \subseteq \mathcal{A}_1$. Similarly, $T \in \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_r$.

Suppose $\mathcal{A}_1 \neq S_{ij}(\mathcal{A}_1)$. Then there is a $P \in \mathcal{A}_1$ with $s_{ij}(P) \notin \mathcal{A}_1$. Now

$$|P \cap \overbrace{T \cap \cdots \cap T}^{r-1}| \geqslant t,$$

for $T \in \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_r$. Suppose $[n] \setminus \{x_1, \ldots, x_t\}$ is a block in P. Since T has exactly $t+1$ blocks, we deduce that $P = T$. This means that $T \in (\mathcal{A}_1)_{ij}$, and $s_{ij}(T) \in S_{ij}(\mathcal{A}_1)$. So $T \notin S_{ij}(\mathcal{A}_1)$, a contradiction.

Suppose $[n] \setminus \{x_1, \ldots, x_t\}$ is not a block in P. Then $\{x_1\}, \{x_2\}, \ldots, \{x_t\}$ are blocks in P. This implies that $P \in S_{ij}(\mathcal{A}_1)$, for $S_{ij}(\mathbf{A})$ is trivial. Since $P \in \mathcal{A}_1$, we must have $s_{ij}(P) \in \mathcal{A}_1$, a contradiction. Hence $\mathcal{A}_1 = S_{ij}(\mathcal{A}_1)$. Similarly $\mathcal{A}_l = S_{ij}(\mathcal{A}_l)$ for $l = 2, \ldots, r$. □

An element $\mathbf{A} \in I(n, r, t)$ is said to be *compressed* if for any $i, j \in [n]$, $i \neq j$, we have $S_{ij}(\mathbf{A}) = \mathbf{A}$. For a set partition P, let $\sigma(P) = \{x : \{x\} \in P\}$ denote the union of its singletons (block of size 1). For a family $\mathcal{A}$ of set partitions, let $\sigma(\mathcal{A}) = \{\sigma(P) : P \in \mathcal{A}\}$. Note that $\sigma(\mathcal{A})$ is a family of subsets of $[n]$. Now for $\mathbf{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, where $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \mathcal{B}(n)$, set $\sigma(\mathbf{A}) = \{\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_r)\}$. We say $\sigma(\mathbf{A})$ is r-cross t-intersecting if $\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_r)$ are r-cross t-intersecting.

Proposition 2.3. *Given an element $\mathbf{A} \in I(n, r, t)$, by repeatedly applying the splitting operations, we eventually obtain a compressed $\mathbf{A}^* \in I(n, r, t)$ with $|\mathbf{A}^*| = |\mathbf{A}|$.*

Proof. Note that if $S_{ij}(\mathbf{A}) \neq \mathbf{A}$, then the (i, j)-splits of some partitions are finer than the originals and therefore will move down in the partition lattice. Eventually this results in a compressed family of partitions. □

For a compressed $\mathbf{A}$, its r-cross t-intersecting property can be transferred to $\sigma(\mathbf{A})$, thus allowing us to access the structure of $\mathbf{A}$ via the structure of $\sigma(\mathbf{A})$.

Proposition 2.4. *If $\mathbf{A} \in I(n,r,t)$ is compressed, then $\sigma(\mathbf{A})$ is r-cross t-intersecting.*

Proof. Let $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\}$. Assume, for a contradiction, that there exist $P_l \in \mathcal{A}_l$, $l = 1, \ldots, r$ such that $|\sigma(P_1) \cap \cdots \cap \sigma(P_l)| < t$. Since $|P_1 \cap \cdots \cap P_r| \geqslant t$, there are $s \geqslant t - |\sigma(P_1) \cap \cdots \cap \sigma(P_l)|$ common blocks of $P_1, \ldots, P_r$ (each of size at least 2), say $M_1, \ldots, M_s$, which are disjoint from $\sigma(P_1) \cup \cdots \cup \sigma(P_r)$. Fix two distinct points x_e, y_e from each M_e. Then $P_1^* = s_{x_s y_s}(\cdots(s_{x_1 y_1}(P_1))\cdots) \in \mathcal{A}_1$, for $\mathbf{A}$ is compressed. Now $|P_1^* \cap P_2 \cap \cdots \cap P_r| < t$, a contradiction. □

3 Proof of main result

Recall that the size of $\mathcal{B}(n)$ is the n-th Bell number, denoted by B_n, and the number of all set partitions of $[n]$ which are singleton-free (i.e. without any singleton) is denoted by $\tilde{B}_n$.

The following identities for B_n and $\tilde{B}_n$ are straightforward.

Lemma 3.1. *Let $n \geqslant 2$. Then*

$$B_n = \sum_{k=0}^{n} \binom{n}{k} \tilde{B}_{n-k}, \tag{1}$$

$$\tilde{B}_n = \sum_{k=1}^{n-1} \binom{n-1}{k} \tilde{B}_{n-1-k}, \tag{2}$$

with the conventions $B_0 = \tilde{B}_0 = 1$.

Note in passing that $\tilde{B}_1 = 0$. By (1) and (2),

$$B_n = \tilde{B}_n + \tilde{B}_{n+1}. \tag{3}$$

Given a real number x, we shall denote the greatest integer less than or equal to x, by $\lfloor x \rfloor$. Note that $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$. Some useful inequalities involving B_n can be found in [12]. However we just need the following inequality.

Lemma 3.2. *There is a positive integer n_0 such that for $n \geqslant n_0$,*

$$\tilde{B}_{n-1} > 8^n \sum_{\lfloor \frac{n}{2} \rfloor \leqslant k \leqslant n} \binom{n}{k} \tilde{B}_{n-k}.$$

Proof. By (2),

$$\begin{aligned} \sum_{\lfloor \frac{n}{2} \rfloor \leqslant k \leqslant n} \binom{n}{k} \tilde{B}_{n-k} &\leqslant \tilde{B}_{n-\lfloor \frac{n}{2} \rfloor + 2} \sum_{\lfloor \frac{n}{2} \rfloor \leqslant k \leqslant n} \binom{n}{k} \\ &\leqslant 2^n \tilde{B}_{n-\lfloor \frac{n}{2} \rfloor + 2}. \end{aligned}$$

So it is sufficient to show that $\tilde{B}_{n-1}/\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+2} > (16)^n$.

Again by (2), for any fixed q, $\tilde{B}_m/\tilde{B}_{m-2} > q$ for sufficiently large m. Therefore

$$\begin{aligned}\frac{\tilde{B}_{n-1}}{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+2}} &\geqslant \left(\frac{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+2u}}{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+2u-2}}\right)\cdots\left(\frac{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+6}}{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+4}}\right)\left(\frac{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+4}}{\tilde{B}_{n-\lfloor\frac{n}{2}\rfloor+2}}\right)\\ &> q^{u-1},\end{aligned}$$

where $u = \lfloor\frac{1}{2}(\lfloor\frac{n}{2}\rfloor - 3)\rfloor$. Clearly $u-1 \geqslant \frac{n}{8}$. So if we choose $q = (16)^8$, then for sufficiently large n, the lemma follows. □

Let $\mathbf{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_r\} \in I(n, r, t)$ be compressed. We say $\sigma(\mathbf{A})$ is *trivial* if there is a fixed t-set, say T, such that $T \subseteq \sigma(P_l)$ for all $P_l \in \mathcal{A}_l$, $l = 1, \ldots, r$.

Theorem 3.3. *Let $\mathbf{A} \in I(n, 2, 1)$ be compressed. If $\sigma(\mathbf{A})$ is non-trivial, then*

$$|\mathbf{A}| < B_{n-1}^2.$$

Proof. Let $\mathbf{A} = \{\mathcal{A}_1, \mathcal{A}_2\}$. For $k \geqslant 1$, let $\mathcal{F}_{lk} = \sigma(\mathcal{A}_l) \cap \binom{[n]}{k}$. If $\mathcal{F}_{l1} \neq \varnothing$ for $l = 1, 2$, then $\sigma(\mathbf{A})$ is trivial. So we may assume that $\mathcal{F}_{21} = \varnothing$. By Proposition 2.4, $\sigma(\mathbf{A})$ is cross-intersecting. Note that $|\mathcal{A}_1| \leqslant \sum_{1\leqslant k\leqslant n} |F_{1k}|\tilde{B}_{n-k}$ and $|\mathcal{A}_2| \leqslant \sum_{2\leqslant k\leqslant n} |F_{2k}|\tilde{B}_{n-k}$. Then

$$\begin{aligned}|\mathcal{A}_1| &\leqslant \sum_{1\leqslant k<\lfloor\frac{n}{2}\rfloor} |F_{1k}|\tilde{B}_{n-k} + \sum_{\lfloor\frac{n}{2}\rfloor\leqslant k\leqslant n} |F_{1k}|\tilde{B}_{n-k}\\ &\leqslant \sum_{1\leqslant k<\lfloor\frac{n}{2}\rfloor} |F_{1k}|\tilde{B}_{n-k} + \sum_{\lfloor\frac{n}{2}\rfloor\leqslant k\leqslant n} \binom{n}{k}\tilde{B}_{n-k},\end{aligned}$$

and

$$|\mathcal{A}_2| \leqslant \sum_{2\leqslant k<\lfloor\frac{n}{2}\rfloor} |F_{2k}|\tilde{B}_{n-k} + \sum_{\lfloor\frac{n}{2}\rfloor\leqslant k\leqslant n} \binom{n}{k}\tilde{B}_{n-k}.$$

Let

$$\begin{aligned}Q &= \sum_{\lfloor\frac{n}{2}\rfloor\leqslant k\leqslant n} \binom{n}{k}\tilde{B}_{n-k}\\ M_1 &= \sum_{1\leqslant k<\lfloor\frac{n}{2}\rfloor} |F_{1k}|\tilde{B}_{n-k}\\ M_2 &= \sum_{2\leqslant k<\lfloor\frac{n}{2}\rfloor} |F_{2k}|\tilde{B}_{n-k}.\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{A}| &\leqslant (M_1 + Q)(M_2 + Q)\\ &= M_1M_2 + M_1Q + M_2Q + Q^2.\end{aligned}$$

Note that by (2) and (3),

$$\begin{aligned}
M_l &\leqslant \tilde{B}_n \sum_{1\leqslant k<\lfloor \frac{n}{2}\rfloor} |F_{lk}| \\
&\leqslant \tilde{B}_n \sum_{1\leqslant k<\lfloor \frac{n}{2}\rfloor} \binom{n}{k} \\
&\leqslant 2^n \tilde{B}_n \\
&\leqslant 2^n B_{n-1}.
\end{aligned}$$

By Lemma 3.2 and (3), $Q \leqslant \frac{1}{8^n}\tilde{B}_{n-1} < \frac{1}{8^n}B_{n-1} < B_{n-1}$. Therefore

$$\begin{aligned}
M_1Q + M_2Q + Q^2 &< (2^n + 2^n + 1)B_{n-1}\left(\frac{\tilde{B}_{n-1}}{8^n}\right) \\
&< \frac{1}{2}B_{n-1}\tilde{B}_{n-1}.
\end{aligned}$$

By Theorem 1.1,

$$\begin{aligned}
M_1M_2 &\leqslant \sum_{\substack{1\leqslant k_1<\lfloor \frac{n}{2}\rfloor,\\ 2\leqslant k_2<\lfloor \frac{n}{2}\rfloor}} \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}\tilde{B}_{n-k_1}\tilde{B}_{n-k_2} \\
&= \left(\sum_{1\leqslant k<\lfloor \frac{n}{2}\rfloor} \binom{n-1}{k-1}\tilde{B}_{n-k}\right)\left(\sum_{2\leqslant k<\lfloor \frac{n}{2}\rfloor} \binom{n-1}{k-1}\tilde{B}_{n-k}\right).
\end{aligned}$$

By (1),

$$\sum_{1\leqslant k<\lfloor \frac{n}{2}\rfloor} \binom{n-1}{k-1}\tilde{B}_{n-k} < B_{n-1}$$

$$\sum_{2\leqslant k<\lfloor \frac{n}{2}\rfloor} \binom{n-1}{k-1}\tilde{B}_{n-k} < B_{n-1} - \tilde{B}_{n-1}.$$

So $M_1M_2 \leqslant (B_{n-1} - \tilde{B}_{n-1})B_{n-1}$, and

$$\begin{aligned}
|\mathbf{A}| &< B_{n-1}^2 - \tilde{B}_{n-1}B_{n-1} + \frac{1}{2}(B_{n-1})\tilde{B}_{n-1} \\
&< B_{n-1}^2.
\end{aligned}$$

□

Proof of Theorem 1.3.

Let $\mathbf{A} \in I(n,2,1)$ be of maximum size. We may assume that $\mathbf{A}$ has size at least B_{n-1}^2. Repeatedly apply the splitting operations until we obtain an $\mathbf{A}^* \in I(n,2,1)$ such that

$\mathbf{A}^*$ is compressed (Proposition 2.3). By Proposition 2.4, $\sigma(\mathbf{A}^*)$ is cross-intersecting. If $\sigma(\mathbf{A}^*)$ non-trivial, by Theorem 3.3, $|\mathbf{A}| < B_{n-1}^2$, a contradiction. So $\sigma(\mathbf{A}^*)$ is trivial. This implies that $\mathbf{A}^*$ is trivial, for $\mathbf{A}$ is of maximum size. It then follows from Proposition 2.2 that $\mathbf{A}$ is trivial.

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