Rainbow Connection of Sparse Random Graphs

Alan Frieze* Charalampos E. Tsourakakis†

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA15213, U.S.A.

alan@random.math.cmu.edu, ctsourak@math.cmu.edu

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Abstract

An edge colored graph G is rainbow edge connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connectivity of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

In this work we study the rainbow connectivity of binomial random graphs at the connectivity threshold $p = \frac{\log n+\omega}{n}$ where $\omega = \omega(n) \to \infty$ and $\omega = o(\log n)$ and of random r-regular graphs where $r \geqslant 3$ is a fixed integer. Specifically, we prove that the rainbow connectivity $rc(G)$ of $G = G(n,p)$ satisfies $rc(G) \sim \max\{Z_1, \mathrm{diam}(G)\}$ with high probability (*whp*). Here Z_1 is the number of vertices in G whose degree equals 1 and the diameter of G is asymptotically equal to $\frac{\log n}{\log\log n}$ *whp*. Finally, we prove that the rainbow connectivity $rc(G)$ of the random r-regular graph $G = G(n,r)$ *whp* satisfies $rc(G) = O(\log^{2\theta_r} n)$ where $\theta_r = \frac{\log(r-1)}{\log(r-2)}$ when $r \geqslant 4$ and $rc(G) = O(\log^4 n)$ *whp* when $r = 3$.

1 Introduction

Connectivity is a fundamental graph theoretic property. Recently, the concept of *rainbow connectivity* was introduced by Chartrand et al. in [7]. An edge colored graph G is rainbow edge connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connectivity $rc(G)$ of a connected graph G is the smallest number of colors that are needed in order to make G rainbow edge connected. Notice,

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that by definition a rainbow edge connected graph is also connected and furthermore any connected graph has a trivial edge coloring that makes it rainbow edge connected, since one may color the edges of a given spanning tree with distinct colors. Other basic facts established in [7] are that $rc(G) = 1$ if and only if G is a clique and $rc(G) = |V(G)| - 1$ if and only if G is a tree. Besides its theoretical interest, rainbow connectivity is also of interest in applied settings, such as securing sensitive information [13], transfer and networking [5].

The concept of rainbow connectivity has attracted the interest of various researchers. Chartrand et al. [7] determine the rainbow connectivity of several special classes of graphs, including multipartite graphs. Caro et al. [4] prove that for a connected graph G with n vertices and minimum degree δ, the rainbow connectivity satisfies $rc(G) \leqslant \frac{\log \delta}{\delta} n(1+f(\delta))$, where $f(\delta)$ tends to zero as δ increases. The following simpler bound was also proved in [4], $rc(G) \leqslant n\frac{4\log n+3}{\delta}$. Krivelevich and Yuster [12] removed the logarithmic factor from the Caro et al. [4] upper bound. Specifically they proved that $rc(G) \leqslant \frac{20n}{\delta}$. Due to a construction of a graph with minimum degree δ and diameter $\frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$ by Caro et al. [4], the best upper bound one can hope for is $rc(G) \leqslant \frac{3n}{\delta}$. Chandran, Das, Rajendraprasad and Varma [6] have subsequently proved an upper bound of $\frac{3n}{\delta+1} + 3$, which is therefore essentially optimal.

As Caro et al. point out, the random graph setting poses several intriguing questions. Specifically, let $G = G(n,p)$ denote the binomial random graph on n vertices with edge probability p [8]. Caro et al. [4] proved that $p = \sqrt{\log n/n}$ is the sharp threshold for the property $rc(G(n,p)) \leqslant 2$. He and Liang [9] studied further the rainbow connectivity of random graphs. Specifically, they obtain the sharp threshold for the property $rc(G) \leqslant d$ where d is constant. For further results and references we refer the interested reader to the recent survey of Li and Sun [13]. In this work we look at the rainbow connectivity of the binomial graph at the connectivity threshold $p = \frac{\log n+\omega}{n}$ where $\omega = o(\log n)$. This range of values for p poses problems that cannot be tackled with the techniques developed in the aforementioned work. Rainbow connectivity has not been studied in random regular graphs to the best of our knowledge.

Let

$$L = \frac{\log n}{\log \log n} \tag{1}$$

and let $A \sim B$ denote $A = (1 + o(1))B$ as $n \to \infty$.
We establish the following theorems:

Theorem 1 *Let $G = G(n,p), p = \frac{\log n+\omega}{n}$, $\omega \to \infty, \omega = o(\log n)$. Also, let Z_1 be the number of vertices of degree 1 in G. Then, with high probability (whp)*[1]

$$rc(G) \sim \max\{Z_1, L\},$$

It is known that *whp* the diameter of $G(n,p)$ is asymptotic to L for p as in the above range, see for example Theorem 10.17 of Bollobás [2]. Theorem 1 gives asymptotically optimal results. Our next theorem is not quite as precise.

[1] An event A_n holds with high probability (*whp*) if $\lim_{n\to+\infty} \mathbf{Pr}[A_n] = 1$.

Theorem 2 *Let $G = G(n,r)$ be a random r-regular graph where $r \geqslant 3$ is a fixed integer. Then, whp*

$$rc(G) = \begin{cases} O(\log^4 n) & r = 3 \\ O(\log^{2\theta_r} n) & r \geqslant 4. \end{cases}$$

where $\theta_r = \frac{\log(r-1)}{\log(r-2)}$.

All logarithms whose base is omitted are natural. It will be clear from our proofs that the colorings in the above two theorems can be constructed in a low order polynomial time. The second theorem, while weaker, contains an unexpected use of a Markov Chain Monte-Carlo (MCMC) algorithm for randomly coloring a graph.

The paper is organized as follows: After giving a sketch of our approach in Section 2, in Sections 3, 4 we prove Theorems 1, 2 respectively. Finally, in Section 5 we conclude by suggesting open problems.

2 Sketch of approach

The general idea in the proofs of both theorems is as follows:

(a) Randomly color the edges of the graph in question. For Theorem 1 we can (in the main) use a uniformly random coloring. The distribution for Theorem 2 is a little more complicated.

(b) To prove that this works, we have to find, for each pair of vertices x, y, a large collection of edge disjoint paths joining them. It will then be easy to argue that at least one of these paths is rainbow colored.

(c) To find these paths we pick a typical vertex x. We grow a regular tree T_x with root x. The depth is chosen carefully. We argue that for a typical pair of vertices x, y, many of the leaves of T_x and T_y can be put into 1-1 correspondence f so that (i) the path P_x from x to leaf v of T_x is rainbow colored, (ii) the path P_y from y to the leaf $f(v)$ of T_y is rainbow colored and (iii) P_x, P_y do not share color.

(d) We argue that from most of the leaves of T_x, T_y we can grow a tree of depth approximately equal to half the diameter. These latter trees themselves contain a bit more than $n^{1/2}$ leaves. These can be constructed so that they are vertex disjoint. Now we argue that each pair of trees, one associated with x and one associated with y, are joined by an edge.

(e) We now have, by construction, a large set of edge disjoint paths joining leaves v of T_x to leaves $f(v)$ of T_y. A simple estimation shows that *whp* for at least one leaf v of T_x, the path from v to $f(v)$ is rainbow colored and does not use a color already used in the path from x to v in T_x or the path from y to $f(v)$ in T_y.

We now fill in the details of both cases.

3 Proof of Theorem 1

Observe first that $rc(G) \geqslant \max\{Z_1, diameter(G)\}$. First of all, each edge incident to a vertex of degree one must have a distinct color. Just consider a path joining two such vertices. Secondly, if the shortest distance between two vertices is ℓ then we need at least ℓ colors. Next observe that *whp* the diameter D is asymptotically equal to L, see for example [2]. We break the proof of Theorem 1 into several lemmas.

Let a vertex be *large* if $\deg(x) \geqslant \log n/100$ and *small* otherwise.

Lemma 1 Whp*, there do not exist two small vertices within distance at most* $3L/4$.

Proof

$$\begin{aligned}
&\mathbf{Pr}\left[\exists x,y\in[n]:\ \deg(x),\deg(y)\leqslant \log n/100 \text{ and } dist(x,y)\leqslant \frac{3L}{4}\right]\\
&\leqslant \binom{n}{2}\sum_{k=1}^{3L/4} n^{k-1}p^k\left(\sum_{i=0}^{\log n/100}\binom{n-1-k}{i}p^i(1-p)^{n-1-k}\right)^2\\
&\leqslant \sum_{k=1}^{3L/4} n(2\log n)^k\left(2\binom{n}{\log n/100}p^{\log n/100}(1-p)^{n-1-\log n/100}\right)^2\\
&\leqslant \sum_{k=1}^{3L/4} n(2\log n)^k\left(2(100e^{1+o(1)})^{\log n/100}n^{-1+o(1)}\right)^2\\
&\leqslant \sum_{k=1}^{3L/4} n(2\log n)^k n^{-1.9}\\
&\leqslant 2n(2\log n)^{3L/4}n^{-1.9}\\
&\leqslant n^{-.1}.
\end{aligned}$$

□

We use the notation $e[S]$ for the number of edges induced by a given set of vertices S. Notice that if a set S satisfies $e[S] \geqslant s+t$ where $t \geqslant 1$, the induced subgraph $G[S]$ has at least $t+1$ cycles.

Lemma 2 *Fix $t \in \mathbb{Z}^+$ and $0 < \alpha < 1$. Then, whp there does not exist a subset $S \subseteq [n]$, such that $|S| \leqslant \alpha tL$ and $e[S] \geqslant |S| + t$.*

Proof For convenience, let $s = |S|$ be the cardinality of the set S.Then,

$$\begin{aligned}
\mathbf{Pr}\left[\exists S : s\leqslant \alpha tL \ \text{ and } \ e[S]\geqslant s+t\right] &\leqslant \sum_{s\leqslant \alpha tL}\binom{n}{s}\binom{\binom{s}{2}}{s+t}p^{s+t}\\
&\leqslant \sum_{s\leqslant \alpha tL}\left(\frac{ne}{s}\right)^s\left(\frac{es^2p}{2(s+t)}\right)^{s+t}
\end{aligned}$$

$$\leqslant \sum_{s\leqslant \alpha tL} (e^{2+o(1)}\log n)^s \left(\frac{es\log n}{n}\right)^t$$
$$\leqslant \alpha tL\left((e^{2+o(1)}\log n)^{\alpha L}\left(\frac{e\alpha t\log^2 n}{n\log\log n}\right)\right)^t$$
$$< \frac{1}{n^{(1-\alpha-o(1))t}}.$$

□

Remark 1 *Let T be a rooted tree of depth at most $4L/7$ and let v be a vertex not in T, but with b neighbors in T. Let S consist of v, the neighbors of v in T plus the ancestors of these neighbors. Then $|S| \leqslant 4bL/7+1 \leqslant 3bL/5$ and $e(S) = |S|+b-2$. It follows from the proof of Lemma 2 with $\alpha = 3/5$ and $t = 8$, that we must have $b \leqslant 10$ with probability $1-o(n^{-3})$.*

Our next lemma shows the existence of the subgraph $G'_{x,y}$ described next and shown in Figure 1 for a given pair of vertices x, y. We first deal with paths between large vertices. Now let

$$\epsilon = \epsilon(n) = o(1) \text{ be such that } \frac{\epsilon\log\log n}{\log 1/\epsilon} \to \infty \text{ and let } k = \epsilon L. \tag{2}$$

Here L is defined in (1) and we could take $\epsilon = 1/(\log\log n)^{1/2}$.

Lemma 3 *Whp, for all pairs of large vertices $x, y \in [n]$ there exists a subgraph $G_{x,y}(V_{x,y}, E_{x,y})$ of G as shown in Figure 1. The subgraph consists of two isomorphic vertex disjoint trees T_x, T_y rooted at x, y each of depth k. T_x and T_y both have a branching factor of $\log n/101$. I.e. each vertex of T_x, T_y has at least $\log n/101$ neighbors, excluding*

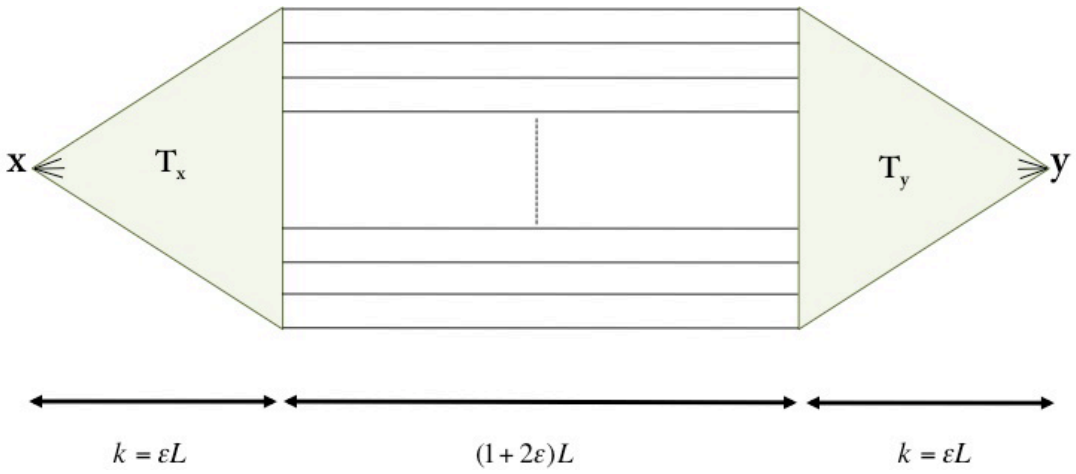

Figure 1: Structure of Lemma 3.

its parent in the tree. Let the leaves of T_x be $x_1, x_2, \ldots, x_\tau$ where $\tau \geqslant n^{4\epsilon/5}$ and those of T_y be $y_1, y_2, \ldots, y_\tau$. Then $y_i = f(x_i)$ where f is a natural isomporphism that preserves the parent-child relation. Between each pair of leaves $(x_i, y_i), i = 1, 2, \ldots, \tau$ there is a path P_i of length $(1 + 2\epsilon)L$. The paths $P_i, i = 1, 2, \ldots, \tau$ are edge disjoint.

Proof Because we have to do this for all pairs x, y, we note without further comment that likely (resp. unlikely) events will be shown to occur with probability $1 - o(n^{-2})$ (resp. $o(n^{-2})$).

To find the subgraph shown in Figure 1 we grow tree structures as shown in Figure 2. Specifically, we first grow a tree from x using BFS until it reaches depth k. Then, we grow a tree starting from y again using BFS until it reaches depth k. Finally, we grow trees from the leaves of T_x and T_y using BFS for depth $\gamma = (\frac{1}{2} + \epsilon)L$. Now we analyze these processes. Since the argument is the same we explain it in detail for T_x and we outline the differences for the other trees. We use the notation $D_i^{(\rho)}$ for the number of vertices at depth i of the BFS tree rooted at ρ.

First we grow T_x. As we grow the tree via BFS from a vertex v at depth i to vertices at depth $i + 1$ certain *bad* edges from v may point to vertices already in T_x. Remark 1 shows with probability $1 - o(n^{-3})$ there can be at most 10 bad edges emanating from v.

Furthermore, Lemma 1 implies that there exists at most one vertex of degree less than $\frac{\log n}{100}$ at each level *whp*. Hence, we obtain the recursion

$$D_{i+1}^{(x)} \geqslant \left(\frac{\log n}{100} - 10\right)(D_i^{(x)} - 1) \geqslant \frac{\log n}{101} D_i^{(x)}. \tag{3}$$

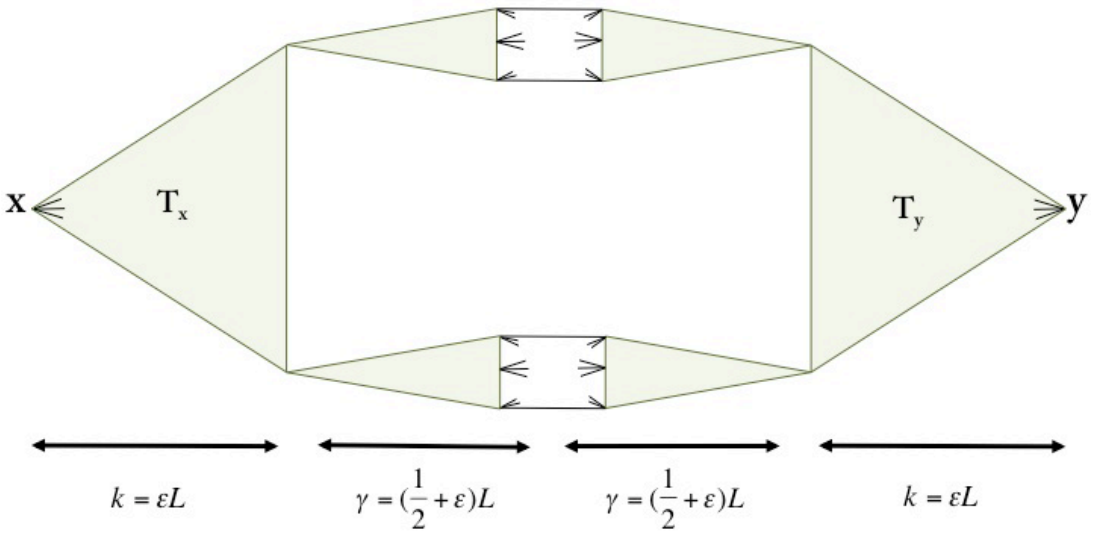

Figure 2: Subgraph found in the proof of Lemma 3.

Therefore the number of leaves satisfies

$$D_k^{(x)} \geqslant \left(\frac{\log n}{101}\right)^{\epsilon L} \geqslant n^{4\epsilon/5}. \tag{4}$$

We can make the branching factor exactly $\frac{\log n}{101}$ by pruning. We do this so that the trees T_x are isomorphic to each other.

With a similar argument

$$D_k^{(y)} \geqslant n^{\frac{4}{5}\epsilon}. \tag{5}$$

The only difference is that now we also say an edge is bad if the other endpoint is in T_x. This immediately gives

$$D_{i+1}^{(y)} \geqslant \left(\frac{\log n}{100} - 20\right)(D_i^{(y)} - 1) \geqslant \frac{\log n}{101} D_i^{(y)}$$

and the required conclusion (5).

Similarly, from each leaf $x_i \in T_x$ and $y_i \in T_y$ we grow trees $\widehat{T}_{x_i}, \widehat{T}_{y_i}$ of depth $\gamma = \left(\frac{1}{2}+\epsilon\right)L$ using the same procedure and arguments as above. Remark 1 implies that there are at most 20 edges from the vertex v being explored to vertices in any of the trees already constructed. At most 10 to T_x plus any trees rooted at an x_i and another 10 for y. The numbers of leaves of each $\widehat{T}_{x_i}$ now satisfies

$$\widehat{D}_\gamma^{(x_i)} \geqslant \frac{\log n}{100}\left(\frac{\log n}{101}\right)^\gamma \geqslant n^{\frac{1}{2}+\frac{4}{5}\epsilon}.$$

Similarly for $\widehat{D}_\gamma^{(y_i)}$.

Observe next that BFS does not condition the edges between the leaves X_i, Y_i of the trees $\widehat{T}_{x_i}$ and $\widehat{T}_{y_i}$. I.e., we do not need to look at these edges in order to carry out our construction. On the other hand we have conditioned on the occurence of certain events to imply a certain growth rate. We handle this technicality as follows. We go through the above construction and halt if ever we find that we cannot expand by the required amount. Let $\mathbf{A}$ be the event that we do not halt the construction i.e. we fail the conditions of Lemmas 1 or 2. We have $\mathbf{Pr}[\mathbf{A}] = 1 - o(1)$ and so,

$$\mathbf{Pr}[\exists i : e(X_i, Y_i) = 0 \mid \mathbf{A}] \leqslant \frac{\mathbf{Pr}[\exists i : e(X_i, Y_i) = 0]}{\mathbf{Pr}(\mathbf{A})} \leqslant 2n^{\frac{4\epsilon}{5}}(1-p)^{n^{1+\frac{8\epsilon}{5}}} \leqslant n^{-n^\epsilon}.$$

We conclude that *whp* there is always an edge between each X_i, Y_i and thus a path of length at most $(1+2\epsilon)L$ between each x_i, y_i. □

Let $q = (1+5\epsilon)L$ be the number of available colors. We color the edges of G randomly. We show that the probability of having a rainbow path between x, y in the subgraph $G_{x,y}$ of Figure 1 is at least $1 - \frac{1}{n^3}$.

Lemma 4 *Color each edge of G using one color at random from q available. Then, the probability of having at least one rainbow path between two fixed large vertices $x, y \in [n]$ is at least $1 - \frac{1}{n^3}$.*

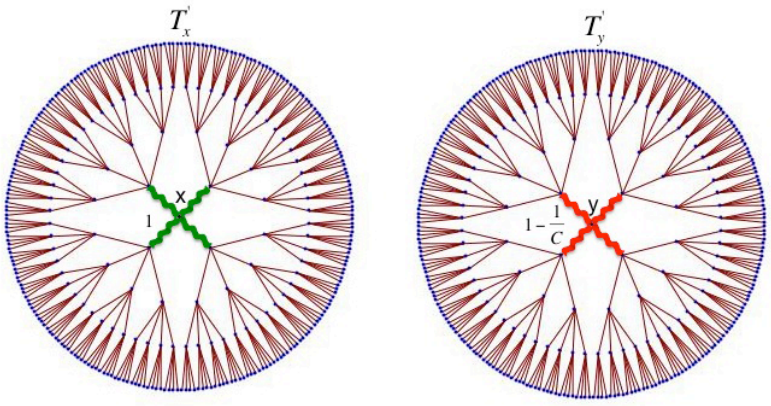

Figure 3: Figure shows $\frac{\log n}{101}$-ary trees T_x, T_y. The two roots are shown respectively at the center of the trees. In our thinking of the random coloring as an evolutionary process, the green edges incident to x survive with probability 1, the red edges incident to y with probability $1-\frac{1}{q}$ and all the other edges with probability $p_0 = \left(1-\frac{2k}{q}\right)^2$ where k is the depth of both trees and q the number of available colors. Our analysis in Lemma 3 using these probabilities gives a lower bound on the number of alive pairs of leaves after coloring T_x, T_y from the root to the leaves respectively.

Proof We show that the subgraph $G_{x,y}$ contains such a path. We break our proof into two steps:

Before we proceed, we provide certain necessary definitions. Think of the process of coloring T_x, T_y as an evolutionary process that colors edges by starting from the two roots $x, f(x) = y$ until it reaches the leaves. In the following, we call a vertex u of T_x (T_y) *alive/living* if the path $P(x,u)$ ($P(y,u)$) from x (y) to u is rainbow, i.e., the edges have received distinct colors. We call a pair of vertices $\{u, f(u)\}$ alive, $u \in T_x, f(u) \in T_y$ if $u, f(u)$ are both *alive* and the paths $P(x,u), P(y,f(u))$ share no color. Define $A_j = |\{(u,f(u)) : (u,f(u)) \text{ is alive and } \text{depth}(u) = j\}|$ for $j = 1,..,k$.

• Step 1: Existence of at least $n^{\frac{4}{5}\epsilon}$ living pairs of leaves

Assume the pair of vertices $\{u, f(u)\}$ is alive where $u \in T_x, f(u) \in T_y$. It is worth noticing that $u, f(u)$ have the same depth in their trees. We are interested in the number of pairs of children $\{u_i, f(u_i)\}_{i=1,..,\log n/101}$ that will be alive after coloring the edges from $\text{depth}(u)$ to $\text{depth}(u)+1$. A living pair $\{u_i, f(u_i)\}$ by definition has the following properties: edges $(u,u_i) \in E(T_x)$ and $(f(u), f(u_i)) \in E(T_y)$ receive two distinct colors, which are different from the set of colors used in paths $P(x,u)$ and $P(y,f(u))$. Notice the latter set of colors has cardinality $2 \times \text{depth}(u) \leqslant 2k$.

Let A_j be the number of living pairs at depth j. We first bound the size of A_1.

$$\mathbf{Pr}\left[A_1 \leqslant \frac{\log n}{200}\right] \leqslant 2^{\log n/101}\left(\frac{1}{q}\right)^{\log n/300} = O(n^{-\Omega(\log\log n)}). \tag{6}$$

Here $2^{\log n/101}$ bounds the number of choices for A_1. For a fixed set A_1 there will be at least $\frac{\log n}{101} - \frac{\log n}{200} \geqslant \frac{\log n}{300}$ edges incident with x that have the same color as their corresponding edges incident with y, under f. The factor $q^{-\log n/300}$ bounds the probability of this event.

For $j > 1$ we see that the random variable equal to the number of living pairs of children of $(u, f(u))$ stochastically dominates the random variable $X \sim \mathrm{Bin}\left(\frac{\log n}{101}, p_0\right)$, where $p_0 = \left(1 - \frac{2k}{q}\right)^2 = \left(\frac{1+3\epsilon}{1+5\epsilon}\right)^2$. The colorings of the descendants of each live pair are independent and so we have using the Chernoff bounds for $2 \leqslant j \leqslant k$,

$$\mathbf{Pr}\left[A_j < \left(\frac{\log n}{200}\right)^j p_0^{j-1} \middle| A_{j-1} \geqslant \left(\frac{\log n}{200}\right)^{j-1} p_0^{j-2}\right] \leqslant \exp\left\{-\frac{1}{2} \cdot \left(\frac{99}{200}\right)^2 \cdot \frac{\log n}{101} \cdot \left(\frac{\log n}{200}\right)^{j-1} p_0^j\right\} = O(n^{-\Omega(\log\log n)}). \quad (7)$$

(6) and (7) justify assuming that $A_k \geqslant \left(\frac{\log n}{200}\right)^k p_0^{k-1} \geqslant n^{\frac{4}{5}\epsilon}$.

• Step 2: Existence of rainbow paths between x, y in $G_{x,y}$

Assuming that there are $\geqslant n^{4\epsilon/5}$ living pairs of leaves (x_i, y_i) for vertices x, y,

$$\mathbf{Pr}(x, y \text{ are not rainbow connected}) \leqslant \left(1 - \prod_{i=0}^{2\gamma-1}\left(1 - \frac{2k+i}{q}\right)\right)^{n^{4\epsilon/5}}.$$

But

$$\prod_{i=0}^{2\gamma-1}\left(1 - \frac{2k+i}{q}\right) \geqslant \left(1 - \frac{2k+2\gamma}{q}\right)^{2\gamma} = \left(\frac{\epsilon}{1+5\epsilon}\right)^{2\gamma}.$$

So

$$\mathbf{Pr}(x, y \text{ are not rainbow connected}) \leqslant \exp\left\{-n^{4\epsilon/5}\left(\frac{\epsilon}{1+5\epsilon}\right)^{2\gamma}\right\} = \exp\left\{-n^{4\epsilon/5 - O(\log(1/\epsilon)/\log\log n)}\right\}. \quad (8)$$

Using (2) and the union bound taking (8) over all large x, y completes the proof of Lemma 4. □

We now finish the proof of Theorem 1 i.e. take care of small vertices.

We showed in Lemma 4 that *whp* for any two large vertices, a random coloring results in a rainbow path joining them. We divide the small vertices into two sets: vertices of degree 1, V_1 and the vertices of degree at least 2, V_2. Suppose that our colors are $1, 2, \ldots, q$ and $V_1 = \{v_1, v_2, \ldots, v_s\}$. We begin by giving the edge incident with v_i the color i. Then we slightly modify the argument in Lemma 4. If x is the neighbor of $v_i \in V_1$ then color i cannot be used in Steps 1 and 2 of that procedure. In terms of analysis this replaces q by $(q-1)$ ($(q-2)$ if y is also a neighbor of V_1) and the argument is essentially unchanged i.e.

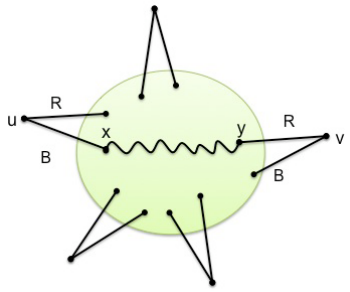

Figure 4: Taking care of small vertices.

whp there will be a rainbow path between each pair of large vertices. Furthermore, any path starting at v_i can only use color i once and so there will be rainbow paths between V_1 and V_1 and between V_1 and the set of large vertices.

The set V_2 is treated by using only two extra colors. Assume that Red and Blue have not been used in our coloring. Then we use Red and Blue to color two of the edges incident to a vertex $u \in V_2$ (the remaining edges are colored arbitrarily). This is shown in Figure **??**. Suppose that $V_2 = \{w_1, w_2, \ldots, w_t\}$. Then if we want a rainbow path joining w_i, w_j where $i < j$ then we use the red edge to go to its neighbor w_i'. Then we take the already constructed rainbow path to w_j'', the neighbor of w_j via a blue edge. Then we can continue to w_j. □

4 Proof of Theorem 2

We first observe that simply randomly coloring the edges of $G = G(n, r)$ with $q = n^{o(1)}$ colors will not do. This is because there will *whp* be $\Omega(nq^{1-r^2}) = \Omega(n^{1-o(1)})$ vertices v where all edges at distance at most two from v have the same color.

We follow a similar strategy to the proof in Theorem 1. We grow small trees T_x from each vertex x. Then for a pair of vertices x, y we build disjoint trees on the leaves of T_x, T_y so that *whp* we can find edge disjoint paths between any set of leaves S_x of T_x and any set of leaves of S_y of the same size. A bounded number of leaves of T_x, T_y will be excluded from this statement. The main difference will come from our procedure for coloring the edges. Because of the similarities, we will give a little less detail in the common parts of our proofs. We are in effect talking about building a structure like that shown in Figure 2. There is one difference, we will have to take care of which leaves of T_x we pair with which leaves of T_y, for a pair of vertices x, y.

Having grown the trees, we have the problem of coloring the edges. Instead of independently and randomly coloring the edges, we use a greedy algorithm that produces a coloring that is guaranteed to color edges differently, if they are close. This will guarantee that the edges of T_x are rainbow, for all vertices x. We then argue that we can find, for each vertex pair x, y, a partial mapping g from the leaves of T_x to the leaves of T_y such that the path from x to leaf v in T_x and the path from y to leaf $g(v)$ in T_y do not share

a color. This assumes that v has an image under the partial mapping g. We will have to argue that g is defined on enough vertices in T_x. Given this, we then consider the colors on a set of edge disjoint paths that we can construct from the leaves of T_x to their g-counterpart in the leaves of T_y.

We will use the configuration model of Bollobás [3] in our proofs, see [11] or [14] for details. Let $W = [2m = rn]$ be our set of *configuration points* and let $W_i = [(i-1)r+1, ir]$, $i \in [n]$, partition W. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing F (i.e. a partition of W into m pairs) we obtain a (multi-)graph G_F with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing F uniformly at random from among all possible pairings Ω_W of the points of W produces a random (multi-)graph G_F. Each r-regular simple graph G on vertex set $[n]$ is equally likely to be generated as G_F. Here simple means without loops of multiple edges. Furthermore, if $r = O(1)$ then G_F is simple with a probability bounded below by a positive value independent of n. Therefore, any event that occurs *whp* in G_F will also occur *whp* in $G(n, r)$.

4.1 Tree building

We will grow a Breadth First Search tree T_x from each vertex. We will grow each tree to depth

$$k = k_r = \begin{cases} \lceil \log_{r-2} \log n \rceil & r \geqslant 4. \\ \lceil 2 \log_2 \log n - 2 \log_2 \log_2 \log n \rceil & r = 3. \end{cases}$$

Observe that

$$T_x \text{ has at most } r(1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{k-1}) = r\frac{(r-1)^k - 1}{r-1} \text{ edges.} \quad (9)$$

It is useful to observe that

Lemma 5 *Whp, no set of $s \leqslant \ell_1 = \frac{1}{10} \log_{r-1} n$ vertices contains more than s edges.*

Proof Indeed,

$$\begin{aligned} \mathbf{Pr}(\exists S \subseteq [n], |S| \leqslant \ell_1, e[S] \geqslant |S| + 1) &\leqslant \sum_{s=3}^{\ell_1} \binom{n}{s} \binom{\binom{s}{2}}{s+1} \left(\frac{r^2}{rn - rs}\right)^{s+1} \quad (10) \\ &\leqslant \frac{r\ell_1}{n} \sum_{s=3}^{\ell_1} \binom{n}{s} \binom{\binom{s}{2}}{s} \left(\frac{r^2}{rn - rs}\right)^{s} \\ &\leqslant \frac{r\ell_1}{n} \sum_{s=3}^{\ell_1} \left(\frac{ne}{s} \cdot \frac{se}{2} \cdot \frac{2r}{n}\right)^{s} \\ &\leqslant \frac{r\ell_1}{n} \cdot \ell_1 \cdot (e^2 r)^{\ell_1} = o(1). \quad (11) \end{aligned}$$

Explanation of (10)**:** The factor $\left(\frac{r^2}{rn-rs}\right)^{s+1}$ can be justified as follows. We can estimate

$$\begin{aligned}\mathbf{Pr}(e_1, e_2, \ldots, e_{s+1} \in E(G_F)) &= \prod_{i=0}^{s} \mathbf{Pr}(e_{i+1} \in E(G_F) \mid e_1, e_2, \ldots, e_i \in E(G_F)) \\ &\leqslant \left(\frac{r^2}{rn-rs}\right)^{s+1}\end{aligned}$$

if we pair up the lowest index endpoint of each e_i in some arbitrary order. The fraction $\frac{r^2}{rn-rs}$ is an upper bound on the probability that this endpoint is paired with the other endpoint, regardless of previous pairings. □

Denote the leaves of T_x by L_x.

Corollary 3 *Whp,* $(r-1)^k \leqslant |L_x| \leqslant r(r-1)^{k-1}$ *for all* $x \in [n]$.

Proof This follows from the fact that *whp* the vertices spanned by each T_x span at most one cycle. This in turn follows from Lemma 5. □

Consider two vertices $x, y \in V(G)$ where $T_x \cap T_y = \emptyset$. We will show that *whp* we can find a subgraph $G'(V', E'), V' \subseteq V, E' \subseteq E$ with similar structure to that shown in Figure 2. Here $k = k_r$ and $\gamma = \left(\frac{1}{2} + \epsilon\right) \log_{r-1} n$ for some small positive constant ϵ.

Remark 2 *In our analysis we expose the pairing* F*, only as necessary. For example the construction of* T_x *involves exposing all pairings involving non-leaves of* T_x *and one pairing for each leaf. There can be at most one exception to this statement, for the rare case where* T_x *contains a unique cycle. In particular, if we expose the point* q *paired with a currently unpaired point* p *of a leaf of* T_x *then* q *is chosen randomly from the remaining unpaired points.*

Suppose that we have constructed $i = O(\log n)$ vertex disjoint trees of depth γ rooted at some of the leaves of T_x. We grow the $(i+1)$st tree $\widehat{T}_z$ via BFS, without using edges that go into y or previously constructed trees. Let a leaf $z \in L_x$ be *bad* if we have to omit a single edge as we construct the first $\ell_1/2$ levels of $\widehat{T}_z$. The previously constructed trees plus y account for $O(n^{1/2+\epsilon})$ vertices and pairings, so the probability that z is bad, given all the pairings we have exposed so far, is at most $O((r-1)^{\ell_1/2} n^{-1/2+\epsilon}) = O(n^{-1/3})$. Here bad edges can only join two leaves. This probability bound holds regardless of whichever other vertices are bad. This follows from the way we build the pairing F, see the final statement of Remark 2. So *whp* there will be at most 3 bad leaves on any T_x. Indeed, $\mathbf{Pr}(\exists x : x \text{ has } \geqslant 4 \text{ bad leaves}) \leqslant n\binom{O(\log n)}{4} n^{-4/3} = o(1)$.

If a leaf is not bad then the first $\ell_1/2$ levels produce $\Theta(n^{1/20})$ leaves. From this, we see that *whp* the next $\gamma - \ell_1$ levels grow at a rate $r - 1 - o(n^{-1/25})$. Indeed, given that a level has L vertices where $n^{1/20} \leqslant L \leqslant n^{3/4}$, the number of vertices in the next level dominates $Bin\left((r-1)L, 1 - O\left(\frac{n^{3/4}}{n}\right)\right)$, after accounting for the configuration points used in building previous trees. Indeed, $(r-1)L$ configuration points associated with good leaves will be unpaired and for each of them, the probability it is paired with a

point associated with a vertex in any of the trees constructed so far is $O(n^{1/2+2\epsilon}/n)$. This probability bound holds regardless of the pairings of the other leaf configuration points. We can thus assert that *whp* we will have that all but at most three of the leaves L_x of T_x are roots of vertex disjoint trees $\widehat{T}_1, \widehat{T}_2, \ldots$, each with $\Theta(n^{1/2+\epsilon/2})$ leaves. Let L_x^* denote these *good* leaves. The same analysis applies when we build trees $\widehat{T}'_1, \widehat{T}'_2, \ldots$, with roots at L_y.

Now the probability that there is no edge joining the leaves of $\widehat{T}_i$ to the leaves of $\widehat{T}'_j$ is at most

$$\left(1 - \frac{(r-1)\Theta(n^{1/2+\epsilon/2})}{rn}\right)^{(r-1)n^{1/2+\epsilon/2}} \leqslant e^{-\Omega(n^\epsilon)}.$$

To summarise,

Remark 3 Whp *we will succeed in finding in G_F and hence in $G = G(n,r)$, for all $x, y \in V(G_F)$, for all $u \in L_x^*, v \in L_y^*$, a path $P_{u,v}$ from u to v of length $O(\log n)$ such that if $u \neq u'$ and $v \neq v'$ then $P_{u,v}$ and $P_{u',v'}$ are edge disjoint. These paths avoid T_x, T_y except at their start and endpoints.*

4.2 Coloring the edges

We now consider the problem of coloring the edges of G. Let H denote the line graph of G and let $\Gamma = H^{2k}$ denote the graph with the same vertex set as H and an edge between vertices e, f of Γ if there there is a path of length at most k between e and f in H. We will construct a proper coloring of Γ using

$$q = 10(r-1)^{2k} \sim 100 \log^{2\theta_r} n \text{ where } \theta_r = \frac{\log(r-1)}{\log(r-2)}$$

colors. We do this as follows: Let $e_1, e_2, \ldots, e_m$ be an arbitrary ordering of the vertices of Γ. For $i = 1, 2, \ldots, m$, color e_i with a random color, chosen uniformly from the set of colors not currrently appearing on any neighbor in Γ. At this point only $e_1, e_2, \ldots, e_{i-1}$ will have been colored.

Suppose then that we color the edges of G using the above method. Fix a pair of vertices x, y of G. We see immediately, that no color appears twice in T_x and no color appears twice in T_y. This is because the distance between edges in T_x is at most $2k$. This also deals with the case where $V(T_x) \cap V(T_y) \neq \emptyset$, for the same reason. So assume now that T_x, T_y are vertex disjoint. We can find lots of paths joining x and y. We know that the first and last k edges of each path will be individually rainbow colored. We will first show that we have many choices of path where these $2k$ edges are rainbow colored when taken together.

4.3 Case 1: $r \geqslant 4$:

We argue now that we can find $\sigma_0 = (r-2)^{k-1}$ leaves $u_1, u_2, \ldots, u_\tau \in T_x$ and σ_0 leaves $v_1, v_2, \ldots, v_\tau \in T_y$ such for each i the T_x path from x to u_i and the T_y path from y to v_i do not share any colors.

Lemma 6 *Let T_1, T_2 be two vertex disjoint copies of an edge colored complete d-ary tree with ℓ levels, where $d \geqslant 3$. Let T_1, T_2 be rooted at x, y respectively. Suppose that the colorings of T_1, T_2 are both rainbow. Let $\kappa = (d-1)^\ell$. Then there exist leaves $u_1, u_2, \ldots, u_\kappa$ of T_1 and leaves $v_1, v_2, \ldots v_\kappa$ of T_2 such that the following is true: If P_i, P_i' are the paths from x to u_i in T_1 and from y to v_i in T_2 respectively, then $P_i \cup P_i'$ is rainbow colored for $i = 1, 2, \ldots, \kappa$.*

Proof Let A_ℓ be the minimum number of rainbow path pairs that we can find in any such pair of edge colored trees. We prove that $A_\ell \geqslant (d-1)^\ell$ by induction on ℓ. This is true trivially for $\ell = 0$. Suppose that x is incident with $x_1, x_2, \ldots, x_d$ and that the sub-tree rooted at x_i is $T_{1,i}$ for $i = 1, 2, \ldots, d$. Define y_i and $T_{2,i}$, $i = 1, 2, \ldots, d$ similarly with respect to y. Suppose that the color of the edge (x, x_i) is c_i for $i = 1, 2, \ldots, d$ and let $Q_x = \{c_1, c_2, \ldots, c_d\}$. Similarly, suppose that the color of the edge (y, y_i) is c_i' for $i = 1, 2, \ldots, d$ and let $Q_y = \{c_1', c_2', \ldots, c_d'\}$. Next suppose that Q_j is the set of colors in Q_x that appear on the edges $E(T_{2,j}) \cup \{(y, y_j)\}$. The sets $Q_1, Q_2, \ldots, Q_d$ are pair-wise disjoint. Similarly, suppose that Q_i' is the set of colors in Q_y that appear on the edges $E(T_{1,i}) \cup \{(x, x_i)\}$. The sets $Q_1', Q_2', \ldots, Q_d'$ are pair-wise disjoint.

Now define a bipartite graph H with vertex set $A + B = [d] + [d]$ and an edge (i, j) iff $c_i \notin Q_j$ and $c_j' \notin Q_i'$. We claim that if $S \subseteq A$ then its neighbor set $N_H(S)$ satisfies the inequality

$$d|S| - |N_H(S)| - |S| \leqslant |S| \cdot |N_H(S)|. \tag{12}$$

Here the LHS of (12) bounds from below, the size of the set $S : N_H(S)$ of edges between S and $N_H(S)$. This is because there are at most $|S|$ edges missing from $S : N_H(S)$ due to $i \in S$ and $j \in N_H(S)$ and $c_i \in Q_j$. At most $|N_H(S)|$ edges are missing for similar reasons. On the other hand, $d|S|$ is the number there would be without these missing edges. The RHS of (12) is a trivial upper bound.

Re-arranging we get that

$$|N_H(S)| - |S| \geqslant \left\lceil \frac{(d - 2 - |S|)|S|}{|S| + 1} \right\rceil \geqslant -1.$$

(We get -1 when $|S| = d$).

Thus H contains a matching M of size $d-1$. Suppose without loss of generality that this matching is $(i, i), i = 1, 2, \ldots, d-1$. We know by induction that for each i we can find paths $(P_{i,j}, \widehat{P}_{i,j})$, $j = 1, 2, \ldots, (d-1)^{\ell-1}$ where $P_{i,j}$ is a root to leaf path in $T_{1,i}$ and $\widehat{P}_{i,j}$ is a root to leaf path in $T_{2,i}$ and that $P_{i,j} \cup \widehat{P}_{i,j}$ is rainbow for all i, j. Furthermore, (i, i) being an edge of H, means that the edge sets $\{(x, x_i)\} \cup E(P_{i,j}) \cup E(\widehat{P}_{i,j}) \cup \{(y, y_i\}$ are all rainbow. □

Let

$$V_1 = \{x : V(T_x) \text{ contains a cycle}\}.$$

When $x, y \notin V_1$ we apply this Lemma to T_x, T_y by deleting one of the r sub-trees attached to each of x, y and applying the lemma directly to the $(r-1)$-ary trees that remain. This will yield $(r-2)^k$ pairs of paths. If $x \in V_1$, we delete $r-2$ sub-trees attached to x leaving

at least two $(r-1)$-ary trees of depth $k-1$ with roots adjacent to x. We can do the same at y. Let c_1, c_2 be the colors of the two edges from x to the roots of these two trees T_1, T_2. Similarly, let c_1', c_2' be the colors of the two analogous edges from y to the trees T_1', T_2'. If color c_1 does not appear in T_1' then we apply the lemma to T_1 and T_1'. Otherwise, we can apply the lemma to T_1 and T_2'. In both cases we obtain $(r-2)^{k-1}$ pairs of paths.

Accounting for bad vertices we put

$$\sigma = \sigma_0 - 6 = (r-2)^{k-1} - 6 \geqslant \frac{\log n}{r-2} - 6$$

and we see from Remark 3 that we can *whp* find σ paths $P_1, P_2, \ldots, P_\sigma$ of length $O(\log n)$ from x to y. Path P_i goes from x to a leaf $u_i \in L_x^*$ via T_x and then traverses $Q_i = P(u_i, v_i)$ where $v_i = \phi(u_i) \in L_y^*$ and then goes from v_i to a y via T_y. Here ϕ is some partial map from L_x^* to L_y^*. It is a random variable that depends on the coloring $\mathcal{C}$ of the edges of T_x and T_y. The paths $P_1, P_2, \ldots, P_\sigma$ depend on the choice of ϕ and hence $\mathcal{C}$ and so we should write $P_i = P_i(\mathcal{C})$.

We fix the coloring $\mathcal{C}$ and hence $P_1, P_2, \ldots, P_\sigma$. Let $\mathcal{R}$ be the event that at least one of the paths $P_1, P_2, \ldots, P_\sigma$ is rainbow colored. We show that $\mathbf{Pr}(\neg\mathcal{R} \mid \mathcal{C})$ is small.

We let $c(e)$ denote the color of edge e in a given coloring. We remark next that for a particular coloring $c_1, c_2, \ldots, c_m$ of the edges $e_1, e_2, \ldots, e_m$ we have

$$\mathbf{Pr}(c(e_i) = c_i,\ i = 1, 2, \ldots, m) = \prod_{i=1}^{m} \frac{1}{a_i}$$

where $q - \Delta \leqslant a_i \leqslant q$ is the number of colors available for the color of the edge e_i given the coloring so far i.e. the number of colors unused by the neighbors of e_i in Γ when it is about to be colored.

Now fix an edge $e = e_i$ and the colors $c_j,\ j \neq i$. Let C be the set of colors not used by the neighbors of e_i in Γ. The choice by e_i of its color under this conditioning is not quite random, but close. Indeed, we claim that for $c, c' \in C$

$$\frac{\mathbf{Pr}(c(e) = c \mid c(e_j) = c_j,\ j \neq i)}{\mathbf{Pr}(c(e) = c' \mid c(e_j) = c_j,\ j \neq i)} \leqslant \left(\frac{q-\Delta}{q-\Delta-1}\right)^{\Delta}.$$

This is because, changing the color of e_i only affects the number of colors available to neighbors of e_i, and only by at most one.

Thus, for $c \in C$, we have

$$\mathbf{Pr}(c(e) = c \mid c(e_j) = c_j,\ j \neq i) \leqslant \frac{1}{q-\Delta}\left(\frac{q-\Delta}{q-\Delta-1}\right)^{\Delta}.$$

Now $\Delta \leqslant (r-1)^{2k} = q/10$ and we deduce that

$$\mathbf{Pr}(c(e) = c \mid c(e_j) = c_j,\ j \neq i) \leqslant \frac{2}{q}.$$

It follows that for $i \in [\sigma]$,

$$\mathbf{Pr}(P_i \text{ is rainbow colored} \mid \mathcal{C},\ \text{coloring of } \bigcup_{j\neq i} Q_j) \geqslant \left(1 - \frac{4(k+\gamma)}{q}\right)^{2\gamma}.$$

This is because when we consider the coloring of Q_i there will always be at most $2k + 2\gamma$ colors forbidden by non-neighboring edges, if it is to be rainbow colored.

It then follows that

$$\begin{aligned}\mathbf{Pr}(\neg\mathcal{R} \mid \mathcal{C}) &\leqslant \left(1 - \left(1 - \frac{4(k+\gamma)}{q}\right)^{2\gamma}\right)^{\sigma}\\ &\leqslant \left(\frac{8\gamma(k+\gamma)}{q}\right)^{\sigma}\\ &\leqslant \left(\frac{(2+10\epsilon)\log_{r-1}^2 n}{10\log^{\theta_r} n}\right)^{\sigma} = o(n^{-2}).\end{aligned}$$

This completes the proof of Theorem 2 when $r \geqslant 4$.

Case 2: $r = 3$**:**

When $r = 3$ we can't use $(r-2)^k$ to any effect. Also, we need to increase q to $\log^4 n$. This necessary for a variety of reasons. One reason is that we will reduce σ to $2^{k/2}$. We want this to be $\Omega(\log n)$ and this will force k to (roughly) double what it would have been if we had followed the recipe for $r \geqslant 4$. This makes Δ close to $\log^4 n$ and we need $q \gg \Delta$.

And we need to modify the argument based on Lemma 6. Instead of inducting on the trees at depth one from the roots x, y, we now induct on the trees at depth two. Assume first that $x, y \notin V_1$. After ignoring one branch for T_x and T_y we now consider the sub-trees $T_{x,i}, T_{y,i}$, $i = 1, 2, 3, 4$ of T_x, T_y whose roots $x_1, \ldots, x_4$ and $y_1, \ldots, y_4$ are at depth two. We cannot necessarily make this construction when $x \in V_1$. Let P_i be the path from x to x_i in T_x and let $\widehat{P}_j$ be the path from y to y_j in T_y. Next suppose that $\widehat{Q}_j$ is the set of colors in Q that appear on the edges $E(T_{y,j}) \cup E(\widehat{P}_j)$. Similarly, suppose that Q'_i is the set of colors in Q' that appear on the edges $\{E(T_{x,i}) \cup E(P_i)\}$.

Re-define H to be the bipartite graph with vertex set $A + B = [4] + [4]$. The edges of H are as before: (i, j) exists iff $c_i \notin Q_j$ and $c'_j \notin \widehat{Q}_i$. This time we can only say that a color is in at most two $\widehat{Q}_i$'s and similarly for the Q'_j's. The effect of this is to replace (12) by

$$4|S| - 2(|N_H(S)| + |S|) \leqslant |S| \cdot |N_H(S)|$$

from which we can deduce that

$$|S| - |N_H(S)| \leqslant \frac{|S| \cdot |N_H(S)|}{2} \leqslant 2|N_H(S)|.$$

It follows that $|N_H(S)| \geqslant \lceil |S|/3 \rceil \geqslant |S| - 2$ and so H contains a matching of size two. An inductive argument then shows that we are able to find $2^{\lfloor k/2 \rfloor}$ rainbow pairs of paths. The proof now continues as in the case $r \geqslant 4$, arguing about the coloring of paths $P_1, P_2, \ldots, P_\sigma$ where now $\sigma = 2^{\lfloor k/2 \rfloor}$.

We finally deal with the vertices in V_1. We classify them according to the size of the cycle C_x that is contained in $V(T_x)$. If T_x contains a cycle C_x then necessarily $|C_x| \leqslant 2k$ and so there are at most $2k$ types in our classification. It follows from Lemma 5 that if $x, y \in V_1$ and $T_x \cap T_y \neq \emptyset$ then $C_x = C_y$ *whp*. Note next that the distance from x to C_x is at most $k - |C_x|/2$. If C is a cycle of length at most $2k$, let $V_C = \{x : C = C_x\}$ and let E_C be the set of edges contained in V_C. We have

$$|V_C| = O(|C|2^{k-|C|/2}) = O(2^k) = O(\log^2 n/\log\log n). \tag{13}$$

We introduce $2k$ new sets $\widehat{Q}_i, i = 3, 4, \ldots, 2k$ of $O(\log^2 n/\log\log n)$ colors, distinct from Q. Thus we introduce $O(\log^2 n)$ new colors overall. We re-color each E_C with the colors from $\widehat{Q}_{|C|}$. It is important to observe that if $|C| = |C'|$ then the graphs induced by V_C and $V_{C'}$ are isomorphic and so we can color them isomorphically. By the latter we mean that we choose some isomorphism f from V_C to $V_{C'}$ and then if e is an edge of V_C then we color e and $f(e)$ with the same color. After this re-coloring, we see that if T_x and T_y are not vertex disjoint, then they are contained in the same V_C. The edges of V_C are rainbow colored and so now we only need to concern ourselves with $x, y \in V_1$ such that T_x and T_y are vertex disjoint. Assume now that $x, y \in V_1$.

Assume first that x, y are of the same type and that they are at the same distance from C_x, C_y respectively. Our aim now is to define binary trees T'_x, T'_y "contained" in T_x, T_y that can be used as in Lemma 6. If we delete an edge $e = (u, v)$ of C_x then the graph that remains on $V(T_x)$ is a tree with at most two vertices u, v of degree two. Now delete one of the three sub-trees of T_x. If there are vertices of degree two, make sure one of them is in this sub-tree. If necessary, shrink the path of length two with the remaining vertex of degree two in the middle to an edge e_x. It has leaves at depth $k-1$ and leaves at depth $k-2$. The resulting binary tree will be our T'_x. The leaves at depth $k-1$ come in pairs. Delete one vertex from each pair and shrink the paths of length two through the vertex at depth $k-2$ to an edge.

The edges that are obtained by shrinking paths of length two will have two colors. Because x, y are at the same distance from their cycles, we can delete $f(e)$ from C_y and do the construction so that T'_x and T'_y will be isomorphically colored.

It is now easy to find 2^{k-2} pairs of paths whose unions are rainbow colored. Each leaf of T_x, T_y can be labelled by a $\{0,1\}$ string of length $k-2$. We pair string $\xi_1\xi_2\cdots\xi_{k-1}\xi_{k-2}$ in T_x with $(1-\xi_1)\xi_2\cdots\xi_{k-1}\xi_{k-2}$ in T_y. The associated paths will have a rainbow union. The proof now continues as in the case $r \geqslant 4$, arguing about the coloring of paths $P_1, P_2, \ldots, P_\sigma$ where now $\sigma = 2^{k-2}$.

If x is further from C_x than y is from C_y then let z be the vertex on the path from x to C_x at the same distance from C_x as y is from C_y. We have a rainbow path from z to y and adding the T_x path from x to z gives us a rainbow path from x to y. This relies on the fact that V_{C_x} and V_{C_y} are isomorphically colored.

If x, y are of a different type, then T_x and T_y are re-colored with distinct colors and we can proceed as as in the case $r \geqslant 4$, arguing about the coloring of paths $P_1, P_2, \ldots, P_\sigma$ where now $\sigma = 2^k$, using Corollary 3.

If $x \in V_1$ and $y \notin V_1$ then we can proceed as if both are not in V_1. This is because of the re-coloring of the edges of T_x. We can proceed as as in the case $r \geqslant 4$, arguing about the coloring of paths $P_1, P_2, \ldots, P_\sigma$ where now $\sigma = 2^k$, using Corollary 3.

This completes our proof of Theorem 2. □

5 Conclusion

In this work we have given an aymptotically tight result on the rainbow connectivity of $G = G(n,p)$ at the connectivity threshold. It is reasonable to conjecture that this could be tightened:

Conjecture: *Whp*, $rc(G) = \max\{Z_1, diameter(G(n,p))\}$.

Our result on random regular graphs is not so tight. It is still reasonable to believe that the above conjecture also holds in this case. (Of course $Z_1 = 0$ here).

It is worth mentioning that if the degree r in Theorem 2 is allowed to grow as fast as $\log n$ then one can prove a result closer to that of Theorem 1.

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