List-coloring graphs on surfaces with varying list-sizes

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In Memory of Herbert S. Wilf, 1931-2012

Abstract

Let G be a graph embedded on a surface S_ε with Euler genus $\varepsilon > 0$, and let $P \subset V(G)$ be a set of vertices mutually at distance at least 4 apart. Suppose all vertices of G have $H(\varepsilon)$-lists and the vertices of P are precolored, where $H(\varepsilon) = \left\lfloor \frac{7+\sqrt{24\varepsilon+1}}{2} \right\rfloor$ is the Heawood number. We show that the coloring of P extends to a list-coloring of G and that the distance bound of 4 is best possible. Our result provides an answer to an analogous question of Albertson about extending a precoloring of a set of mutually distant vertices in a planar graph to a 5-list-coloring of the graph and generalizes a result of Albertson and Hutchinson to list-coloring extensions on surfaces.

Keywords: list-coloring; Heawood number; graphs on surfaces

1 Introduction

For a graph G the distance between vertices x and y, denoted $dist(x, y)$, is the number of edges in a shortest x-y-path in G, and we denote by $dist(P)$ the least distance between

two vertices of P. In [1] M. O. Albertson asked if there is a distance $d > 0$ such that every planar graph with a 5-list for each vertex and a set of precolored vertices P with $dist(P) \geqslant d$ has a list-coloring that is an extension of the precoloring of P. In that paper he proved such a result for 5-coloring with $d \geqslant 4$, answering a question of C. Thomassen. There have been some preliminary answers to Albertson's question in [4, 8, 11]; initially Tuza and Voigt [17] showed that $d > 4$. Kawarabayashi and Mohar [11] have shown that when P contains k vertices, there is a function $d_k > 0$ that suffices for such list-coloring. Then recently Dvořák, Lidický, Mohar and Postle [9] have announced a complete solution, answering Albertson's question in the affirmative, independent of the size of P.

Let S_ε denote a surface of Euler genus $\varepsilon > 0$. Its Heawood number is given by

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor$$

and gives the best possible bound on the chromatic number of S_ε except for the Klein bottle whose chromatic number is 6. (For all basic chromatic and topological graph theory results, see [10, 13].) In many instances results for list-coloring graphs on surfaces parallel classic results on surface colorings. Early on it was noted that the Heawood number also gives the list-chromatic number for surfaces; see [10] for history. Also Dirac's Theorem [7] has been generalized to list-coloring by Böhme, Mohar and Stiebitz for most surfaces; the missing case, $\varepsilon = 3$, was completed by Král' and Škrekovski. This result informs and eases much of our work.

Theorem 1.1 ([5, 12])**.** *If G embeds on S_ε, $\varepsilon > 0$, then G can be $(H(\varepsilon) - 1)$-list-colored unless G contains $K_{H(\varepsilon)}$.*

Analogously to Albertson's question on the plane, we and others (see [11]) ask related list-coloring questions for surfaces. In this paper we ask if there is a distance $d > 0$ such that every graph on S_ε, $\varepsilon > 0$, with $H(\varepsilon)$-lists on each vertex and a set of precolored vertices P with $dist(P) \geqslant d$ has a list-coloring that is an extension of the precoloring of P. In [3] Albertson and Hutchinson proved the following result; the main result of this paper generalizes this theorem to list-coloring.

Theorem 1.2 ([3])**.** *For each $\varepsilon > 0$, except possibly for $\varepsilon = 3$, if G embeds on a surface of Euler genus ε and if P is a set of precolored vertices with $dist(P) \geqslant 6$, then the precoloring extends to an $H(\varepsilon)$-coloring of G.*

Others have studied similar extension questions with k-lists on vertices for $k \geqslant 5$. For example, see [16], Thm. 4.4, for $k \geqslant 6$ and [11], Thm. 6.1, for $k = 5$; however, in both results the embedded graphs must satisfy constraints depending on the Euler genus and the number of precolored vertices. Our main result is Thm. 1.3, which shows that there is a constant bound on the distance between precolored vertices that ensures list-colorability for all graphs embedded on all surfaces when vertices have $H(\varepsilon)$-lists. It improves on Thm. 1.2 by removing the possible exception for $\varepsilon = 3$, reducing the distance of the precolored vertices from 6 to 4, and broadening the results to list-coloring.

Theorem 1.3. *Let G embed on S_ε, $\varepsilon > 0$, and let $P \subset V(G)$ be a set of vertices with $dist(P) \geqslant 4$. Then if the vertices of P each have a 1-list and all other vertices have an $H(\varepsilon)$-list, G can be list-colored. The distance bound of 4 is best possible.*

When G is embedded on S_ε, let the *width* [2] denote the length of a shortest non-contractible cycle of G; this is also known as *edge-width*. For list-coloring we have the following corollary of Thms. 1.1 and 1.3.

Corollary 1.4. *If G embeds on S_ε, $\varepsilon > 0$, with width at least 4, if the vertices of $P \subset V(G)$ have 1-lists and all other vertices have $H(\varepsilon)$-lists, then G is list-colorable when $dist(P) \geqslant 3$. The distance bound of 3 is best possible.*

Given that graphs embedded with very large width can be 5-list-colored as proved in [6], it is straightforward to deduce a 6-list-coloring extension result for such graphs. When G embeds on S_ε, $\varepsilon > 0$, with width at least $2^{O(\varepsilon)}$, if a set of vertices P with $dist(P) \geqslant 3$ have 1-lists and all others have 6-lists, then after the vertices of P are deleted and the color of each $x \in P$ is deleted from the lists of x's neighbors, the remaining graph has 5-lists, large width, and so is list-colorable. Thus G is list-colorable, but only when embedded with large width whose size increases with the Euler genus of the surface.

A consequence of Thomassen's proof of 5-list-colorability of planar graphs [15] is that if all vertices of a graph in the plane have 5-lists except that the vertices of one face have 3-lists, then the graph can be list-colored. For surfaces, we offer as a related result another corollary of Thm. 1.3.

Corollary 1.5. *If G embeds on S_ε, $\varepsilon > 0$, and contains a set of faces each pair of which is at distance at least two apart, with all vertices on these faces having $(H(\varepsilon) - 1)$-lists and all other vertices having $H(\varepsilon)$-lists, then G can be list-colored.*

The paper concludes with related questions.

2 Background results on surfaces, Euler genus and the Heawood formula

Let S_ε denote a surface of Euler genus $\varepsilon > 0$. If ε is odd, then S_ε is the nonorientable surface with ε crosscaps, but when ε is even, S_ε may be orientable or not. We let $\mathcal{T}$ denote the torus, the orientable surface of Euler genus 2, and $\mathcal{K}$ the Klein bottle, the nonorientable surface of Euler genus 2.

The Heawood number $H(\varepsilon)$, defined above, gives the largest n for which K_n embeds on a surface S_ε of Euler genus ε, as well as the chromatic number of S_ε, except that K_6 is the largest complete graph embedding on $\mathcal{K}$ and 6 is its chromatic number.

The least Euler genus ε for which K_n embeds on S_ε is given by the inverse function

$$\varepsilon = I(n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

ε	$H(\varepsilon)$	e	f	Largest Face	ε	$H(\varepsilon)$	e	f	Largest Face
1	6	15	10	3	13	12	66	43	6
2	7	21	14	3	14	12	66	42	9
3	7	21	13	6	15	13	78	52	3
4	8	28	18	5	16	13	78	51	6
5	9	36	24	3	17	13	78	50	9
6	9	36	23	6	18	13	78	49	12
7	10	45	30	3	19	14	91	60	5
8	10	45	29	6	20	14	91	59	8
9	10	45	28	9	21	14	91	58	11
10	11	55	36	5	22	15	105	70	3
11	11	55	35	8	23	15	105	69	6
12	12	66	44	3	24	15	105	68	9

Table 1: Embedding parameters for $K_{H(\varepsilon)}$

Each K_n, $n \geqslant 5$, of course, has a minimum value of $\varepsilon > 0$ for which it embeds on S_ε, called the *Euler genus of* K_n, but for $\varepsilon \geqslant 2$ more than one surface S_ε may have the same maximum K_n that embeds on it. For example, both S_5 and S_6 have Heawood number 9 with K_9 being the largest complete graph that embeds there. Embedding patterns of $K_{H(\varepsilon)}$ depend on the congruence class of $H(\varepsilon)$ modulo 3 for $\varepsilon \geqslant 1$. In Table 1, which gives values of ε and $H(\varepsilon)$ for $\varepsilon = 1, \ldots, 24$, e is the number of edges in $K_{H(\varepsilon)}$, $f = 2 - \varepsilon - v + e$ is the number of faces in a 2-cell embedding of $K_{H(\varepsilon)}$ on S_ε, and the final column gives the size of the largest possible face when $K_{H(\varepsilon)}$ is so embedded. That largest face size is three more than the difference $2e - 3f$.

For our results we need to know when $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on S_ε. When K_n embeds on S_ε, but not on $S_{\varepsilon-1}$, then K_n necessarily embeds with a 2-cell embedding. When K_n embeds in addition on $S_{\varepsilon+1}$, $\ldots$, $S_{\varepsilon+i}$ with $i > 0$, then it may not have a 2-cell embedding on the latter surfaces. For example, on surfaces S_1, $\mathcal{T}$, S_4, and S_5, the complete graphs K_6, K_7, K_8 and K_9 have 2-cell embeddings, respectively, but K_6, K_7 and K_9 may or may not have 2-cell embeddings on $\mathcal{K}$, S_3 and S_6, respectively.

If f is a face of an embedded graph G, let $V(f)$ and $E(f)$ denote the incident vertices and edges of f. We say that $V(f) \cup E(f)$ is the *boundary* of f and that the *closure* of f is the union of f and its boundary. Each edge of $E(f)$ either lies on another face besides f or it might lie just on f. For example, Fig. 1 shows two graphs embedded on the torus, $\mathcal{T}$. In the first graph, edges 2-3 and 4-7 each border two faces, but edges 3-6 and 8-9 each border only one face. The size s of a face f is determined by counting, with multiplicity, the number of edges on its boundary, and we then call f an s-region. In other words, when s_1 edges of $E(f)$ lie on another face of G besides f and s_2 edges lie only on f, then we call f an s-region where $s = s_1 + 2s_2$. When f is a 2-cell, $E(f)$ forms a single facial walk W_f, and the size of the face equals the length of the facial walk, counting multiplicity

of repeated edges. Since an s-region f may have repeated edges and repeated vertices, we indicate $|V(f)| = t$ by calling f also a t-vertex-region where $t \leqslant s$. Hence the shaded region in the first graph in Fig. 1 is a 13-region and a 9-vertex-region, since two edges and four vertices are repeated; the shaded region in the second graph, with no repeated vertices or edges, is a 13-region and a 13-vertex-region.

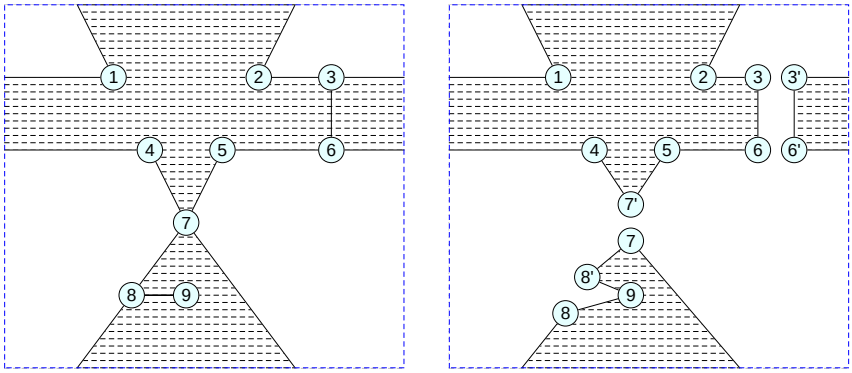

Figure 1: A 2-cell region in a graph embedded on the torus, $\mathcal{T}$, before and after vertex- and edge-duplication

Here in summary are statistics on 2-cell embeddings of $K_{H(\varepsilon)}$. The patterns presented are visible from Table 1 and are easily derived from Euler's formula and the function $I(n)$, given above.

Lemma 2.1. *Let $\varepsilon \geqslant 1$ and suppose $K_{H(\varepsilon)}$ has a 2-cell embedding on S_ε (but $S_\varepsilon \neq \mathcal{K}$). Set $i = \left\lfloor \frac{H(\varepsilon)-3}{3} \right\rfloor$ so that $H(\varepsilon) = 3i+3, 3i+4$ or $3i+5$ with $i \geqslant 1$.*

1. *If $H(\varepsilon) = 3i+3$, then $\varepsilon = (3i^2-i)/2, (3i^2-i+2)/2, \ldots,$ or $(3i^2+i-2)/2$. The number of faces of the embedding is given by $f = 3i^2+5i+2, 3i^2+5i+1, \ldots,$ or $3i^2+4i+3$, respectively, and the largest possible face is an s-region with $s = 3, 6, \ldots,$ or $3i$, resp.*

2. *If $H(\varepsilon) = 3i+4$, then $\varepsilon = (3i^2+i)/2, (3i^2+i+2)/2, \ldots,$ or $(3i^2+3i)/2$. The number of faces of the embedding is given by $f = 3i^2+7i+4, 3i^2+7i+3, \ldots,$ or $3i^2+6i+4$, respectively, and the largest possible face is an s-region with $s = 3, 6, \ldots,$ or $3i+3$, resp.*

3. *If $H(\varepsilon) = 3i+5$, then $\varepsilon = (3i^2+3i+2)/2, (3i^2+3i+4)/2, \ldots,$ or $(3i^2+5i)/2$. The number of faces of the embedding is given by $f = 3i^2+9i+6, 3i^2+9i+5, \ldots,$ or $3i^2+8i+7$, respectively, and the largest possible face is an s-region with $s = 5, 8, \ldots,$ or $3i+2$, resp.*

From the point of view of the genus, given $\varepsilon > 0$, we can determine directly whether or not $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on S_ε. $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding if and only if $\varepsilon = (3i^2-i)/2$ or $(3i^2+i)/2$ or $(3i^2+3i+2)/2$ for some

value of $i > 0$. Thus given $\varepsilon > 0$, we compute $H(\varepsilon)$ and set $i = \lfloor H(\varepsilon)/3 \rfloor - 1$ so that $H(\varepsilon) = 3i + 3, 3i + 4$, or $3i + 5$. Then $K_{H(\varepsilon)}$ necessarily embeds with a 2-cell embedding if $I(H(\varepsilon)) = \varepsilon$; that is, S_ε is the genus surface for $K_{H(\varepsilon)}$.

In the results of Table 1 we do not claim that every 2-cell embedding of $K_{H(\varepsilon)}$ achieves the maximum face size when that size is greater than three. For example when $K_{H(\varepsilon)}$ has a largest face being a 5- or 6-region, it might embed as a near-triangulation with one 5- or 6-region, respectively, or it might be a triangulation except for two 4-regions or a triangulation except for a 4- and a 5-region, resp. (An embedding is a *near-triangulation* if at most one region is not 3-sided.)

We note from Table 1 and Lemma 2.1 that there are some instances of ε when $K_{H(\varepsilon)}$ embeds possibly with an $(H(\varepsilon) - 1)$-region which might allow for the embedding of two different (not disjoint, but distinct) copies of $K_{H(\varepsilon)}$ on S_ε, as explained in the next lemma.

Lemma 2.2. *Let $K_{H(\varepsilon)}$ have a 2-cell embedding on S_ε, $\varepsilon > 0$.*

1. *The largest possible face in the embedding is an $(H(\varepsilon) - 1)$-region. If there is an $(H(\varepsilon) - 1)$-region, there is just one, and the embedding is a near-triangulation.*

2. *If every face of the embedding is at most an $(H(\varepsilon) - 2)$-region, then no additional copy of $K_{H(\varepsilon)}$ can simultaneously embed on S_ε.*

3. *When $K_{H(\varepsilon)}$ can embed with an $(H(\varepsilon) - 1)$-region that is also an $(H(\varepsilon) - 1)$-vertex-region, then two different copies of $K_{H(\varepsilon)}$ can embed, by adding a vertex adjacent to all vertices of that region, and then the two complete graphs share a copy of $K_{H(\varepsilon)-1}$. Such an embedding is possible only if $H(\varepsilon) = 3i + 4$ and $\varepsilon = (3i^2 + 3i)/2$, and the resulting embedding is a triangulation.*

We call the latter graph $DK_{H(\varepsilon)}$; it is also $K_{H(\varepsilon)+1} \setminus \{e\}$ for some edge e.

Proof. Suppose that $K_{H(\varepsilon)}$ has a 2-cell embedding with at least one s-region where $s \geqslant H(\varepsilon) - 1$. Then Euler's formula plus a count of edges on faces with multiplicities leads to a contradiction to Lemma 2.1 in all cases except when there is precisely one $(H(\varepsilon) - 1)$-region, $H(\varepsilon) = 3i + 4$, $\varepsilon = (3i^2 + 3i)/2$, and all other faces are 3-regions.

Suppose $K_{H(\varepsilon)}$ embeds on S_ε with every face having at most $H(\varepsilon) - 2$ sides. No two additional vertices in different faces of $K_{H(\varepsilon)}$ can be adjacent. For $2 \leqslant k \leqslant 4$, k mutually adjacent, additional vertices cannot form $K_{H(\varepsilon)}$ together with $H(\varepsilon) - k$ vertices on the boundary of a face.

Proofs of remaining parts follow easily from Euler's Formula and Lemma 2.1. □

If $V' \subseteq V(G)$, we denote by $G[V']$ the induced subgraph on the vertices in V'; for $E' \subseteq E(G)$, we denote by $G[E']$ the induced subgraph on the edge set E'. When f is a face of an embedded G, we may also call the subgraph $G[E(f)]$ the boundary of f; that is, it may be convenient at times to think of the boundary of a face f as a set $V(f) \cup E(f)$ and at other times as the subgraph $G[E(f)]$.

We restate two very useful corollaries of Thm. 6 in [5]. The first involves a case that is not covered in that theorem, but which follows easily from their proof. If f is the

infinite face of a connected plane graph, we call the boundary of f the *outer boundary* of G, and when $G[E(f)]$ is a cycle, we call it the *outer cycle*. Without loss of generality we may suppose that for a connected plane graph the outer boundary is a cycle.

Corollary 2.3. *([5]) Let G be a connected plane graph with outer cycle C that is a k-cycle with $k \leqslant 6$. If every vertex of G has a list of size at least 6, then a precoloring of C extends to all of G unless $k = 6$, there is a vertex in $V(G) \setminus V(C)$ that is adjacent to all vertices of $V(C)$, and its list consists of six colors that appear on the precolored C.*

Then the results of Thm. 6 in [5] together with Cor. 2.3 give the next corollary.

Corollary 2.4 ([5])**.** *Let G be a connected plane graph with outer cycle C that is a k-cycle with $3 \leqslant k \leqslant 6$. If every vertex of G has a list of size at least $max(5, k+1)$, then a precoloring of C extends to all of G.*

The next lemma is used repeatedly in the proof of Thms. 3.3 and 4.3. It is an extension of the similar result for 5-list-colorings in [5]. The parameters are motivated by the "Largest Face" and $H(\varepsilon)$-list sizes from Table 1.

Lemma 2.5. *Let H be a connected graph with a 2-cell embedding on S_ε, $\varepsilon > 0$, and let f be a 2-cell k-region of H, $k \geqslant 3$. Let G be a plane graph embedded within f and let G_f be a simple, connected graph that consists of G, $H[E(f)]$, and edges joining $V(G)$ and $V(f)$ so that G_f is embedded in the closure of f. Let $P = \{v_1, \ldots, v_j\}$ be a subset of $V(G_f)$ satisfying $dist(P) \geqslant 3$. Then if every vertex of G_f has an ℓ-list except that the vertices of P each have a 1-list, every proper precoloring of $H[E(f)]$ extends to a list-coloring of G_f provided that no vertex of P is adjacent to a vertex of $V(f)$ with the same color as its 1-list, and*

1. *$k = 3$ and $\ell \geqslant 6$,*
2. *$k \geqslant 4$ and $\ell \geqslant k + 2$, or*
3. *$k = 6$ or $k \geqslant 9$, $\ell = k + 1$, and there is no vertex x adjacent to $k + 1$ vertices of $V(f) \cup \{v_i\}$, for some $i = 1, \ldots, j$, with x's list consisting of $\ell = k + 1$ colors that all appear on $V(f) \cup \{v_i\}$.*

Proof. Note that $G_f[E(f)] = H[E(f)]$. Also note that the condition $dist(P) \geqslant 3$ guarantees that no vertex of G_f is adjacent to more than one v_i. For $v_i \in P \setminus V(f)$, we say that we *excise* v_i if we delete it and delete its color from the list of colors for each neighbor that is not precolored. The proof has three cases that together prove parts 1-3 of the lemma.

Case A. Assume $k = 3$ and $\ell \geqslant 6$, $4 \leqslant k \leqslant 6$ and $\ell \geqslant k + 2$, or $k = 6$ and $\ell = 7$. In these cases first we excise the vertices of $P \setminus V(f)$ so that every remaining vertex of G has a list of size at least 5 for $k = 3$, of size at least $k + 1$ for $k = 4, 5, 6$, or else of size at least 6 when $k = 6$.

In the following we may need to do some surgery, perhaps repeatedly, on the face f and its boundary, so that we can apply Cor. 2.4. First, more easily, when f is a 2-cell

k-region on which lies no repeated vertex, then G_f is a plane graph with outer cycle a k-cycle, $k \leqslant 6$. By Cor. 2.4 a precoloring of $G_f[E(f)]$ extends to $G_f \setminus P$ and this coloring extends to all of G_f unless there is a vertex x with a 6-list, adjacent to six vertices of $V(f)$ with the six colors of x's list. If x's list was decreased to a 6-list, x was adjacent to some vertex v_i, but this situation is disallowed by hypothesis in part 3.

Otherwise in a traversal of W_f we visit a vertex more than once and may travel along an edge twice. In the former case, each time we revisit a vertex x, we can split that vertex in two, into x_1 and x_2, and similarly divide the edges incident with x so that the face f is expanded to become the new face f', still a k-region, and the graph G_f becomes $G_{f'}$ which is naturally embedded in the closure of f' and contains the same adjacencies. Now there is one more vertex in $V(f')$ and the same set of edges $E(f') = E(f)$ on the boundary and in the boundary subgraph $G_{f'}[E(f')]$. A precoloring of $G_f[E(f)]$ gives a precoloring of $G_{f'}[E(f')]$ in which vertices x_1 and x_2 receive the same color; we call this procedure *vertex-duplication*. In the latter case, when we revisit an edge $e = (y, y')$, we may visit both of its endpoints twice or one endpoint twice and the other just once. We similarly duplicate the edge $e = (y, y')$ by duplicating one or both of its endpoints and splitting e into two new edges e_1 and e_2. Then we divide the other edges incident with e so that G_f becomes $G_{f'}$ which is naturally embedded in the closure of the new face f', still a k-region, but now with one or two more vertices in $V(f')$, the same number of edges in $E(f')$ and in $G_{f'}[E(f')]$, and with one less duplicated edge in $W_{f'}$. A precoloring of $G_f[E(f)]$ gives a precoloring of $G_{f'}[E(f')]$ in which duplicated vertices receive the same color; we call this procedure *edge-duplication*. We note that in both duplications there cannot be a vertex x that is adjacent to both copies of a duplicated vertex (since G_f is a simple graph). As an example, the first graph in Fig. 1 shows a 2-cell face that is a 13-region, in which vertices 3, 6, 7, and 8, are repeated, and edges 3-6 and 8-9 are repeated. Vertex- and edge-duplication produces the second graph, which has a new face that is a 13-region and whose facial walk is a cycle given by 1-8-9-8′-7-2-3-6-5-7′-4-6′-3′-1.

In all cases after vertex- and edge-duplication, the 2-cell k-region f becomes a 2-cell k-region f^* with no repeated vertex or edge on the outer boundary. G_f has been transformed into a plane graph G_{f^*} with outer cycle, $G_{f^*}[E(f^*)]$, of length $k \leqslant 6$. The precoloring of $G_f[E(f)]$ has become a precoloring of $G_{f^*}[E(f^*)]$ with duplicated vertices receiving the same color. Then by Cor. 2.4, the precoloring of $G_{f^*}[E(f^*)]$ extends to $G_{f^*} \setminus P$ and so the precoloring of $G_f[E(f)]$ extends to $G_f \setminus P$ and to all of G_f since the exceptional case of part 3 cannot occur. (Since f^* is at most a 6-region and has a duplicated vertex, it is a t-vertex-region for some $t < 6$, and there cannot be a vertex adjacent to six vertices of $V(f^*)$.)

Case B. Suppose $k \geqslant 7$ and $\ell \geqslant k + 2$ so that in all cases $\ell \geqslant 9$. For $v \in V(G)$, let $E_f(v)$ denote the set of edges joining v with a vertex of $V(f)$. Suppose there is a vertex x of $V(G)$ that is adjacent to at least $k - 3$ vertices of $V(f)$. If $x = v_i$ for some $i, 1 \leqslant i \leqslant j$, then $G_f[E(f) \cup E_f(v_i)]$ can be properly colored by assumption. If $x \neq v_i$ for any $i, 1 \leqslant i \leqslant j$, then x is adjacent to either one or no vertex v_i, and since x has an ℓ-list, $\ell \geqslant k + 2$, the coloring of $G_f[E(f) \cup E_f(v_i)]$ (respectively, $G_f[E(f)]$) extends to x. In all cases $G_f[E(f) \cup E_f(x)]$ divides f into regions of size at most 6, and the coloring

of $G_f[E(f) \cup E_f(x)]$ extends to the interior of each s-region, $3 \leqslant s \leqslant 6$, by Case A since interior vertices, other than the v_i, have 9-lists.

Otherwise every vertex x in G is adjacent to at most $k-4$ vertices of $V(f)$. For each such vertex x we delete from its list the colors of $V(f)$ to which it is adjacent. This may reduce the list for x to one of size six or more. Next we excise the vertices of P in $G \setminus V(f)$, resulting in the planar graph $G \setminus P$ with every vertex having a list of size at least five, which can be list-colored by [15]. This list-coloring is compatible with the precoloring of $G_f[E(f)]$ and extends to P and so to all of G_f.

Case C. The case of $k = 6, \ell = 7$ was covered in Case A. Suppose that $k \geqslant 9$ and $\ell = k+1 \geqslant 10$. Suppose there is a vertex x of $V(G)$ that is adjacent to at least $k-4$ vertices of $V(f)$. As before, if $x = v_i$ for some $i, 1 \leqslant i \leqslant j$, then $G_f[E(f) \cup E_f(v_i)]$ can be properly colored by assumption. If $x \neq v_i$ for any $i, 1 \leqslant i \leqslant j$, then x is adjacent to one or no vertex v_i, and the coloring of $G_f[E(f) \cup E_f(v_i)]$ (resp., $G_f[E(f)]$) extends to x in all cases unless (since $\ell = k+1$) x is adjacent to all vertices of $V(f) \cup \{v_i\}$ for some $i, 1 \leqslant i \leqslant j$, and x's list consists of ℓ colors all appearing on $V(f) \cup \{v_i\}$. We have disallowed this case. Now $G_f[E(f) \cup E_f(x)]$ forms a graph that consists of triangles and s-regions with $s \leqslant 7$. The coloring of $G_f[E(f) \cup E_f(x)]$ extends to the interior of each region by the previous cases, since $\ell \geqslant 10$.

Otherwise every vertex x of G is adjacent to at most $k-5$ vertices of $V(f)$, and we proceed as in the proof of Case B by decreasing the lists of vertices adjacent to $V(f)$ and excising all the v_i to create a planar graph with every vertex having at least a 5-list. The resulting graph is list-colorable with a coloring compatible with that of $G_f[E(f)]$ and extending to G_f. □

3 Results on K_n genus surfaces

Most parts of the proof of the next lemma are clear; these results are used repeatedly in the proof of the main results.

Lemma 3.1. *1. Suppose at most one vertex of K_n has a 1-list, at least one vertex has an n-list, and the remaining vertices have $(n-1)$-lists or n-lists. Then K_n can be list-colored.*

2. If one vertex of DK_n has a 1-list and all other vertices have n-lists, then DK_n can be list-colored.

3. If at most six vertices of DK_n, $n \geqslant 7$, have lists of size $n-1$ and all others have n-lists, then DK_n can be list-colored.

Proof. We include the proof of part 3. Suppose that one of the two vertices of degree $n-1$, say x, has an n-list. Then $K_n = DK_n \setminus \{x\}$ has at most six vertices with $(n-1)$-lists and can be list-colored since $n \geqslant 7$. This coloring extends to x which has an n-list and is adjacent to $n-1$ vertices of the colored K_n. Otherwise both vertices of degree $n-1$, say x and y, have $(n-1)$-lists, $L(x)$ and $L(y)$ respectively. Suppose there is a common color

c in $L(x)$ and $L(y)$. Then coloring x with c extends to a coloring of $K_n = DK_n \setminus \{y\}$ after which y can also be colored with c. Otherwise $L(x)$ and $L(y)$ are disjoint. Suppose that when $DK_n \setminus \{y\}$ is list-colored, the colors on $K_{n-1} = DK_n \setminus \{x, y\}$ are precisely the $n-1$ colors of $L(y)$ so that the coloring does not extend. If there is some vertex z of K_{n-1} with an n-list that contains a color not in $L(y)$ and different from the color c_x used on x, we use c_x on z, freeing up the previous color of z for y. Otherwise, for every z with an n-list, that list equals $L(y) \cup \{c_x\}$. Besides these vertices of K_{n-1} with prescribed n-lists, there are at most four others in K_{n-1} which have $n-1$ lists. These four vertices might be colored with colors from $L(x)$, but that still leaves at least one color $c'_x \neq c_x$ in $L(x)$ that has not been used. We change the color of x to c'_x and the color of one of the n-list vertices of K_{n-1} to c_x, thus freeing up that vertex's previous color to be used on y. □

Theorem 3.2. *Suppose G embeds on S_ε, $\varepsilon > 0$, and does not contain $K_{H(\varepsilon)}$. Then when every vertex of G has an $H(\varepsilon)$-list except that the j vertices of $P = \{v_1, \ldots, v_j\}$, $j \geqslant 0$, have 1-lists and $dist(P) \geqslant 3$, then G is list-colorable.*

Proof. Let G embed on S_ε, $\varepsilon > 0$, and suppose G does not contain $K_{H(\varepsilon)}$. We excise the vertices of $P = \{v_1, \ldots, v_j\}$, if present, leaving a graph with all vertices having at least $(H(\varepsilon) - 1)$-lists since $dist(P) \geqslant 3$. By [5, 12], the smaller graph can be list-colored, and that list-coloring extends to G. □

In particular this result holds for all graphs on the Klein bottle since K_7 does not embed there. The first value not covered by the next theorem is $\varepsilon = 3$ with $H(\varepsilon) = 7$.

Theorem 3.3. *Suppose G has a 2-cell embedding on S_ε, $\varepsilon > 0$, and contains $K_{H(\varepsilon)}$. Then when every vertex of G has an $H(\varepsilon)$-list except that the j vertices of $P = \{v_1, \ldots, v_j\}$, $j \geqslant 0$, have 1-lists, G is list-colorable provided that ε is of the form $\varepsilon = (3i^2 - i)/2$, $(3i^2 + i)/2$, or $(3i^2 + 3i + 2)/2$, for some $i \geqslant 1$, and $dist(P) \geqslant 4$.*

Proof. We know that $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on S_ε for $\varepsilon = 1, 4$ as does K_7 on $\mathcal{T}$. (K_6 and K_7 may or may not have 2-cell embeddings on $\mathcal{K}$ and on S_3, respectively.)

The values $\varepsilon = (3i^2 - i)/2$, $(3i^2 + i)/2$, or $(3i^2 + 3i + 2)/2$ for some $i \geqslant 1$ are those for which $K_{H(\varepsilon)}$ necessarily has a 2-cell embedding on S_ε; they give the value of the genus surface of $K_{H(\varepsilon)}$ for each of the modulo 3 classes of $H(\varepsilon)$. Since $dist(P) \geqslant 4$, at most one vertex $v_k \in P$ is in or is adjacent to a vertex of $K_{H(\varepsilon)}$ (but not both), and in the latter case v_k is adjacent to at most $H(\varepsilon) - 1$ vertices of the complete graph since $K_{H(\varepsilon)+1}$ does not embed on S_ε. Thus in all cases $K_{H(\varepsilon)} \cup P$ can be list-colored by Lemma 3.1.1. When $\varepsilon = 1$, $H(\varepsilon) = 6$, and K_6 embeds as a triangulation on S_1. When $\varepsilon > 1$, if $\varepsilon = (3i^2 - i)/2$ or $(3i^2 + i)/2$, $K_{H(\varepsilon)}$ embeds as a triangulation, and if $\varepsilon = (3i^2 + 3i + 2)/2$, $K_{H(\varepsilon)}$ embeds with the largest face size at most five, and in all cases $H(\varepsilon) \geqslant 7$. Hence we apply Lemma 2.5 for $\varepsilon \geqslant 1$ to see that the list-coloring of $K_{H(\varepsilon)}$ extends to the interior of each of its faces and so G is list-colorable. □

A similar proof would show that when the orientable surface S_ε with ε even is the orientable genus surface for $K_{H(\varepsilon)}$ (i.e., when ε is even and gives the least Euler genus

such that $K_{H(\varepsilon)}$ embeds on orientable S_ε), then for every G with a 2-cell embedding on orientable S_ε and containing $K_{H(\varepsilon)}$ the same list-coloring result holds. The first corollary of Section 1 also follows easily.

Proof of Cor. 1.4. Suppose $H(\varepsilon) = 3i + 3, i \geqslant 1$. If $\varepsilon = (3i^2 - i)/2$, then $K_{H(\varepsilon)}$ embeds with $f = (i + 1)(3i + 2)$ faces by Lemma 2.1.1. $K_{H(\varepsilon)}$ contains $(3i + 3)(3i + 2)(3i + 1)/6$ 3-cycles, more than the number of faces so that $K_{H(\varepsilon)}$ embeds with a noncontractible 3-cycle. Thus in this case G cannot contain $K_{H(\varepsilon)}$ and by Thm. 3.2, G can be list-colored. If $\varepsilon = (3i^2 - i + 2)/2$, ..., or $(3i^2 + i - 2)/2$, then $K_{H(\varepsilon)}$ embeds with fewer than $f = (i + 1)(3i + 2)$ faces and so the same result holds.

When $H(\varepsilon) = 3i+4$ or $3i+5, i \geqslant 1$, an analogous proof shows that G cannot contains $K_{H(\varepsilon)}$ and so is list-colorable.

To see that distance at least 3 is best possible for the precolored vertices, take a vertex x with a k-list $L(x)$ and attach k pendant edges to vertices, precolored with each of the colors of $L(x)$. □

4 All surfaces

First we explore some topology of surfaces and non-2-cell faces of embedded graphs. Cycles on surfaces (i.e., simple closed curves on the surface), for both orientable and nonorientable surfaces, are of three types: contractible and surface-separating, noncontractible and surface-separating, and noncontractible and surface-nonseparating. (When the meaning is clear, we suppress the prefix "surface.") A non-2-cell face of an embedded graph must contain a noncontractible surface cycle within its interior. For example, in the second graph in Fig. 1, the shaded region is a 2-cell face, and the unshaded region is a non-2-cell face that contains a noncontractible and nonseparating cycle. (For a more detailed discussion see Chapters 3 and 4 of [13].)

Suppose f is a non-2-cell face of $K_{H(\varepsilon)}$ embedded on S_ε. We repeatedly "cut" along simple noncontractible surface cycles that lie wholly within the face f until the "derived" face or faces become 2-cells. Each "cut" is replaced with one or two disks, creating a new surface, and with each "cut" $K_{H(\varepsilon)}$ stays embedded on a surface $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon$. Below we explain this surface surgery and count the number of newly created faces, called *derived* faces in the surgery.

Lemma 4.1. *Suppose $K_{H(\varepsilon)}$ embeds on S_ε, $\varepsilon > 0$. Then the largest possible 2-cell face in the embedding is an $(H(\varepsilon) - 1)$-region.*

Proof. Suppose the embedded $K_{H(\varepsilon)}$ has a non-2-cell k-region f; initially there are no derived faces. In f we can find a simple noncontractible cycle C, disjoint from its boundary, $V(f) \cup E(f)$. If C is surface-separating, it is necessarily 2-sided. We replace C by two copies of itself, C and C', and insert in each copy a disk, producing surfaces $S(1)$ and $S'(1)$, each with Euler genus that is positive and less than ε. Since $K_{H(\varepsilon)}$ is connected, it is embedded on one of these surfaces, say $S(1)$. The face f of $K_{H(\varepsilon)}$ on S_ε becomes the derived face f_1 of $K_{H(\varepsilon)}$ on $S(1)$ and retains the same set of boundary vertices $V(f_1) = V(f)$

and edges $E(f_1) = E(f)$ so that f_1 is also a k-region. Initially f is not a derived face, f_1 becomes a derived face and the Euler genus decreases by at least 1. If, later on in the process, f is a derived face, then f_1 is also a derived face, the number of derived faces does not increase, and the Euler genus decreases by at least 1.

If C is not surface-separating and is 2-sided, we duplicate it and sew in two disks, as above, to create one new surface $S(1)$ of lower and positive Euler genus on which $K_{H(\varepsilon)}$ is embedded. If C was not separating within the face f, then the derived face f_1 keeps the same set of boundary vertices and edges as f and remains a k-region. As above, the number of derived faces increases by at most 1 and the Euler genus decreases by at least 2. If C was separating within the face f, then f splits into two derived faces f_1 and f_1'. Each vertex of $V(f)$ and each edge of $E(f)$ appears on one of these derived faces or possibly two when it was a repeat on f. More precisely, if f_1 is a k_1-region and f_1' is a k_1'-region, then necessarily $k_1 + k_1' = k$. In this case the Euler genus decreases by 2 and number of derived faces increases by at most 2, increasing by 2 only when the face being cut was an original face of $K_{H(\varepsilon)}$. If C is not surface-separating and is 1-sided, we replace C by a cycle DC of twice the length of C and insert a disk within DC, producing a surface $S(1)$ with Euler genus that is less than ε. $K_{H(\varepsilon)}$ remains embedded on $S(1)$, necessarily with positive Euler genus, and the derived face f_1 keeps the same boundary vertices and edges as f, remaining a k-region. Thus the number of derived faces increases by at most 1 and the Euler genus decreases by at least 1.

Now we prove the lemma by induction on the number of non-2-cell faces of the embedded $K_{H(\varepsilon)}$. We know the conclusion holds when there are no non-2-cell faces by Lemma 2.2. Otherwise let f be a non-2-cell k-region. We repeatedly cut along simple noncontractible cycles within f and its derived faces, creating surfaces $S(1)$, $S(2)$, ... on which $K_{H(\varepsilon)}$ remains embedded. We continue until every derived face of f is a 2-cell. Then $K_{H(\varepsilon)}$ is embedded on, say, $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon$ and has fewer non-2-cell faces. By induction each 2-cell face has size at most $H(\varepsilon) - 1$ and thus every original 2-cell face, which has not been affected by the surgery, also has size at most $H(\varepsilon) - 1$. □

We have purposefully proved more within the previous proof.

Corollary 4.2. *Suppose $K_{H(\varepsilon)}$ has a non-2-cell embedding on S_ε, and suppose that after cutting along noncontractible cycles in non-2-cell faces, $K_{H(\varepsilon)}$ has a 2-cell embedding on $S_{\varepsilon'}$, $\varepsilon' < \varepsilon$. Then the number of faces in the latter embedding that are derived from faces in the original embedding is at most $\varepsilon - \varepsilon'$.*

Proof. In the previous proof we saw that with some cuts the number of derived faces is increased by at most 1 and the Euler genus is decreased by at least 1; let c_0 denote the number of cuts in which there is no increase in the number of derived faces and c_1 the number of cuts in which there is an increase of 1 in the number of derived faces. If the increase is always at most 1, then the result follows. The number of derived faces is increased by 2 precisely when the cutting cycle C within a face f' is 2-sided, is not surface-separating, is separating within f', and f' is an original face of the embedding. In that case the Euler genus is decreased by 2 also; let c_2 denote the number of such cuts.

Then the decrease in the Euler genus, $\varepsilon - \varepsilon'$ is at least $c_0 + c_1 + 2c_2 \geqslant c_1 + 2c_2$, which equals the number of derived faces. □

Theorem 4.3. *Given $\varepsilon > 0$ and G a graph on n vertices that has a 2-cell embedding on S_ε, suppose that G contains $K_{H(\varepsilon)}$. If $P \subset V(G)$ satisfies $dist(P) \geqslant 4$, then if the vertices of P each have a 1-list and every other vertex of G has an $H(\varepsilon)$-list, then G can be list-colored.*

Proof. The proof is by induction on ε and on n. We know the theorem holds for G with a 2-cell embedding on S_ε for $1 \leqslant \varepsilon \leqslant 2$ by Thm. 3.3. Consider graphs with 2-cell embeddings on S_{ε^*} for $\varepsilon^* \geqslant 3$. For each such embedded graph, the subgraph $K_{H(\varepsilon^*)}$ inherits an embedding on S_{ε^*}, and $H(\varepsilon^*) \geqslant 7$.

Since $dist(P) \geqslant 4$ we know that at most one vertex of P lies in or is adjacent to a vertex of $K_{H(\varepsilon^*)}$. If there is one, call it v_i^* and if not, ignore reference to v_i^* in the following. By Lemma 3.1.1 we know that $G[V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}]$ can be list-colored since v_i^* is adjacent to at most $H(\varepsilon^*) - 1$ vertices of $K_{H(\varepsilon^*)}$ (because $K_{H(\varepsilon^*)+1}$ does not embed on S_{e^*}). If G contains a vertex x in neither $V(K_{H(\varepsilon^*)})$ nor P, then $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$ can be list-colored by first coloring $K_{H(\varepsilon^*)}$ and then coloring x, which has an $H(\varepsilon^*)$-list and is adjacent to at most $H(\varepsilon^*) - 1$ vertices of $K_{H(\varepsilon^*)}$.

Thus on surface S_{ε^*} we know the result holds for every graph on n vertices with $n \leqslant H(\varepsilon^*) + 1$. Let G have n^* vertices, $n^* > H(\varepsilon^*) + 1$, and have a 2-cell embedding on S_{ε^*}.

Let f be a k-region in the inherited embedding of $K_{H(\varepsilon^*)}$ with incident vertices $V(f)$ and edges $E(f)$, and let G_f denote the subgraph of G lying in the closure of f, $f \cup V(f) \cup E(f)$. Suppose f is a 2-cell face of $K_{H(\varepsilon^*)}$ in whose interior lie vertices of $V(G) \setminus \{V(f) \cup \{v_i^*\}\}$; call these interior vertices U_f. Then after deleting the vertices of U_f, $G \setminus U_f$ has a 2-cell embedding on S_{ε^*} with fewer than n^* vertices, contains $K_{H(\varepsilon^*)}$, and contains vertices of $P' \subseteq P$ with $dist(P') \geqslant 4$. By induction $G \setminus U_f$ is list-colorable. By Lemma 4.1 $k \leqslant H(\varepsilon^*) - 1$. We claim that the resulting list-coloring of $G[V(f) \cup \{v_i^*\}]$ extends to G_f.

If $k \leqslant H(\varepsilon^*) - 2$, then the coloring extends by Lemma 2.5.1 and 2.5.2. Otherwise $k = H(\varepsilon^*) - 1$ and the coloring then extends by Lemma 2.5.3, unless there is a vertex x of G_f that has an $H(\varepsilon^*)$-list, is adjacent to v_i^*, not in $V(f)$, and to all vertices of $V(f)$, and its $H(\varepsilon^*)$-list consists of $H(\varepsilon^*)$ colors that appear on its neighbors. Then $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$ forms $DK_{H(\varepsilon^*)}$, which triangulates S_{ε^*} and does not contain another vertex of P since $dist(P) \geqslant 4$. Since v_i^* is adjacent to at most three vertices of $DK_{H(\varepsilon^*)}$ (the vertices of a 3-region), $G[V(DK_{H(\varepsilon^*)}) \cup \{v_i^*\}]$ can be list-colored by Lemma 3.1.3. Then the list-coloring extends to the graph in the interior of each 3-region by Lemma 2.5.1 since $H(\varepsilon^*) \geqslant 7$.

Thus we can assume that every vertex of $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$ lies in a non-2-cell region of the embedding of $K_{H(\varepsilon^*)}$ on S_{ε^*}. We claim there are two vertices of $K_{H(\varepsilon^*)}$ that lie only on its 2-cell faces; we prove that below. One of these might lie in P or be adjacent to v_i^*, but the other, say x^*, has an $H(\varepsilon^*)$-list and is adjacent only to vertices of $K_{H(\varepsilon^*)}$, precisely $H(\varepsilon^*) - 1$ of these.

In that case we consider $G \setminus \{x^*\}$. If $G \setminus \{x^*\}$ does not contain $K_{H(\varepsilon^*)}$, it can be list-colored by Thm. 3.2. Otherwise $G \setminus \{x^*\}$ does contain $K_{H(\varepsilon^*)}$. $G \setminus \{x^*\}$ might have a 2-cell embedding on S_{ε^*} or it might not. In the former case, by induction on n it can be list-colored. Suppose that $G \setminus \{x^*\}$ does not have a 2-cell embedding on S_{ε^*}. Then the face f^* that was formed by deleting x^* is the one and only non-2-cell face of that embedding since no other face of G has been changed by the deletion of x^*. Then we cut along noncontractible cycles within f^*, as described in Lemma 4.1, until every face, derived from f^*, is a 2-cell in $G \setminus \{x^*\}$ now embedded on $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon^*$. We have $H(\varepsilon') = H(\varepsilon^*)$ since $G \setminus \{x^*\}$ contains $K_{H(\varepsilon^*)}$. Thus $G \setminus \{x^*\}$ can be list-colored by induction on the Euler genus, and in all cases that coloring extends to G since x^* has a list of size $H(\varepsilon^*)$ which is larger than its degree.

We return to the claim that there are two vertices of $K_{H(\varepsilon^*)}$ that lie only on 2-cell faces of its embedding on S_{ε^*}, given that every vertex of $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$ lies in a non-2-cell face of the embedded $K_{H(\varepsilon^*)}$. Since the number of vertices of G, n^*, is greater than $H(\varepsilon^*) + 1$, there are some non-2-cell faces containing other vertices of G. We count the maximum number of vertices of $K_{H(\varepsilon^*)}$ that lie on these non-2-cells to show that number is at most $H(\varepsilon^*) - 2$.

As in Lemma 4.1 we repeatedly cut each non-2-cell face of the embedded $K_{H(\varepsilon^*)}$ until all remaining faces, the original and the derived, are 2-cells; suppose $K_{H(\varepsilon^*)}$ is then embedded on $S_{\varepsilon'}$ with $\varepsilon' < \varepsilon^*$. We know that every vertex originally on a non-2-cell face of $K_{H(\varepsilon^*)}$ is represented on at least one derived face and we show below that the total number of vertices on derived faces is at most $H(\varepsilon^*) - 2$. We also know that $\varepsilon' \geqslant I(H(\varepsilon^*))$. Let $n_1 = \varepsilon' - I(H(\varepsilon^*))$, which is nonnegative, and $n_2 = \varepsilon^* - \varepsilon'$, which is positive. The variable n_1 will determine the face sizes in the 2-cell embedding of $K_{H(\varepsilon^*)}$ on $S_{\varepsilon'}$ (see Table 1), and n_2 will determine the maximum number of derived faces that have been created.

We consider the modulo 3 class of $H(\varepsilon^*)$, and we begin with the case of $H(\varepsilon^*) = 3i+4$, $i \geqslant 1$. We know that $\varepsilon^* \in \{(3i^2 + i)/2, \ldots, (3i^2 + 3i)/2\} = \{I(3i + 4), \ldots, I(3i + 4) + i\}$ so that $n_1 + n_2 \leqslant i$ by Lemma 2.1. By Cor. 4.2 the number of derived faces is at most n_2. We can determine the possible face sizes of a 2-cell embedding of $K_{H(\varepsilon^*)}$ on $S_{\varepsilon'}$ with $\varepsilon' = I(3i + 4) + n_1$. A 2-cell embedding on $S_{I(3i+4)}$ is necessarily a triangulation. A 2-cell embedding on $S_{I(3i+4)+1}$ consists of triangles except possibly for one 6-region, or triangles plus two faces whose sizes sum to 9, or triangles plus three faces whose sizes sum to 12 (necessarily three 4-regions). More generally when $\varepsilon' = I(3i + 4) + n_1$, then the embedding might consist of triangles plus one $(3n_1 + 3)$-region, or triangles plus two faces whose sizes sum to $3n_1 + 6$, or triangles plus three faces whose sizes sum to $3n_1 + 9$, etc. And if we choose n_2 faces, all the derived faces, the sum of their sizes can be at most $3n_1 + 3n_2 \leqslant 3i < 3i + 2 = H(\varepsilon^*) - 2$.

For $i \geqslant 1$, the same calculation holds when $H(\varepsilon^*) = 3i + 3$, and when $H(\varepsilon^*) = 3i + 5$, a similar count will work. In the latter case we have $n_1 + n_2 \leqslant i - 1$, though the face sizes may be slightly larger. A 2-cell embedding of $K_{H(\varepsilon^*)}$ on $S_{I(3i+5)}$ may have triangles plus a 5-region or triangles plus two 4-regions. In general a 2-cell embedding of $K_{H(\varepsilon^*)}$ on $S_{I(3i+5)+n_1}$ might have triangles plus one $(3n_1 + 5)$-region or triangles plus two regions whose sizes sum to $3n_1 + 8$, etc. With n_2 faces, all the derived faces, their sum of sizes

can be at most $3n_1 + 3n_2 + 2 \leqslant 3i - 1 < 3i + 3 = H(\varepsilon^*) - 2$. □

We now complete the proof our main result, Thm. 1.3.

Proof of Thm. 1.3. If G has a non-2-cell embedding on S_ε that contains $K_{H(\varepsilon)}$, we can perform surgery on the non-2-cell faces, as we did in the proof of Lemma 4.1 and Thm. 4.3, to obtain a 2-cell embedding of G on a surface of Euler genus $\varepsilon' < \varepsilon$ that still contains $K_{H(\varepsilon)}$, and hence $H(\varepsilon') = H(\varepsilon)$. We can thus apply Thm. 4.3 to G on $S_{\varepsilon'}$. This shows that the result holds for every embedding, 2-cell or non-2-cell, and Thm. 1.3 follows. □

The distance bound of 4 in Thms. 1.3 and 4.3 is best possible, for consider $K_{H(\varepsilon)}$ with a pendant edge attaching a degree-1 vertex to each vertex of $K_{H(\varepsilon)}$. Give each degree-1 vertex the list $\{1\}$ and place that vertex in the set P. When every other vertex has an identical $H(\varepsilon)$-list that contains 1, the graph is not list-colorable and $dist(P) = 3$.

The second corollary of Section 1 now follows easily.

Proof of Cor. 1.5. Let $f_1, \ldots, f_j$ be the faces with vertices with smaller lists. Add a vertex x_i to f_i and make it adjacent to all vertices of $V(f_i)$. Give each x_i a 1-list $\{\alpha\}$ where α appears in no list of a vertex of G, and add α to the list of each vertex of $V(f_i)$, now the neighbors of x_i. Then $G \cup \{x_1, \ldots, x_j\}$ can be list-colored by Thm. 1.3 since with $P = \{x_1, \ldots, x_j\}$, $dist(P) \geqslant 4$, and this coloring is a list-coloring of G. □

5 Concluding Questions

1. Škrekovski [14] has shown the extension of Dirac's theorem that if G is embedded on S_ε, $\varepsilon \geqslant 5$, $\varepsilon \neq 6, 9$, and does not contain $K_{H(\varepsilon)-1}$ or $K_{H(\varepsilon)-4} + C_5$, then G can be $(H(\varepsilon) - 2)$-colored. Is the same true for list-coloring?

2. If G embeds on S_ε and does not contain one of the two graphs of Question 1, if the vertices of one face have at least $(H(\varepsilon) - 2)$-lists, and if all other vertices have at least $H(\varepsilon)$-lists, can G be list-colored?

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