Generalized Alcuin's Sequence

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Abstract

We introduce a new family of sequences $\{t_k(n)\}_{n=-\infty}^{\infty}$ for given positive integer $k \geqslant 3$. We call these new sequences as *generalized Alcuin's sequences* because we get Alcuin's sequence which has several interesting properties when $k = 3$. Also, $\{t_k(n)\}_{n=0}^{\infty}$ counts the number of partitions of $n-k$ with parts being k, $(k-1), 2(k-1), 3(k-1), \ldots, (k-1)(k-1)$. We find an explicit linear recurrence equation and the generating function for $\{t_k(n)\}_{n=-\infty}^{\infty}$. For the special case $k = 4$ and $k = 5$, we get a simpler formula for $\{t_k(n)\}_{n=-\infty}^{\infty}$ and investigate the period of $\{t_k(n)\}_{n=-\infty}^{\infty}$ modulo a fixed integer. Also, we get a formula for $p_5(n)$ which is the number of partitions of n into exactly 5 parts.

Keywords: Alcuin's sequence, integer partition.

1 Introduction

Alcuin of York (c. 740-804) lived over four hundred years before Fibonacci. Like Fibonacci, Alcuin has a sequence of integers named after him. Alcuin's sequence $\{t_3(n)\}_{n=0}^{\infty}$ has several interesting properties and it can be defined as the number of incongruent integer triangles of perimeter n. It is also related to the solutions to the flask-sharing problem (see pages 150 to 165 of [4]) and to $p_3(n)$, the number of partitions of n into exactly 3 parts. If n is even then $t_3(n) = p_3\left(\frac{n}{2}\right)$, else $t_3(n) = t_3(n+3)$; see [3]. So we can write,

$$t_3(n) = \begin{cases} p_3\left(\dfrac{n}{2}\right), & \text{if } n \equiv 0 \pmod 2, \\ p_3\left(\dfrac{n+3}{2}\right), & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

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We now introduce a new sequence $\{t_k(n)\}_{n=0}^{\infty}$ for given positive integer $k \geqslant 3$ that is a generalization of the Alcuin's sequence:

$$t_k(n) = p_k\left(\frac{n+a_i}{k-1}\right), \quad \text{if } n \equiv i \pmod{k-1},$$

where if $i = 0$ then $a_i = 0$ else $a_i = k(k-1-i)$ and $p_k(x)$ is the number of partitions of x into exactly k parts. We call $\{t_k(n)\}_{n=0}^{\infty}$ a *generalized Alcuin's sequence*. Later on, we extend this definition so that $t_k(n)$ ranges from $n = -\infty$ to ∞.

In Section 2, we find an explicit linear recurrence equation and the generating function for $\{t_k(n)\}_{n=-\infty}^{\infty}$. From the generating function, we can see that $t_k(n)$ counts the number of partitions of $n-k$ with parts being k, $(k-1)$, $2(k-1)$, $\ldots$, $(k-1)(k-1)$. In Sections 3 and 4, we obtain simpler formulae for $\{t_k(n)\}_{n=-\infty}^{\infty}$ with $k = 4$ and $k = 5$, respectively, and using these formulae we study the period of $\{t_k(n)\}_{n=-\infty}^{\infty}$ modulo a fixed integer for the special cases $k = 4$ and $k = 5$. For $k = 3$ the Alcuin sequence modulo m has least period $12m$ [3]. Motivated by this result we study the least period of generalized Alcuin's sequences for bigger k. For any integer $m \geqslant 2$, the sequence $\{t_4(n) \pmod{m}\}_{n=-\infty}^{\infty}$ is periodic with the least period $36m$. Similarly, for $k = 5$, the least period is either $240m$ for odd m, or $480m$ for even m. As a by-product, we also obtain a formula for $p_5(n)$, the number of partitions of n into exactly 5 parts. Conclusions and further work are given in Section 5.

2 Generalization of Alcuin's Sequence

Definition 1 (Generalized Alcuin's Sequence)**.** Let us define the sequence $\{t_k(n)\}_{n=0}^{\infty}$

$$t_k(n) = p_k\left(\frac{n+a_i}{k-1}\right), \quad \text{if } n \equiv i \pmod{k-1},$$

where if $i = 0$ then $a_i = 0$ else $a_i = k(k-1-i)$ and $p_k(x)$ is the number of partitions of x into exactly k parts.

As we will see later, $t_k(n)$ is the number of partitions of $n-k$ such that the parts are $k, (k-1), 2(k-1), 3(k-1), \ldots, (k-1)(k-1)$.

If we take $k = 3$ in Definition 1, we get Alcuin's sequence :

$$t_3(n) = \begin{cases} p_3\left(\dfrac{n}{2}\right), & \text{if } n \equiv 0 \pmod{2}, \\ p_3\left(\dfrac{n+3}{2}\right), & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Obviously, we can get

$$t_k(n(k-1)) = p_k(n) \tag{1}$$

and, for $1 \leqslant i \leqslant k-2$,

$$\begin{aligned} t_k\left(n\left(k-1\right)+i\right) &= p_k\left(\frac{n\left(k-1\right)+i+a_i}{k-1}\right) = p_k\left(\frac{nk-n+i+k^2-k-ki}{k-1}\right) \\ &= p_k\left(\frac{\left(n+k-i\right)\left(k-1\right)}{k-1}\right) = p_k\left(n+k-i\right). \end{aligned} \tag{2}$$

We use these facts in the following.

Theorem 2. *The generating function of the sequence* $\{t_k(n)\}_{n=0}^{\infty}$ *is*

$$F_k\left(x\right) = \frac{x^k}{\left(1-x^k\right)\prod_{i=1}^{k-1}\left(1-x^{i(k-1)}\right)}.$$

Proof. The generating function for $\{t_k(n)\}_{n=0}^{\infty}$ is

$$F_k\left(x\right) = \sum_{n=0}^{\infty} t_k\left(n\right)x^n.$$

Then we can write, using Equations (1) and (2),

$$\begin{aligned} F_k\left(x\right) &= \sum_{n=0}^{\infty}\sum_{i=0}^{k-2} t_k\left(n\left(k-1\right)+i\right)x^{n(k-1)+i} \\ &= \sum_{n=0}^{\infty}\left(p_k\left(n\right)x^{n(k-1)} + \sum_{i=1}^{k-2} p_k\left(n+k-i\right)x^{n(k-1)+i}\right). \end{aligned} \tag{3}$$

Formulas for the generating functions of the sequences $\{p_k(n)\}$ are well known (see [2], for example):

$$G_k\left(x\right) = \sum_{n=0}^{\infty} p_k\left(n\right)x^n = \frac{x^k}{\prod_{i=1}^{k}\left(1-x^i\right)}.$$

Then, for $0 \leqslant j \leqslant k-1$, we derive

$$G_k^j\left(x\right) = \sum_{n=0}^{\infty} p_k\left(n+j\right)x^n = \frac{x^{k-j}}{\prod_{i=1}^{k}\left(1-x^i\right)}.$$

Substituting in Equation (3), we get

$$\begin{aligned} F_k\left(x\right) &= G_k^0\left(x^{k-1}\right) + \sum_{i=1}^{k-2} x^i G_k^{k-i}\left(x^{k-1}\right) \\ &= \frac{x^k\left(1+x^k+x^{2k}+\cdots+x^{(k-2)k}\right)}{\prod_{i=1}^{k}\left(1-x^{i(k-1)}\right)}. \end{aligned}$$

The proof is complete using the fact that

$$1-x^{k(k-1)} = \left(1-x^k\right)\left(1+x^k+x^{2k}+\cdots+x^{(k-2)k}\right).$$

□

It is now clear that $t_k(n)$ counts the number of partitions of $n-k$ into parts $k, (k-1)$, $2(k-1), 3(k-1), \ldots, (k-1)(k-1)$.

Let $p(x) = \left(1-x^k\right)\prod_{i=1}^{k-1}\left(1-x^{i(k-1)}\right)$ and c_i be the coefficient of the term x^i in the expansion of $p(x)$. Let ℓ be the degree of $p(x)$; then $\ell = \dfrac{k\left(k^2-2k+3\right)}{2}$.

Theorem 3. *The sequence* $\{t_k(n)\}_{n=0}^{\infty}$ *is determined by the linear recurrence relation of order* ℓ

$$t_k(n) = -\sum_{i=1}^{\ell} c_i t_k(n-i),$$

for $n \geqslant \ell$, *where* c_i*'s are the coefficients of* $p(x)$ *defined above, and the initial values of* $t_k(n)$ *for* $0 \leqslant n \leqslant \ell-1$ *are obtained from Definition 1.*

Proof. By the generating function of the sequence $\{t_k(n)\}_{n=0}^{\infty}$, we have

$$\begin{aligned} x^k &= p(x)\sum_{n=0}^{\infty} t_k(n)\,x^n \\ &= \sum_{n=0}^{\infty} t_k(n)\left(x^n + c_1x^{n+1} + c_2x^{n+2} + \cdots + c_\ell x^{n+\ell}\right). \end{aligned}$$

The linear recurrence relation of order ℓ can be read off by equating coefficients of x^n for $n \geqslant \ell$. □

Let us use the same recurrence relation in Theorem 3 to define $t_k(n)$ for $n < 0$, so this extends the ranges from $n = -\infty$ *to* ∞.

Corollary 4. *If we take* $k = 3$ *in the sequence* $\{t_k(n)\}_{n=-\infty}^{\infty}$ *we get as special case the Alcuin's sequence in [3]. Also, substituting* $k = 3$ *in Theorem 2 and 3, we get Theorem 2 and Theorem 3 in [3], respectively.*

3 The case $k = 4$

Let us take $k = 4$ in the generalized Alcuin's sequences for $\{t_k(n)\}_{n=0}^{\infty}$ given in Definition 1. We obtain the sequence $\{t_4(n)\}_{n=0}^{\infty}$

$$t_4(n) = \begin{cases} p_4\left(\dfrac{n}{3}\right), & \text{if } n \equiv 0 \pmod 3, \\ p_4\left(\dfrac{n+8}{3}\right), & \text{if } n \equiv 1 \pmod 3, \\ p_4\left(\dfrac{n+4}{3}\right), & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

If we substitute $k = 4$ in Theorem 2 and Theorem 3, we get the generating function

$$F_4(x) = \frac{x^4}{\left(1-x^4\right)\left(1-x^3\right)\left(1-x^6\right)\left(1-x^9\right)} \tag{4}$$

and the linear recurrence relation of order 22

$$\begin{aligned} t_4(n) &= t_4(n-3)+t_4(n-4)+t_4(n-6)-t_4(n-7)-t_4(n-10)-t_4(n-12) \quad (5)\\ &\quad -t_4(n-15)+t_4(n-16)+t_4(n-18)+t_4(n-19)-t_4(n-22) \end{aligned}$$

for $n \geqslant 22$, together with the initial values $0,0,0,0,1,0,0,1,1,0,2,1,1,3,2,1,5,3,2,6,5,3$ of $t_4(n)$ for $0 \leqslant n \leqslant 21$.

Example 5. From the generating function $F_4(x)$, $t_4(n)$ is the number of partitions of $n-4$ with parts $3,4,6$ and 9. For example, $t_4(16)=5$ and 12 can be partitioned in five distinct ways:

$$3+3+3+3, \quad 4+4+4, \quad 3+3+6, \quad 6+6 \quad \text{and} \quad 3+9.$$

Theorem 6. *Let $\|x\|$ be the nearest integer to x. Then for all $n \geqslant 0$,*

$$p_4(n) = \begin{cases} \left\| \dfrac{(n+1)^3}{144} - \dfrac{(n+1)}{48} \right\|, & \text{if } n \text{ is even,} \\ \\ \left\| \dfrac{(n+1)^3}{144} - \dfrac{(n+1)}{12} \right\|, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The generating function of the numbers $p_4(n)$ is

$$G_4(x) = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

Now, by partial fractions we obtain

$$\begin{aligned} G_4(x) &= -\frac{13}{288(x-1)^2} + \frac{1}{24(x-1)^3} + \frac{1}{32(x+1)^2} + \frac{1}{24(x-1)^4} \quad (6)\\ &\quad + \frac{1}{8(x^2+1)} - \frac{1}{9(x^2+x+1)}. \end{aligned}$$

Using the general binomial theorem (see [2], for example), the first four terms of $G_4(x)$ can be written as

$$-\frac{13}{288}\sum_{n=0}^{\infty}(n+1)x^n - \frac{1}{24}\sum_{n=0}^{\infty}\binom{n+2}{n}x^n + \frac{1}{32}\sum_{n=0}^{\infty}(n+1)(-1)^n x^n + \frac{1}{24}\sum_{n=0}^{\infty}\binom{n+3}{n}x^n.$$

Thus the first four terms give the following coefficient of x^n :

$$\frac{2n^3+6n^2-9n+9n(-1)^n+9(-1)^n-13}{288}.$$

The last two terms in Equation (6) can be handled by Maple using Taylor expansions and their contribution to the coefficient of x^n is $c/72$, where c is given in the following table.

n mod 12	0	1	2	3	4	5	6	7	8	9	10	11
c	1	8	-9	-8	17	0	-17	8	9	-8	-1	0

Combining all the contributions, we obtain the formula $p_4(n) = \frac{n^3}{144} + \frac{n^2}{48} - \frac{c-1}{72}$, for n even, and $p_4(n) = \frac{n^3}{144} + \frac{n^2}{48} - \frac{n}{16} + \frac{2c-11}{144}$, for n odd. This can be represented as in the statement of the theorem. □

Remark 7. A search in internet [5] gives the previous formula for $p_4(n)$, but we could not find any proof of it. For the sake of completeness we give such a proof in the previous theorem using generating functions.

Plugging the formula for $p_4(n)$ into the definition of $t_4(n)$ we obtain the following corollary.

Corollary 8. *Let $C = 1/3888$ and $\|x\|$ be the nearest integer to x. Then for all $n \geqslant 0$,*

$$t_4(n) = \begin{cases} \|C(n^3+9n^2-54)\|, & \text{if } n \equiv 0 \pmod 6, \\ \|C(n^3+33n^2+255n+143)\|, & \text{if } n \equiv 1 \pmod 6, \\ \|C(n^3+21n^2+120n+154)\|, & \text{if } n \equiv 2 \pmod 6, \\ \|C(n^3+9n^2-81n-297)\|, & \text{if } n \equiv 3 \pmod 6, \\ \|C(n^3+33n^2+336n+1034)\|, & \text{if } n \equiv 4 \pmod 6, \\ \|C(n^3+21n^2+39n-413)\|, & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Let us use the *same* recurrence relation in (5) to define $\{t_4(n)\}_{n=-\infty}^{\infty}$ for $n < 0$. Then we get

$$\begin{aligned} t_4(n) &= t_4(n+3) + t_4(n+4) + t_4(n+6) - t_4(n+7) - t_4(n+10) - t_4(n+12) \\ &\quad - t_4(n+15) + t_4(n+16) + t_4(n+18) + t_4(n+19) - t_4(n+22) \end{aligned} \tag{7}$$

together with the initial values of $t_4(n)$ for $0 \leqslant n \leqslant 21$. Recall these initial values are $0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 2, 1, 1, 3, 2, 1, 5, 3, 2, 6, 5, 3$. From this equation we see that recurrence relation (5) also holds for any integer n.

By using Equation (7) we can construct the following table of values of $t_4(n)$ when $-22 \leqslant n \leqslant 8$:

n	-22	-21	-20	-19	-18	-17	-16	-15	...	1	2	3	4	5	6	7	8
$t_4(n)$	-1	-1	0	0	-1	0	0	0	...	0	0	0	1	0	0	1	1

Obviously, the sequence is palindromic from the table if we ignore signs; we can easily see that

$$t_4(-n) = -t_4(n-14). \tag{8}$$

Note that $\ell - 2k = 22 - 8 = 14$ for the sequence $\{t_4(n)\}_{n=-\infty}^{\infty}$, where $k = 4$ and $\ell = 22$.

Theorem 9. *For any integer $m \geqslant 2$, the sequence $\{t_4(n)\,(\mathrm{mod}\, m)\}_{n=-\infty}^{\infty}$ is periodic with least period $36m$.*

Proof. Let L be the least period of $\{t_4(n)\,(\mathrm{mod}\, m)\}_{n=-\infty}^{\infty}$. First we prove that $36m$ is a period of the sequence and thus L divides $36m$. That is, for each $m \geqslant 2$,

$$t_4\,(n+36m) \equiv t_4\,(n) \pmod m \quad \text{for any integer } n. \tag{9}$$

In order to show this, we break it into 5 cases.

(1) $n \geqslant 0$: For $n \equiv 0 \pmod 6$, and by Corollary 8,

$$\begin{aligned} t_4\,(n+36m) &= \left\| \frac{n^3}{3888} + \frac{n^2}{432} - \frac{1}{72} \right\| + \frac{n^2}{36}m + \frac{n}{6}m + nm^2 + 12m^3 + 3m^2 \\ &\equiv t_4\,(n) \pmod m\,. \end{aligned}$$

In a similar way one shows that, Equation (9) is also satisfied for $n \equiv 1, 2, 3, 4, 5 \pmod 6$.

(2) $-14 < n < 0$: Equation (9) is also satisfied for $n = -1, -2, -3, \ldots, -13$ by using Corollary 8 and Equation (8).

(3) $-36m < n \leqslant -14$: For $n \equiv 0 \pmod 6$, and using again Corollary 8 and Equation (8),

$$\begin{aligned} t_4\,(n+36m) &= -\left\| -\frac{n^3}{3888} - \frac{n^2}{432} + \frac{1}{72} \right\| + \frac{n^2}{36}m + \frac{n}{6}m + nm^2 + 12m^3 + 3m^2 \\ &\equiv -t_4\,(-n-14) \pmod m \\ &\equiv t_4\,(n) \pmod m\,. \end{aligned}$$

Again, in a similar way, Equation (9) is also satisfied for $n \equiv 1, 2, 3, 4, 5 \pmod 6$.

(4) $-36m - 14 < n \leqslant -36m$: Since $n = -36m - a$ for $0 \leqslant a \leqslant 13$, we have

$$t_4\,(n+36m) = t_4\,(-36m - a + 36m) = t_4\,(-a) \equiv 0 \pmod m$$

and

$$\begin{aligned} t_4\,(n) &= t_4\,(-36m-a) = t_4\,(36m+a-14) \quad \text{(by Equation (8))} \\ &\equiv 0 \pmod m\,, \end{aligned}$$

so that $t_4\,(n+36m) \equiv t_4\,(n) \pmod m$ for $-36m - 14 < n \leqslant -36m$.

(5) $n \leqslant -36m-14$: For $n \equiv 0 \pmod 6$, and using Corollary 8 and Equation (8),

$$\begin{aligned} t_4(n+36m) &= -t_4(-n-36m-14) \\ &= -\left\| -\frac{n^3}{3888} - \frac{n^2}{432} + \frac{1}{72} \right\| + \frac{n^2}{36}m + \frac{n}{6}m + nm^2 + 12m^3 + 3m^2 \\ &\equiv -t_4(-n-14) \pmod m \\ &\equiv t_4(n) \pmod m. \end{aligned}$$

As before, Equation (9) is also satisfied for $n \equiv 1,2,3,4,5 \pmod 6$.

Hence, Equation (9) is satisfied for any integer n. Therefore, $36m$ is a period of the sequence and L divides $36m$.

Secondly, we prove $L = 36m$. This can be verified by a computer program for small $m \leqslant 10$. Thus, here we prove $L = 36m$ for only $m > 10$. Let $\lambda = L, L-1$ or $L-2$ such that $\lambda \equiv 0 \pmod 3$. First of all, we assume that λ is even. Since $t_4(-2) = t_4(-1) = t_4(0) = t_4(1) = t_4(2) = t_4(3) = 0$, we have

$$t_4(\lambda) \equiv t_4(\lambda+1) \equiv t_4(\lambda+2) \equiv t_4(\lambda+3) \equiv 0 \pmod m.$$

Hence, $m|M$ where $M = [t_4(\lambda+1) - t_4(\lambda+2)] + [t_4(\lambda) - t_4(\lambda+3)]$.

We observe that $\{t_4(n) \pmod m\}_{n=-\infty}^{\infty}$ is an aperiodic pattern for $-188 \leqslant n \leqslant 174$ with the aid of a computer program for $m > 1495$. In addition to this, we can also show using a computer that $L \geqslant 362$ for $10 < m \leqslant 1495$. Therefore, we have $L \geqslant 362$ which means that $\lambda \geqslant 360$.

By the definition of nearest integer function, we have

$$\|x \mp y\| = \|x\| \mp \|y\| + \alpha, \quad \alpha = 0, 1 \text{ or } -1 \tag{10}$$

and

$$\|x\| + \|y\| < x + y + 1 \tag{11}$$

for any x and y. From (10), we get

$$\begin{aligned} t_4(\lambda) + t_4(\lambda+1) &= \left\| \frac{2\lambda^3}{3888} + \frac{5\lambda^2}{432} + \frac{\lambda}{12} + \frac{7}{72} \right\| + \alpha_1, \quad -1 \leqslant \alpha_1 \leqslant 1 \\ &= \left\| \frac{2\lambda^3}{3888} \right\| + \left\| \frac{5\lambda^2}{432} \right\| + \left\| \frac{\lambda}{12} \right\| + \left\| \frac{7}{72} \right\| + \alpha_2, \quad -4 \leqslant \alpha_2 \leqslant 4 \end{aligned}$$

and

$$\begin{aligned} t_4(\lambda+2) + t_4(\lambda+3) &= \left\| \frac{2\lambda^3}{3888} + \frac{5\lambda^2}{432} + \frac{\lambda}{18} + \frac{1}{72} \right\| + \beta_1, \quad -1 \leqslant \beta_1 \leqslant 1 \\ &= \left\| \frac{2\lambda^3}{3888} \right\| + \left\| \frac{5\lambda^2}{432} \right\| + \left\| \frac{\lambda}{18} \right\| + \left\| \frac{1}{72} \right\| + \beta_2, \quad -4 \leqslant \beta_2 \leqslant 4. \end{aligned}$$

Hence,

$$\begin{aligned} M &= \left\|\frac{\lambda}{12}\right\| - \left\|\frac{\lambda}{18}\right\| + \gamma_1, \quad -8 \leqslant \gamma_1 \leqslant 8 \\ &= \left\|\frac{\lambda}{36}\right\| + \gamma_2, \quad -9 \leqslant \gamma_2 \leqslant 9 \end{aligned}$$

Since $\lambda \geqslant 360$, we have $M > 1$.

Moreover, from (11), we have

$$t_4(\lambda+1) - t_4(\lambda+2) < \frac{\lambda^2}{432} + \frac{\lambda}{36} - \frac{1}{72} + 1$$

and

$$t_4(\lambda) - t_4(\lambda+3) < -\frac{\lambda^2}{432} + \frac{7}{72} + 1,$$

so we get $M < \frac{\lambda}{36} + \frac{6}{72} + 2 = \frac{\lambda+75}{36}$. This means that $m < \frac{\lambda+75}{36}$ since $m|M$ and $M \neq 0$, and it follows that

$$36m < \lambda + 75 \leqslant L + 75. \tag{12}$$

Because L is a divisor of $36m$, we conclude by inequality (12) that $L = 36m$ for any $m > 10$. Hence, $L = 36m$ for λ even. Similarly we can also show $L = 36m$ for λ odd, and the proof is complete.

□

4 The Case $k = 5$

Let us take $k = 5$ in the generalized Alcuin's sequences for $\{t_k(n)\}_{n=0}^{\infty}$ given in Definition 1. We obtain the sequence $\{t_5(n)\}_{n=0}^{\infty}$

$$t_5(n) = \begin{cases} p_5\left(\frac{n}{4}\right), & \text{if } n \equiv 0 \pmod 4, \\ p_5\left(\frac{n+15}{4}\right), & \text{if } n \equiv 1 \pmod 4, \\ p_5\left(\frac{n+10}{4}\right), & \text{if } n \equiv 2 \pmod 4. \\ p_5\left(\frac{n+5}{4}\right), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

If we substitute $k = 5$ in Theorem 2 and Theorem 3, we get the generating function

$$F_5(x) = \frac{x^5}{(1-x^5)(1-x^4)(1-x^8)(1-x^{12})(1-x^{16})} \tag{13}$$

and the linear recurrence relation of order 45

$$\begin{aligned} t_5(n) &= t_5(n-4)+t_5(n-5)+t_5(n-8)-t_5(n-9)-t_5(n-13) \\ &\quad -2t_5(n-20)+2t_5(n-25)+t_5(n-32)+t_5(n-36)-t_5(n-37) \\ &\quad -t_5(n-40)-t_5(n-41)+t_5(n-45) \end{aligned} \tag{14}$$

for $n \geqslant 45$, together with the values of $t_5(n)$ for $0 \leqslant n \leqslant 44$ that can be easily computed.

Theorem 10. *Let $C = 1/86400$. For all $n \geqslant 0$, we have*

$$p_5(n) = \begin{cases} \|C(30n^4+300n^3+300n^2-3600n-5224)\|, & \text{if } n \text{ is even,} \\ \|C(30n^4+300n^3+300n^2-900n+1526)\|, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The generating function of $p_5(n)$ is

$$G_5(x) = \frac{x^5}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

Now, by partial fractions we obtain that $G_5(x)$ is

$$\begin{aligned} &-\frac{1}{64(x+1)^2}-\frac{3}{128(x+1)}-\frac{5}{288(x-1)^2}-\frac{431}{86400(x-1)}-\frac{1}{120(x-1)^5} \\ &+\frac{5}{288(x-1)^3}-\frac{2x+1}{27(x^2+x+1)}+\frac{x^3+2x^2+3x+4}{25(x^4+x^3+x^2+x+1)}+\frac{x-1}{16(x^2+1)}. \end{aligned} \tag{15}$$

Using the general binomial theorem, the first six terms of $G_5(x)$ can be written as

$$\begin{aligned} &-\frac{1}{64}\sum_{n=0}^{\infty}(n+1)(-1)^n x^n-\frac{3}{128}\sum_{n=0}^{\infty}(-1)^n x^n-\frac{5}{288}\sum_{n=0}^{\infty}(n+1)x^n-\frac{431}{86400}\sum_{n=0}^{\infty}x^n \\ &+\frac{1}{120}\sum_{n=0}^{\infty}\binom{n+4}{n}x^n-\frac{5}{288}\sum_{n=0}^{\infty}\binom{n+2}{n}x^n. \end{aligned}$$

Thus the first six terms give the following coefficient of x^n:

$$\frac{30n^4+300n^3+300n^2-2250n-1350n(-1)^n-3375(-1)^n-1849}{86400}.$$

As in the case of Theorem 6, the last three terms in Equation (15) can be handled by Maple using Taylor expansions and their contribution to the coefficient of x^n is $c/10800$, where c follows now a pattern modulo 60 by taking the values $c = 653, -157, 1043, -1507, 3203, -307, 2003$ and 1853. Combining all contributions, we get the desired result. □

Plugging the formula for $p_5(n)$ into the definition of $t_5(n)$ we obtain the following corollary.

Corollary 11. *Let $C = 1/11059200$. For all $n \geqslant 0$,*

$$t_5(n) = \begin{cases} \|C\left(15n^4 + 600n^3 + 2400n^2 - 115200n - 668672\right)\|, & \text{if } n \equiv 0 \pmod 8, \\ \|C\left(15n^4 + 1500n^3 + 49650n^2 + 564300n + 927703\right)\|, & \text{if } n \equiv 1 \pmod 8, \\ \|C\left(15n^4 + 1200n^3 + 29400n^2 + 259200n + 897328\right)\|, & \text{if } n \equiv 2 \pmod 8, \\ \|C\left(15n^4 + 900n^3 + 13650n^2 - 38700n - 1100297\right)\|, & \text{if } n \equiv 3 \pmod 8, \\ \|C\left(15n^4 + 600n^3 + 2400n^2 - 28800n + 195328\right)\|, & \text{if } n \equiv 4 \pmod 8, \\ \|C\left(15n^4 + 1500n^3 + 49650n^2 + 650700n + 3087703\right)\|, & \text{if } n \equiv 5 \pmod 8, \\ \|C\left(15n^4 + 1200n^3 + 29400n^2 + 172800n - 830672\right)\|, & \text{if } n \equiv 6 \pmod 8, \\ \|C\left(15n^4 + 900n^3 + 13650n^2 + 47700n + 195703\right)\|, & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Similar to Section 3, we use the *same* recurrence relation in (14) to define $\{t_5(n)\}_{n=-\infty}^{\infty}$ for $n < 0$, obtaining

$$t_5(-n) = t_5(n - 35). \tag{16}$$

We note that $\ell - 2k = 45 - 10 = 35$ for the sequence $\{t_5(n)\}_{n=-\infty}^{\infty}$ where $k = 5$ and $\ell = 45$.

Lemma 12. *For any integer n, $4n^3 \pm 15n^2 + 5n \equiv 0 \pmod 6$ and $4n^3 + 5n \equiv 0 \pmod 3$.*

Proof. Since

$$4n^3 \pm 15n^2 + 5n \equiv n^2 - n \equiv n(n-1) \equiv 0 \pmod 2$$

and

$$4n^3 \pm 15n^2 + 5n \equiv n^3 - n \equiv n(n-1)(n-2) \equiv 0 \pmod 3,$$

we get

$$4n^3 \pm 15n^2 + 5n \equiv 0 \pmod 6.$$

Similarly,

$$4n^3 + 5n \equiv n^3 - n \equiv n(n-1)(n-2) \equiv 0 \pmod 3.$$

□

Theorem 13. *For any odd integer $m \geqslant 3$, the sequence $\{t_5(n) \pmod m\}_{n=-\infty}^{\infty}$ is periodic with least period $240m$. For any even integer $m \geqslant 2$, the sequence $\{t_5(n) \pmod m\}_{n=-\infty}^{\infty}$ is periodic with least period $480m$.*

Proof. Let L be the least period of $\{t_5(n) \pmod m\}_{n=-\infty}^{\infty}$. First we prove that if m is odd then $240m$ is a period of the sequence else $480m$ is a period of the sequence so that if m is odd then L divides $240m$ else L divides $480m$. That is for each $m \geqslant 2$ and any integer n, if m is odd then

$$t_5(n + 240m) \equiv t_5(n) \pmod m, \tag{17}$$

and else

$$t_5(n + 480m) \equiv t_5(n) \pmod m. \tag{18}$$

In order to prove this, as before, we break it into 5 cases. For simplicity, we only show the case m is odd; the even case is analogous.

(1) $n \geqslant 0:$ For $n \equiv 0 \pmod 8$, using Corollary 11 and m odd, we have

$$\begin{aligned}
&t_5(n+240m) \\
&= \left\| \frac{n^4}{737280} + \frac{n^3}{18432} + \frac{n^2}{4608} - \frac{n}{96} - \frac{653}{10800} \right\| + \frac{n^3}{768}m + \frac{5n^2}{128}m + \frac{5n}{48}m \\
&\quad + \frac{15n^2}{32}m^2 + \frac{5(5m-1)}{2}m + \frac{75n}{8}m^2 + 75nm^3 + 750m^3 + 4500m^4 \\
&= t_5(n) + \frac{4t^3}{6}m + \frac{15t^2}{6}m + \frac{5t}{6}m + 30t^2m^2 + \frac{5(5m-1)}{2}m + 75tm^2 \\
&\quad + 600tm^3 + 750m^3 + 4500m^4 \quad \text{(by substituting } n = 8t\text{)} \\
&\equiv t_5(n) \pmod m \quad \text{(by Lemma 12 and } 5(5m-1) \equiv 0 \pmod 2\text{)}.
\end{aligned}$$

In a similar way, we show that Equation (17) is also satisfied for $n \equiv 1, 2, 3, 4, 5, 6, 7 \pmod 8$.

(2) $-35 < n < 0:$ If m is odd then Equation (17) is immediately satisfied for this range by using Corollary 11 and Equation (16).

(3) $-240m < n \leqslant -35:$ For $n \equiv 0 \pmod 8$, by using Corollary 11, Equation (16) and m is odd, we have

$$\begin{aligned}
&t_5(n+240m) \\
&= \left\| \frac{n^4}{737280} + \frac{n^3}{18432} + \frac{n^2}{4608} - \frac{n}{96} - \frac{653}{10800} \right\| + \frac{n^3}{768}m + \frac{5n^2}{128}m + \frac{5n}{48}m \\
&\quad + \frac{15n^2}{32}m^2 + \frac{5(5m-1)}{2}m + \frac{75n}{8}m^2 + 75nm^3 + 750m^3 + 4500m^4 \\
&= t_5(-n-35) + \frac{4t^3}{6}m + \frac{15t^2}{6}m + \frac{5t}{6}m + 30t^2m^2 + \frac{5(5m-1)}{2}m + 75tm^2 \\
&\quad + 600tm^3 + 750m^3 + 4500m^4 \quad \text{(by substituting } n = 8t\text{)} \\
&\equiv t_5(n) \pmod m \quad \text{(by Lemma 12 and } 5(5m-1) \equiv 0 \pmod 2\text{)}.
\end{aligned}$$

Again, Equation (17) is also satisfied for $n \equiv 1, 2, 3, 4, 5, 6, 7 \pmod 8$.

(4) $-240m - 35 < n \leqslant -240m:$

That is, $n = -240m - a$ for $0 \leqslant a \leqslant 34$ then

$$t_5(n+240m) = t_5(-240m - a + 240m) = t_5(-a) \equiv 0 \pmod m$$

and

$$\begin{aligned}
t_5(n) &= t_5(-240m-a) = t_5(240m + a - 35) \quad \text{(by Equation (16))} \\
&\equiv 0 \pmod m,
\end{aligned}$$

and so $t_5(n+240m) \equiv t_5(n) \pmod m$.

(5) $n \leqslant -240m - 35$: For $n \equiv 0 \pmod 8$, using Corollary 11, Equation (16) and m odd, we have

$$\begin{aligned}
&t_5(n+240m)\\
&\quad = t_5(-n-240m-35)\\
&\quad = \left\| \frac{n^4}{737280} + \frac{n^3}{18432} + \frac{n^2}{4608} - \frac{n}{96} - \frac{653}{10800} \right\| + \frac{n^3}{768}m + \frac{5n^2}{128}m + \frac{5n}{48}m\\
&\qquad + \frac{15n^2}{32}m^2 + \frac{5(5m-1)}{2}m + \frac{75n}{8}m^2 + 75nm^3 + 750m^3 + 4500m^4\\
&\quad = t_5(-n-35) + \frac{4t^3}{6}m + \frac{15t^2}{6}m + \frac{5t}{6}m + 30t^2m^2 + \frac{5(5m-1)}{2}m + 75tm^2\\
&\qquad + 600tm^3 + 750m^3 + 4500m^4 \quad (\text{by substituting } n = 8t)\\
&\quad \equiv t_5(n) \pmod m \quad (\text{by Lemma 12 and } 5(5m-1) \equiv 0 \pmod 2).
\end{aligned}$$

As before, Equation (17) is also satisfied for $n \equiv 1, 2, 3, 4, 5, 6, 7 \pmod 8$.

If m is odd then Equation (17) is satisfied for any integer n from (1), (2), (3), (4) and (5). Similarly, one can show that if m is even then Equation (18) is satisfied for any integer n by using Lemma 12, Corollary 11 and Equation (16). Therefore, if m is odd $240m$ is a period of the sequence and L divides $240m$, else $480m$ is a period of the sequence and L divides $480m$.

Secondly, we prove that if m is odd then $L = 240m$ else $L = 480m$. This can be verified by a computer program for small $m \leqslant 10$. So here, we prove $L = 36m$ for only $m > 10$. Let $\lambda = L, L-1, L-2, L-3, L-4, L-5, L-6$ or $L-7$ so that $\lambda \equiv 0 \pmod 8$.

Since $t_5(-7) = t_5(-6) = \cdots = t_5(4) = 0$, we have

$$t_5(\lambda) \equiv t_5(\lambda+1) \equiv t_5(\lambda+2) \equiv t_5(\lambda+4) \equiv 0 \pmod m.$$

So, we get $m|M$ where

$$M = [t_5(\lambda+4) - t_5(\lambda)] + [t_5(\lambda+1) - t_5(\lambda+2)] + [t_5(\lambda+4) - t_5(\lambda+2)].$$

$$t_5(\lambda) = \left\| \frac{\lambda^4}{737280} + \frac{\lambda^3}{18432} + \frac{\lambda^2}{4608} - \frac{\lambda}{96} - \frac{653}{10800} \right\|$$

$$t_5(\lambda+1) = \left\| \frac{\lambda^4}{737280} + \frac{13\lambda^3}{92160} + \frac{113\lambda^2}{23040} + \frac{29\lambda}{480} + \frac{1507}{10800} \right\|$$

$$t_5(\lambda+2) = \left\| \frac{\lambda^4}{737280} + \frac{11\lambda^3}{92160} + \frac{77\lambda^2}{23040} + \frac{17\lambda}{480} + \frac{1507}{10800} \right\|$$

$$t_5(\lambda+4) = \left\| \frac{\lambda^4}{737280} + \frac{7\lambda^3}{92160} + \frac{23\lambda^2}{23040} + \frac{\lambda}{480} + \frac{157}{10800} \right\|$$

Hence, from Equation (11), we can write

$$\begin{aligned}
t_5(\lambda+4) - t_5(\lambda) &< \frac{\lambda^3}{46080} + \frac{\lambda^2}{1280} + \frac{\lambda}{80} + \frac{3}{40} + 1 \\
t_5(\lambda+1) - t_5(\lambda+2) &< \frac{\lambda^3}{46080} + \frac{\lambda^2}{640} + \frac{\lambda}{40} + 1 \\
t_5(\lambda+4) - t_5(\lambda+2) &< -\frac{\lambda^3}{23040} - \frac{3\lambda^2}{1280} - \frac{\lambda}{30} - \frac{1}{80} + 1,
\end{aligned}$$

and it follows that

$$M = [t_5(\lambda+4) - t_5(\lambda)] + [t_5(\lambda+1) - t_5(\lambda+2)] + [t_5(\lambda+4) - t_5(\lambda+2)] < \frac{\lambda}{240} - \frac{1}{20} + 3.$$

Similar to Section 3, we can conclude that $L \geqslant 2880$ for $m > 10$ by observing an aperiodic pattern for $\{t_5(n) \pmod m\}_{n=-\infty}^{\infty}$ with the aid of a computer program. That is, we have $L \geqslant 2880$ so that $\lambda \geqslant 2880$. From (10), we get

$$\begin{aligned}
t_5(\lambda+1) + t_5(\lambda) &= \left\| \frac{8\lambda^3}{92160} + \frac{108\lambda^2}{23040} + \frac{34\lambda}{480} + \frac{2160}{10800} \right\| + \alpha_1, \quad -1 \leqslant \alpha_1 \leqslant 1 \\
&= \left\| \frac{8\lambda^3}{92160} \right\| + \left\| \frac{108\lambda^2}{23040} \right\| + \left\| \frac{34\lambda}{480} \right\| + \left\| \frac{2160}{10800} \right\| + \alpha_2, \quad -4 \leqslant \alpha_2 \leqslant 4
\end{aligned}$$

and

$$\begin{aligned}
&2(t_5(\lambda+2) + t_5(\lambda+3)) \\
= \; &2\left(\left\| \frac{4\lambda^3}{92160} + \frac{54\lambda^2}{23040} + \frac{16\lambda}{480} + \frac{1350}{10800} \right\| + \beta_1 \right), \quad -1 \leqslant \beta_1 \leqslant 1 \\
= \; &\left\| \frac{8\lambda^3}{92160} + \frac{108\lambda^2}{23040} + \frac{32\lambda}{480} + \frac{2700}{10800} \right\| + \beta_2, \quad -3 \leqslant \beta_2 \leqslant 3 \\
= \; &\left\| \frac{8\lambda^3}{92160} \right\| + \left\| \frac{108\lambda^2}{23040} \right\| + \left\| \frac{32\lambda}{480} \right\| + \left\| \frac{2700}{10800} \right\| + \beta_3, \quad -6 \leqslant \beta_3 \leqslant 6.
\end{aligned}$$

Hence,

$$\begin{aligned}
M &= \left\| \frac{34\lambda}{480} \right\| - \left\| \frac{32\lambda}{480} \right\| + \gamma_1, \quad -10 \leqslant \gamma_1 \leqslant 10 \\
&= \left\| \frac{\lambda}{240} \right\| + \gamma_2, \quad -11 \leqslant \gamma_2 \leqslant 11
\end{aligned}$$

Since $\lambda \geqslant 2880$, we have $M > 1$. We have $M < \frac{\lambda}{240} - \frac{1}{20} + 3 = \frac{\lambda + 708}{240}$. This means that $m < \frac{\lambda+708}{240}$ (since $m|M$ and $M \neq 0$), and it follows that

$$240m < \lambda + 708 \leqslant L + 708. \tag{19}$$

If m is odd then L is a divisor of $240m$, and so $L = 240m$ by Inequality (19) for any $m > 10$. Hence, if m is odd then $L = 240m$. If m is even then L is a divisor of $480m$, so $L = 240m$ or $480m$. However, we can get

$$\begin{aligned} t_5(240m) &= \left\| 4500m^4 + 750m^3 + \frac{25m^2}{2} - \frac{5m}{2} - \frac{653}{10800} \right\| \\ &= 4500m^4 + 750m^3 + \frac{5m}{2}(5m-1) \\ &\equiv \frac{5m}{2} \pmod m \\ &\not\equiv 0 \pmod m, \end{aligned}$$

when m is even. Therefore, if m is even then L can not be $240m$. This means that if m is even then $L = 480m$ and the theorem is proved. $\square$

5 Conclusion

In this paper we introduce a new sequence $\{t_k(n)\}_{n=-\infty}^{\infty}$ for any given positive integer $k \geqslant 3$ that is a generalization of Alcuin's sequence. We find explicitly a linear recurrence equation and the generating function for $\{t_k(n)\}_{n=-\infty}^{\infty}$. The case $k = 3$ is Alcuin's sequence. For the special case $k = 4$ and $k = 5$, we get simpler formulas for $\{t_k(n)\}_{n=-\infty}^{\infty}$ and investigate the period of $\{t_k(n)\}_{n=-\infty}^{\infty}$ modulo a fixed integer. Also, we get a formula for $p_5(n)$, the number of partitions of n into exactly 5 parts.

In [1], Andrews discussed the geometric interpretation of the sequence $\{t_3(n)\}_{n=0}^{\infty}$ corresponding to Alcuin's sequence. For $k = 4$, let a, b, c and d be integers satisfying the condition

$$a+b+c+d = 3n,\ a+b+c > 2n,\ a+b+d > 2n,\ b+c+d > 2n \text{ and } a+c+d > 2n. \quad (20)$$

If this is case $a, b, c, d < n$, and since

$$t_4(3n) = p_4(n)$$

via the bijection

$$(a, b, c, d) \longleftrightarrow (n-a, n-b, n-c, n-d),$$

we conclude that $t_4(3n)$ counts the number of different a, b, c, d integers which satisfy Condition (20). We wonder what kind of geometric interpretation does $\{t_4(n)\}_{n=-\infty}^{\infty}$, or more generally $\{t_k(n)\}_{n=-\infty}^{\infty}$ for $k \geqslant 4$, represent?

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