Identifying codes of lexicographic product of graphs

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Abstract

Let G be a connected graph and H be an arbitrary graph. In this paper, we study the identifying codes of the lexicographic product $G[H]$ of G and H. We first introduce two parameters of H, which are closely related to identifying codes of H. Then we provide the sufficient and necessary condition for $G[H]$ to be identifiable. Finally, if $G[H]$ is identifiable, we determine the minimum cardinality of identifying codes of $G[H]$ in terms of the order of G and these two parameters of H.

Key words: identifying code; lexicographic product.

1 Introduction

In this paper, we only consider finite undirected simple graphs with at least two vertices. For a given graph G, we often write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. For any two vertices u and v of G, let $d_G(u,v)$ denote the distance between u and v in G. Given a vertex $v \in V(G)$, define $B_G(v) = \{u | u \in V(G), d_G(u,v) \leqslant 1\}$. A *code* C is a nonempty set of vertices. We say that a code C *covers* v if $B_G(v) \cap C \neq \emptyset$; we say that C *separates* two distinct vertices x and y if $B_G(x) \cap C \neq B_G(y) \cap C$. An *identifying code* of G is a code which covers all the vertices of G and separates any pair of distinct vertices of G. If G admits at least one identifying code, we say G is *identifiable* and denote the minimum cardinality of all identifying codes of G by $I(G)$.

The concept of identifying codes was introduced by Karpovsky et al. [12] to model a fault-detection problem in multiprocessor systems. It was noted in [4, 5] that determining the identifying code with the minimum cardinality in a graph is an NP-complete problem.

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Many researchers have focused on the study of identifying codes in some restricted classes of graphs, for example, paths [2], cycles [2, 7, 16], hypercubes [3, 11, 13, 14] and infinite grids [1, 6, 10].

Gravier et al. [8] investigated the identifying codes of Cartesian product of two cliques. Rall and Wash [15] studied the identifying codes of the direct product of two cliques. In this paper, we study the identifying codes of the lexicographic product $G[H]$ of a connected graph G and an arbitrary graph H. In Section 2, we introduce two parameters of a graph which are closely related to identifying codes, and compute these two parameters of the path P_n and the cycle C_n, respectively. In Section 3, we first provide the sufficient and necessary condition for $G[H]$ to be identifiable, then determine $I(G[H])$ in terms of the order of G and the two parameters of H when $G[H]$ is identifiable. In particular, the values of $I(G[P_n])$ and $I(G[C_n])$ are determined.

2 Two parameters

For a graph H, let $C' \subseteq V(H)$ be a code which separates any pair of distinct vertices of H, we use $I'(H)$ to denote the minimum cardinality of all possible C'. This code was studied in [3]. Let $C'' \subseteq V(H)$ be a code which separates any pair of distinct vertices of H and satisfies $C'' \nsubseteq B_H(v)$ for every $v \in V(H)$, we use $I''(H)$ to denote the minimum cardinality of all possible C''.

The two parameters $I'(H)$ and $I''(H)$ are used to compute the minimum cardinality of identifying codes of $G[H]$ of graphs G and H (see Theorem 3.4). In this section we shall compute the two parameters for paths and cycles, respectively.

Given an integer $n \geqslant 3$, suppose

$$V(P_n) = \{0, 1, \ldots, n-1\},\ E(P_n) = \{ij | j = i+1, i = 0, \ldots, n-2\};$$
$$V(C_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\},\ E(C_n) = \{ij | j = i+1, i \in \mathbb{Z}_n\}.$$

Example 2.1 $I'(P_3) = 2$ and $I''(P_3)$ is not well defined; $I'(P_4) = 3$ and $I''(P_4) = 4$; $I'(P_5) = I''(P_5) = 3$; $I'(P_6) = 3$ and $I''(P_6) = 4$.

For P_4, $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{P_4}(1)$ and $\{0, 1, 3\}$ can not separate 0 and 1. For P_5, $\{0, 2, 4\}$ separates any pair of distinct vertices. For P_6, $\{1, 2, 3\}$ separates any pair of distinct vertices, but $\{1, 2, 3\} \subseteq B_{P_6}(2)$.

Example 2.2 $I'(C_4) = 3$ and $I''(C_4) = 4$; $I'(C_5) = 3$ and $I''(C_5) = 4$; $I'(C_6) = I''(C_6) = 3$; $I'(C_7) = I''(C_7) = 4$; $I'(C_9) = I''(C_9) = 6$; $I'(C_{11}) = I''(C_{11}) = 6$.

For C_4, $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{C_4}(1)$. For C_5, $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{C_5}(1)$ and $\{0, 1, 3\}$ can not separate 0 and 1. For C_6, both $\{3, 4, 5\}$ and $\{0, 2, 4\}$ separate any pair of distinct vertices. For C_7, $\{3, 4, 5, 6\}$ separates any pair of distinct vertices. For C_9, both $\{3, 4, 5, 6, 7, 8\}$ and $\{0, 2, 4, 6, 7, 8\}$ separate any pair of distinct vertices. For C_{11}, $\{3, 4, 5, 8, 9, 10\}$ separates any pair of distinct vertices.

The minimum cardinality of identifying codes of a path or a cycle was computed in [2, 7].

Proposition 2.1 ([2, 7]) (i) *For* $n \geqslant 3$*,* $I(P_n) = \lfloor \frac{n}{2} \rfloor + 1$*;*

(ii) *For* $n \geqslant 6$*,* $I(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ \frac{n+3}{2}, & n \text{ is odd}. \end{cases}$

In order to compute the two parameters for paths and cycle, we need the following useful lemma.

Lemma 2.2 *Let* H *be an identifiable graph.*

(i) $I(H) - 1 \leqslant I'(H) \leqslant I(H)$*;*

(ii) *If* $\Delta(H) \leqslant |V(H)| - 2$*, then* $I(H) - 1 \leqslant I'(H) \leqslant I''(H) \leqslant I(H) + 1$*, where* $\Delta(H)$ *is the maximum degree of* H*.*

Proof. Let C' be a code which separates any pair of distinct vertices of H.

(i) Since there exists at most one vertex v not covered by C', $C' \cup \{v\}$ is an identifying code of H. Then $I(H) \leqslant I'(H) + 1$, as desired.

(ii) Note that there exists at most one vertex v such that $C' \subseteq B_H(v)$. Since $\Delta(H) \leqslant |V(H)| - 2$, there exists $v_0 \in V(H) \backslash B_H(v)$ such that $C'' = C' \cup \{v_0\}$ is a code which separates any pair of distinct vertices of H and satisfies $C'' \nsubseteq B_H(w)$ for every $w \in V(H)$. It follows that $I'(H) \leqslant I''(H) \leqslant I'(H) + 1$. By (i), (ii) holds. $\square$

Proposition 2.3 *For* $n \geqslant 7$*,* $I'(P_n) = I''(P_n) = \lfloor \frac{n}{2} \rfloor + 1$*.*

Proof. Combining Proposition 2.1 and Lemma 2.2, we have $I'(P_n) = \lfloor \frac{n}{2} \rfloor + 1$ or $\lfloor \frac{n}{2} \rfloor$. Suppose $I'(P_n) = \lfloor \frac{n}{2} \rfloor$. Then there exists a code W' of size $\lfloor \frac{n}{2} \rfloor$ such that W' separates any pair of distinct vertices of P_n and $B_{P_n}(i) \cap W' = \emptyset$ for a unique vertex i.

Case 1. $i \neq 0$ and $i \neq n-1$. Then $i-1, i, i+1 \notin W'$, and $i-2, i-3, i-4, i+2, i+3, i+4 \in W'$, so $4 \leqslant i \leqslant n-5$. If we delete the six vertices $i-1, i, i+1, i+2, i+3, i+4$, and connect $i-2$ by an edge to $i+5$, then we get an identifying code of P_{n-6}. Hence $|W'| \geqslant 3 + I(P_{n-6}) = \lfloor \frac{n}{2} \rfloor + 1$, a contradiction.

Case 2. $i = 0$ or $n-1$. Without loss of generality, assume that $i = n-1$. Then $n-1, n-2 \notin W'$, and $n-3, n-4, n-5 \in W'$. If $\{0, 1, \ldots, n-6\} \subseteq W'$, then $|W'| = n-2 \geqslant \lfloor \frac{n}{2} \rfloor + 1$, a contradiction. Now suppose $\{0, 1, \ldots, n-6\} \nsubseteq W'$. Take the smallest $k \geqslant 6$ such that $n-k \notin W'$. If $k = n-1$ or $k = n$, then $|W'| \geqslant n-3 \geqslant \lfloor \frac{n}{2} \rfloor + 1$, a contradiction. It is clear that $k \neq n-2$. For $k \leqslant n-3$, by deleting the vertices $n-k, n-k+1, \ldots, n-1$, we get an identifying code of P_{n-k}. It follows that $|W'| \geqslant k - 3 + I(P_{n-k}) \geqslant \lfloor \frac{n}{2} \rfloor + 1$, a contradiction.

Therefore, $I'(P_n) = \lfloor \frac{n}{2} \rfloor + 1$. Since $I''(P_n) = I'(P_n)$ for $I'(P_n) \geqslant 4$, the desired result follows. $\square$

Proposition 2.4 $I'(C_n) = I''(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even and } n \geqslant 8, \\ \frac{n+3}{2}, & n \text{ is odd and } n \geqslant 13. \end{cases}$

Proof. Note that $I''(C_n) = I'(C_n)$ for $I'(C_n) \geqslant 4$. Combining Proposition 2.1 and Lemma 2.2, we only need to prove $I'(C_n) \geqslant I(C_n)$. It is routine to show that $I'(C_n) \geqslant I(C_n)$ for $n = 8$ or $n = 10$. Next, we consider $n \geqslant 12$. Let W' be a code of size $I'(C_n)$ such that W' separates any pair of distinct vertices of C_n. If W' is an identifying code, then $I'(C_n) = |W'| \geqslant I(C_n)$. Now suppose that W' is not an identifying code. Then there exists a unique vertex $i \in V(C_n)$ such that $\{i-1, i, i+1\} \cap W' = \emptyset$, which implies that $\{i-2, i-3, i-4, i+2, i+3, i+4\} \subseteq W'$. If we delete the six vertices $i-1, i, i+1, i+2, i+3, i+4$, and connect $i-2$ by an edge to $i+5$, then we get an identifying code of C_{n-6}. Therefore $I'(C_n) = |W'| \geqslant 3 + I(C_{n-6}) = I(C_n)$, as desired. $\square$

3 Main results

We always assume that G is a connected graph and H is an arbitrary graph. In this section, we first provide the sufficient and necessary condition for $G[H]$ to be identifiable. Moreover, if $G[H]$ is identifiable, we determine the minimum cardinality of identifying codes of $G[H]$ in terms of the order of G and the two parameters of H given in Section 2.

The *lexicographic product* $G[H]$ of graphs G and H is the graph with the vertex set $\{(u,v)|u \in V(G), v \in V(H)\}$, and the edge set $\{\{(u_1,v_1),(u_2,v_2)\}|d_G(u_1,u_2) = 1,$ or $u_1 = u_2$ and $d_H(v_1,v_2) = 1\}$. For any two distinct vertices (u_1,v_1), (u_2,v_2) of $G[H]$, we observe that

$$d_{G[H]}((u_1,v_1),(u_2,v_2)) = \begin{cases} 1, & \text{if } u_1 = u_2, d_H(v_1,v_2) = 1, \\ 2, & \text{if } u_1 = u_2, d_H(v_1,v_2) \geqslant 2, \\ d_G(u_1,u_2), & \text{if } u_1 \neq u_2. \end{cases} \quad (1)$$

For $u \in V(G)$, let $N_G(u) = B_G(u) \backslash \{u\}$. For any $u_1, u_2 \in V(G)$, define $u_1 \equiv u_2$ if and only if $B_G(u_1) = B_G(u_2)$ or $N_G(u_1) = N_G(u_2)$. Hernando et al. [9] proved that " $\equiv$ " is an equivalent relation and the equivalence class of a vertex is of three types: a class of size 1, a clique of size at least 2, an independent set of size at least 2. Denote all equivalence classes by

$$W_1, \ldots, W_p, U_1, \ldots, U_k, V_1, \ldots, V_l, \quad (2)$$

where

(i) $|W_q| = 1$, $q = 1, \ldots, p$;

(ii) for any $u_1, u_2 \in U_i, i = 1, \ldots, k$, $B_G(u_1) = B_G(u_2)$;

(iii) for any $u_1, u_2 \in V_j, j = 1, \ldots, l$, $N_G(u_1) = N_G(u_2)$.

Denote $s(G) = |U_1| + \cdots + |U_k| - k$, $t(G) = |V_1| + \cdots + |V_l| - l$.

For $u \in V(G)$ and $C \subseteq V(H)$, let $C^u = \{(u,v)|(u,v) \in V(G[H]), v \in C\}$. For $S \subseteq V(G[H])$, let $S_u = \{v|v \in V(H), (u,v) \in S\}$. Note that $(S_u)^u = H^u \cap S$, where $H^u = (V(H))^u$. By (1), we have

$$B_{G[H]}((u,v)) = (B_H(v))^u \cup \bigcup_{w \in N_G(u)} H^w, \quad (3)$$

$$B_{G[H]}((u,v)) \cap S = ((B_H(v)) \cap S_u)^u \cup \bigcup_{w \in N_G(u)} (S_w)^w. \quad (4)$$

Theorem 3.1 *Let G be a connected graph and H be an arbitrary graph. Then the lexicographic product $G[H]$ of G and H is identifiable if and only if*

(i) *H is identifiable and $\Delta(H) \leqslant |V(H)| - 2$; or*

(ii) *both G and H are identifiable.*

Proof. Suppose $G[H]$ is identifiable. If H is not identifiable, then there exist two distinct vertices v_1, v_2 of H with $B_H(v_1) = B_H(v_2)$. By (3), $B_{G[H]}((u, v_1)) = B_{G[H]}((u, v_2))$ for $u \in V(G)$. This contradicts the condition that $G[H]$ is identifiable.

If $\Delta(H) = |V(H)| - 1$ and G is not identifiable, then there exist $v \in V(H)$ and two distinct vertices u_1, u_2 of G such that

$$B_H(v) = V(H) \text{ and } B_G(u_1) = B_G(u_2).$$

By (3), we have

$$B_{G[H]}((u_1, v)) = H^{u_1} \cup \bigcup_{u \in N_G(u_1)} H^u = \bigcup_{u \in B_G(u_1)} H^u = \bigcup_{u \in B_G(u_2)} H^u = B_{G[H]}((u_2, v)),$$

which contradicts the condition that $G[H]$ is identifiable.

Therefore, (i) or (ii) holds.

Conversely, suppose (i) or (ii) holds. Assume that $G[H]$ is not identifiable. Therefore, there exist two distinct vertices (u_1, v_1) and (u_2, v_2) such that $B_{G[H]}((u_1, v_1)) = B_{G[H]}((u_2, v_2))$. If $u_1 \neq u_2$, then $d_G(u_1, u_2) = 1$. It follows that $B_G(u_1) = B_G(u_2)$ and $B_H(v_1) = B_H(v_2) = V(H)$, contrary to (i) and (ii). If $u_1 = u_2$, by (3), one gets $B_H(v_1) = B_H(v_2)$, contrary to the condition that H is identifiable. □

Remark 3.1 Let r be a positive integer and Γ be a graph. Given a vertex $v \in V(\Gamma)$, define $B_\Gamma^{(r)}(v) = \{u | u \in V(\Gamma), d_\Gamma(u, v) \leqslant r\}$. An *$r$-identifying code* of Γ is a code which r-covers all the vertices of Γ and r-separates any pair of distinct vertices of Γ (see [12] for details). Identifying codes in this paper are 1-identifying codes. If $r \geqslant 2$, then $G[H]$ does not admit any r-identifying code. Indeed, by (1), $B_{G[H]}^{(r)}((u, v_1)) = B_{G[H]}^{(r)}((u, v_2))$ for $r \geqslant 2$.

Lemma 3.2 *Let G be a connected graph and H be an arbitrary graph. If S is an identifying code of $G[H]$, then for any vertex u of G, S_u separates any pair of distinct vertices of H. Moreover, with reference to (2), the following hold.*

(i) *If $k \neq 0$, then there exists at most one vertex $u \in U_i$ satisfying $S_u \subseteq B_H(v)$ for a vertex v of H, where $i = 1, \ldots, k$;*

(ii) *If $l \neq 0$, then there exists at most one vertex $u \in V_j$ satisfying $S_u \cap B_H(v) = \emptyset$ for a vertex v of H, where $j = 1, \ldots, l$.*

Proof. Assume that there exist $u_0 \in V(G)$ and two distinct vertices v_1, v_2 of H such that $S_{u_0} \cap B_H(v_1) = S_{u_0} \cap B_H(v_2)$. By (4), $B_{G[H]}((u_0, v_1)) \cap S = B_{G[H]}((u_0, v_2)) \cap S$, contrary to the condition that S is an identifying code of $G[H]$.

(i) Assume that there exist two distinct vertices $u_1, u_2 \in U_i$ such that $S_{u_1} \subseteq B_H(v_1)$ and $S_{u_2} \subseteq B_H(v_2)$. Since $B_G(u_1) = B_G(u_2)$, by (4) we have

$$B_{G[H]}((u_1,v_1)) \cap S = (S_{u_1})^{u_1} \cup \bigcup_{u \in N_G(u_1)} (S_u)^u = \bigcup_{u \in B_G(u_2)} (S_u)^u = B_{G[H]}((u_2,v_2)) \cap S.$$

Since S is an identifying code of $G[H]$, we have $(u_1,v_1) = (u_2,v_2)$, a contradiction.

(ii) Assume that there exist two different vertices $u_1, u_2 \in V_j$ such that $S_{u_1} \cap B_H(v_1) = S_{u_2} \cap B_H(v_2) = \emptyset$. Since $N_G(u_1) = N_G(u_2)$, by (4) we have

$$B_{G[H]}((u_1,v_1)) \cap S = \bigcup_{u \in N_G(u_1)} (S_u)^u = \bigcup_{u \in N_G(u_2)} (S_u)^u = B_{G[H]}((u_2,v_2)) \cap S.$$

Since S is an identifying code of $G[H]$, we obtain $(u_1,v_1) = (u_2,v_2)$, a contradiction. $\square$

In equivalence classes (2) of $V(G)$, choose $\overline{u}_i \in U_i, i = 1,\ldots,k$, and $\overline{v}_j \in V_j, j = 1,\ldots,l$. Let $\overline{W}_0 = \cup_{q=1}^{p} W_q \cup \{\overline{u}_1,\ldots,\overline{u}_k,\overline{v}_1,\ldots,\overline{v}_l\}$ and $\overline{U}_i = U_i \backslash \{\overline{u}_i\}, i = 1,\ldots,k$, $\overline{V}_j = V_j \backslash \{\overline{v}_j\}, j = 1,\ldots,l$. Therefore, we have a partition of $V(G)$:

$$\overline{W}_0, \overline{U}_1, \ldots, \overline{U}_k, \overline{V}_1, \ldots, \overline{V}_l. \tag{5}$$

Lemma 3.3 *Let C be an identifying code of graph H, and let C', C'' be two codes which separate any pair of distinct vertices of H and $C'' \not\subseteq B_H(v)$ for every vertex v of H. With reference to (5),*

$$S = \bigcup_{u \in \overline{W}_0} (C')^u \cup \bigcup_{i=1}^{k} \bigcup_{u \in \overline{U}_i} (C'')^u \cup \bigcup_{i=1}^{l} \bigcup_{u \in \overline{V}_i} C^u$$

is an identifying code of $G[H]$.

Proof. For any $u \in V(G)$, we have

$$S_u = \begin{cases} C', & \text{if } u \in \overline{W}_0, \\ C'', & \text{if } u \in \cup_{i=1}^{k} \overline{U}_i, \\ C, & \text{if } u \in \cup_{j=1}^{l} \overline{V}_j. \end{cases}$$

Since G is connected, there exists a vertex w adjacent to u. By (1), S covers all vertices of $G[H]$. Hence, we only need to show that, for any two distinct vertices (u_1,v_1) and (u_2,v_2) of $G[H]$,

$$B_{G[H]}((u_1,v_1)) \cap S \neq B_{G[H]}((u_2,v_2)) \cap S. \tag{6}$$

If $u_1 = u_2$, the fact that S_{u_1} separates v_1 and v_2 implies $B_H(v_1) \cap S_{u_1} \neq B_H(v_2) \cap S_{u_1} = B_H(v_2) \cap S_{u_2}$, so (6) holds by (4). Now suppose that $u_1 \neq u_2$. In order to prove (6), it is sufficient to show that there exists $(u_0,v_0) \in S$ such that

$$d_{G[H]}((u_0,v_0),(u_1,v_1)) \leqslant 1,\ d_{G[H]}((u_0,v_0),(u_2,v_2)) \geqslant 2 \tag{7}$$

or

$$d_{G[H]}((u_0,v_0),(u_2,v_2)) \leqslant 1,\ d_{G[H]}((u_0,v_0),(u_1,v_1)) \geqslant 2. \tag{8}$$

Case 1. $u_1 \not\equiv u_2$. Then there exists $u_0 \in V(G)\backslash\{u_1,u_2\}$ such that $d_G(u_1,u_0)=1$ and $d_G(u_2,u_0) \geqslant 2$, or $d_G(u_1,u_0) \geqslant 2$ and $d_G(u_2,u_0)=1$. Take $v_0 \in S_{u_0}$. Then $(u_0,v_0) \in S$. By (1), (7) or (8) holds.

Case 2. $B_G(u_1)=B_G(u_2)$. Then u_1 and u_2 are adjacent and fall into some U_i. It follows that $u_1 \in \overline{U}_i$ or $u_2 \in \overline{U}_i$. Without loss of generality, suppose $u_1 \in \overline{U}_i$. Pick $u_0=u_1$. Since $C'' \nsubseteq B_H(v_1)$, there exists $v_0 \in C''$ such that $(u_0,v_0) \in S$ and $d_H(v_0,v_1) \geqslant 2$. Then (8) holds by (1).

Case 3. $N_G(u_1)=N_G(u_2)$. Then u_1 and u_2 are at distance 2 and fall into some V_j. It follows that $u_1 \in \overline{V}_j$ or $u_2 \in \overline{V}_j$. Without loss of generality, suppose $u_1 \in \overline{V}_j$. Pick $u_0=u_1$. Since C covers v_1, there exists $v_0 \in C$ such that $(u_0,v_0) \in S$ and $d_H(v_0,v_1) \leqslant 1$. By (1), (7) holds. □

Theorem 3.4 *Let G be a connected graph and H be an arbitrary graph. Suppose* (i) *or* (ii) *holds in Theorem 3.1.*

(i) *If $\Delta(H) \leqslant |V(H)|-2$, then*

$$I(G[H]) = (|V(G)|-s(G)-t(G))I'(H)+s(G)I''(H)+t(G)I(H); \tag{9}$$

(ii) *If $\Delta(H)=|V(H)|-1$, then*

$$I(G[H]) = (|V(G)|-t(G))I'(H)+t(G)I(H). \tag{10}$$

Proof. (i) By Theorem 3.1, $I(H)$ and $I'(H)$ are well defined. Since $V(H)$ separates any pair of distinct vertices of H and $V(H) \nsubseteq B_H(v)$ for every $v \in V(H)$, $I''(H)$ is well defined. Let S be an identifying code of $G[H]$ with the minimum cardinality, by Lemma 3.2,

$$\begin{aligned} I(G[H]) &= |S| = \sum_{i=1}^{p}\sum_{u\in W_i}|S_u| + \sum_{i=1}^{k}\sum_{u\in U_i}|S_u| + \sum_{i=1}^{l}\sum_{u\in V_i}|S_u| \\ &\geqslant (p+k+l)I'(H) + (\sum_{i=1}^{k}|U_i|-k)I''(H) + (\sum_{i=1}^{l}|V_i|-l)I(H) \\ &= (|V(G)|-s(G)-t(G))I'(H)+s(G)I''(H)+t(G)I(H). \end{aligned}$$

By Lemma 3.3 we can construct an identifying code of $G[H]$ with cardinality $(|V(G)|-s(G)-t(G))I'(H)+s(G)I''(H)+t(G)I(H)$. Therefore, (9) holds.

(ii) By Theorem 3.1, both G and H are identifiable. So $I(H)$ and $I'(H)$ are well defined. Owing to $B_G(u_1) \neq B_G(u_2)$ for any two distinct vertices u_1,u_2 of G, we get $k=0$ in (2) and (5). Similar to the proof of (i), (10) holds. □

Combining Propositions 2.1, 2.3, 2.4 and Theorem 3.4, we have the following results.

Corollary 3.5 *Let G be a connected graph of order m ($m \geqslant 2$).*

(i) *For $n \geqslant 7$, $I(G[P_n]) = m(\lfloor \frac{n}{2} \rfloor + 1)$;*

(ii) *For $n \geqslant 12$, $I(G[C_n]) = \begin{cases} \frac{mn}{2}, & n \text{ is even,} \\ \frac{m(n+3)}{2}, & n \text{ is odd.} \end{cases}$*

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