Uniquely K_r-Saturated Graphs

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Abstract

A graph G is *uniquely K_r-saturated* if it contains no clique with r vertices and if for all edges e in the complement, $G + e$ has a unique clique with r vertices. Previously, few examples of uniquely K_r-saturated graphs were known, and little was known about their properties. We search for these graphs by adapting orbital branching, a technique originally developed for symmetric integer linear programs. We find several new uniquely K_r-saturated graphs with $4 \leqslant r \leqslant 7$, as well as two new infinite families based on Cayley graphs for $\mathbb{Z}_n$ with a small number of generators.

1 Introduction

A graph G is *uniquely H-saturated* if there is no subgraph of G isomorphic to H, and for all edges e in the complement of G there is a unique subgraph in $G+e$ isomorphic to H[2]. Uniquely H-saturated graphs were introduced by Cooper, Lenz, LeSaulnier, Wenger, and West [9] where they classified uniquely C_k-saturated graphs for $k \in \{3, 4\}$; in each case there is a finite number of graphs. Wenger [26, 27] classified the uniquely C_5-saturated graphs and proved that there do not exist any uniquely C_k-saturated graphs for $k \in \{6, 7, 8\}$.

In this paper, we focus on the case where $H = K_r$, the complete graph of order r. Usually K_r is the first graph considered for extremal and saturation problems. However, we find that classifying all uniquely K_r-saturated graphs is far from trivial, even in the case that $r = 4$.

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[2]A technicality: for all $t < n(H)$, the complete graph K_t is trivially uniquely H-saturated. We adopt the convention that always $n(G) \geqslant n(H)$.

Previously, few examples of uniquely K_r-saturated graphs were known, and little was known about their properties. We adapt the computational technique of orbital branching into the graph theory setting to search for uniquely K_r-saturated graphs. Orbital branching was originally introduced by Ostrowski, Linderoth, Rossi, and Smriglio [20] to solve symmetric integer programs. We further extend the technique to use augmentations that are customized to this problem. By executing this search, we found several new uniquely K_r-saturated graphs for $r \in \{4, 5, 6, 7\}$ and we provide constructions of these graphs to understand their structure. One of the graphs we discovered is a Cayley graph, which led us to design a search for Cayley graphs that are uniquely K_r-saturated. Motivated by these search results, we construct two new infinite families of uniquely K_r-saturated Cayley graphs.

Erdős, Hajnal, and Moon [10] studied the minimum number of edges in a K_r-saturated graph. They proved that the only extremal examples are the graphs formed by adding $r-2$ dominating vertices to an independent set; these graphs are also uniquely K_r-saturated. However, if G is uniquely K_r-saturated and has a dominating vertex, then deleting that vertex results in a uniquely K_{r-1}-saturated graph. To avoid the issue of dominating vertices, we define a graph to be *r-primitive* if it is uniquely K_r-saturated and has no dominating vertex. Understanding which r-primitive graphs exist is fundamental to characterizing uniquely K_r-saturated graphs.

Since $K_3 \cong C_3$, the uniquely K_3-saturated graphs were proven by Cooper et al. [9] to be stars and Moore graphs of diameter two. While stars are uniquely K_3-saturated, they are not 3-primitive. The Moore graphs of diameter two are exactly the 3-primitive graphs; Hoffman and Singleton [14] proved there are a finite number of these graphs.

David Collins and Bill Kay discovered the only previously known infinite family of r-primitive graphs, that of complements of odd cycles: $\overline{C_{2r-1}}$ is r-primitive. Collins and Cooper discovered two more 4-primitive graphs of orders 10 and 12 [8]. These two graphs are described in detail in Section 5.

One feature of all previously known r-primitive graphs is that they are all regular. Since proving regularity has been instrumental in previous characterization proofs (such as [9, 14]), there was a hope that r-primitive graphs are regular. However, we present a counterexample: a 5-primitive graph on 16 vertices with minimum degree 8 and maximum degree 9.

The major open question in this area concerns the number of r-primitive graphs for a fixed r.

Conjecture 1 (Cooper [8])**.** For each $r \geqslant 3$, there are a finite number of r-primitive graphs.

This conjecture is true for $r = 3$ [14] and otherwise completely open. Before this work, it was not even known if there was more than one r-primitive graph for any $r \geqslant 5$. After we discovered the graphs in this work (which lack any common structure and sometimes appear very strange), we are unsure the conjecture holds even for $r = 4$.

In Section 2, we briefly summarize our results, including our computational method,

the new sporadic[3] r-primitive graphs, and our new algebraic constructions.

2 Summary of results

Our results have three main components. First, we develop a computational method for generating uniquely K_r-saturated graphs. Then, based on one of the generated examples, we construct two new infinite families of uniquely K_r-saturated graphs. Finally, we describe all known uniquely K_r-saturated graphs, including the nine new sporadic graphs found using the computational method.

2.1 Computational method

In Section 3, we develop a new technique for exhaustively searching for uniquely K_r-saturated graphs on n vertices. The search is based on the technique of *orbital branching* originally developed for use in symmetric integer programs by Ostrowski, Linderoth, Rossi, and Smriglio [20, 21]. We focus on the case of constraint systems with variables taking value in $\{0, 1\}$. Orbital branching is based on the standard branch-and-bound technique where an unassigned variable is selected and the search branches into cases for each possible value for that variable. In a symmetric constraint system, the automorphisms of the variables that preserve the constraints and variable values generate orbits of variables. Orbital branching selects an orbit of variables and branches in two cases. The first branch selects an arbitrary representative variable is selected from the orbit and set to zero. The second branch sets all variables in the orbit to one.

We extend this technique to be effective to search for uniquely K_r-saturated graphs. We add an additional constraint to partial graphs: if a pair v_i, v_j is a non-edge in G, then there is a unique set $S_{i,j}$ containing $r-2$ vertices so that $S_{i,j}$ is a clique and every edge between $\{v_i, v_j\}$ and $S_{i,j}$ is included in G. This guarantees that there is at least one copy of K_r in $G + v_iv_j$ for all assignments of edges and non-edges to the remaining unassigned pairs. The orbital branching method is customized to enforce this constraint, which leads to multiple edges being added to the graph in every augmentation step. By executing this algorithm, we found 10 new r-primitive graphs.

2.2 New r-primitive graphs

For $r \in \{4, 5, 6, 7, 8\}$, we used this method to exhaustively search for uniquely K_r-saturated graphs of order at most N_r, where $N_4 = 20$, $N_5 = N_6 = 16$, and $N_7 = N_8 = 17$. Table 1 lists the r-primitive graphs that were discovered in this search. Most graphs do not fit a short description and are labeled $G_N^{(i)}$, where N is the number of vertices and $i \in \{A, B, C\}$ distinguishes between graphs of the same order.

[3]We call a graph *sporadic* if it has not yet been extended to an infinite family. Therefore, even though our search found 10 new graphs, one extended to an infinite family and so is not sporadic.

n	13	15	16	16	17	18
r	4	6	5	6	7	4
Graphs	G_{13}, Paley(13)	$G_{15}^{(A)}, G_{15}^{(B)}$	$G_{16}^{(A)}, G_{16}^{(B)}$	$G_{16}^{(C)}$	$\overline{C}(\mathbb{Z}_{17}, \{1,4\})$	$G_{18}^{(A)}, G_{18}^{(B)}$

Table 1: Newly discovered r-primitive graphs.

In all, ten new graphs were discovered to be uniquely K_r-saturated by this search. Explicit constructions of these graphs are given in Section 5. Two graphs found by computer search are vertex-transitive and have a prime number of vertices. Observe that vertex-transitive graphs with a prime number of vertices are Cayley graphs. One vertex-transitive 4-primitive graph is the Paley graph of order 13 (see [22]). The other vertex-transitive graph is 7-primitive on 17 vertices and is 14 regular. However, it is easier to understand its complement, which is the Cayley graph for $\mathbb{Z}_{17}$ generated by 1 and 4. This graph is listed as $\overline{C}(\mathbb{Z}_{17}, \{1,4\})$ in Table 1 and is the first example of our new infinite families, described below.

2.3 Algebraic Constructions

For a finite group Γ and a generating set $S \subseteq \Gamma$, let $C(\Gamma, S)$ be the *undirected Cayley graph* for Γ generated by S: the vertex set is Γ and two elements $x, y \in \Gamma$ are adjacent if and only if there is a $z \in S$ where $x = yz$ or $x = yz^{-1}$. When $\Gamma \cong \mathbb{Z}_n$, the resulting graph is also called a *circulant graph*. The cycle C_n can be described as the Cayley graph of $\mathbb{Z}_n$ generated by 1. Since $\overline{C_{2r-1}}$ is r-primitive and we discovered a graph on 17 vertices whose complement is a Cayley graph with two generators, we searched for r-primitive graphs when restricted to complements of Cayley graphs with a small number of generators.

For a finite group Γ and a set $S \subseteq \Gamma$, the *Cayley complement* $\overline{C}(\Gamma, S)$ is the complement of the Cayley graph $C(\Gamma, S)$. We restrict to the case when $\Gamma = \mathbb{Z}_n$ for some n, and the use of the complement allows us to use a small number of generators while generating dense graphs.

We search for r-primitive Cayley complements by enumerating all small generator sets S, then iterate over n where $n \geqslant 2 \max S + 1$ and build $\overline{C}(\mathbb{Z}_n, S)$. If $\overline{C}(\mathbb{Z}_n, S)$ is r-primitive for any r, it must be for $r = \omega(\overline{C}(\mathbb{Z}_n, S)) + 1$, so we compute this r using Niskanen and Östergård's *cliquer* library [19]. Also using *cliquer*, we count the number of r-cliques in $\overline{C}(\mathbb{Z}_n, S) + \{0, i\}$ for all $i \in S$. Since $\overline{C}(\mathbb{Z}_n, S)$ is vertex-transitive, this provides sufficient information to determine if $\overline{C}(\mathbb{Z}_n, S)$ is r-primitive. The successful parameters for r-primitive Cayley complements with g generators are given in Tables 1(a) ($g = 2$), 1(b) ($g = 3$), and 1(c) ($g \geqslant 4$).

For two and three generators, a pattern emerged in the generating sets, and interpolating the values of n and r resulted in two infinite families of r-primitive graphs:

Theorem 2. *Let $t \geqslant 2$ and set $n = 4t^2 + 1, r = 2t^2 - t + 1$. Then, $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.*

(a) Two Generators

t	S	r	n
2	$\{1,4\}$	7	17
3	$\{1,6\}$	16	37
4	$\{1,8\}$	29	65
5	$\{1,10\}$	46	101
6	$\{1,12\}$	67	145

(b) Three Generators

t	S	r	n
2	$\{1,5,6\}$	9	31
3	$\{1,8,9\}$	22	73
4	$\{1,11,12\}$	41	133
5	$\{1,14,15\}$	66	211
6	$\{1,17,18\}$	97	307

(c) Sporadic Cayley Complements

g	S	r	n
3	$\{1,3,4\}$	4	13
4	$\{1,5,8,34\}$ $\{1,11,18,34\}$	28	89
5	$\{1,5,14,17,25\}$	19	71
5	$\{1,6,14,17,36\}$	27	101
6	$\{1,6,16,22,35,36\}$	21	97
6	$\{1,8,23,26,43,64\}$	54	185
7	$\{1,20,23,26,30,32,34\}$	15	71
8	$\{1,8,12,18,22,27,33,47\}$	20	97
9	$\{1,4,10,16,25,27,33,40,64\}$	28	133

Table 2: Cayley complement parameters for r-primitive graphs over $\mathbb{Z}_n$.

Theorem 3. *Let $t \geqslant 2$ and set $n = 9t^2-3t+1, r = 3t^2-2t+1$. Then, $\overline{C}(\mathbb{Z}_n, \{1, 3t-1, 3t\})$ is r-primitive.*

An important step to proving these Cayley complements are r-primitive is to compute the clique number. Computing the clique number or independence number of a Cayley graph is very difficult, as many papers study this question [12, 16], including in the special cases of circulant graphs [2, 5, 15, 28] and Paley graphs [1, 3, 4, 7]. Our enumerative approach to Theorem 2 and discharging approach to Theorem 3 provide a new perspective on computing these values.

It remains an open question if an infinite family of Cayley complements $\overline{C}(\mathbb{Z}_n, S)$ exist for a fixed number of generators $g = |S|$ where $g \geqslant 4$. For all known constructions with $g \neq 4$, observe that the generators are roots of unity in $\mathbb{Z}_n$ with $x^{2g} \equiv 1 \pmod{n}$ for each generator x. Being roots of unity is not a sufficient condition for the Cayley complement to be r-primitive, but this observation may lead to algebraic techniques to build more infinite families of Cayley complements.

Determining the maximum density of a clique and independent set for infinite Cayley graphs (i.e., $\overline{C}(\mathbb{Z}, S)$, where S is finite) would be useful for providing bounds on the finite graphs. Further, such bounds could be used by algorithms to find and count large cliques and independent sets in finite Cayley graphs.

3 Orbital branching using custom augmentations

In this section, we describe a computational method to search for uniquely K_r-saturated graphs. We shall build graphs piece-by-piece by selecting pairs of vertices to be edges or non-edges.

To store partial graphs, we use the notion of a *trigraph*, defined by Chudnovsky [6] and used by Martin and Smith [17]. A *trigraph* T is a set of n vertices $v_1, \ldots, v_n$ where every pair v_iv_j is colored black, white, or gray. The black pairs represent edges, the white edges represent non-edges, and the gray edges are unassigned pairs. A graph G is a *realization* of a trigraph T if all black pairs of T are edges of G and all white pairs of T are non-edges of G. Essentially, a realization is formed by assigning the gray pairs to be edges or non-edges. In this way, we consider a graph to be a trigraph with no gray pairs.

Non-edges play a crucial role in the structure of uniquely K_r-saturated graphs. Given a trigraph T and a pair v_iv_j, a set S of $r-2$ vertices is a *K_r-completion* for v_iv_j if every pair in $S \cup \{v_i, v_j\}$ is a black edge, except for possibly v_iv_j. Observe that a K_r-free graph is uniquely K_r-saturated if and only if every non-edge has a unique K_r-completion.

We begin with a completely gray trigraph and build uniquely K_r-saturated graphs by adding black and white pairs. If we can detect that no realization of the current trigraph can be uniquely K_r-saturated, then we backtrack and attempt a different augmentation. The first two constraints we place on a trigraph T are:

(C1) There is no black r-clique in T.
(C2) Every vertex pair has at most one black K_r-completion.

It is clear that a trigraph failing either of these conditions will fail to have a uniquely K_r-saturated realization.

We use the symmetry of trigraphs to reduce the number of isomorphic duplicates. The *automorphism group* of a trigraph T is the set of permutations of the vertices that preserve the colors of the pairs. These automorphisms are computed with McKay's *nauty* library [13, 18] through the standard method of using a layered graph.

3.1 Orbital Branching

Ostrowski, Linderoth, Rossi, and Smriglio introduced the technique of *orbital branching* for symmetric integer programs with 0-1 variables [20] and for symmetric constraint systems [21]. Orbital branching extends the standard branch-and-bound strategy of combinatorial optimization by exploiting symmetry to reduce the search space. We adapt this technique to search for graphs by using trigraphs in place of variable assignments.

Given a trigraph T, compute the automorphism group and select an orbit $\mathcal{O}$ of gray pairs. Since every representative pair in $\mathcal{O}$ is identical in the current trigraph, assigning any representative to be a white pair leads to isomorphic trigraphs. Hence, we need only attempt assigning a single pair in $\mathcal{O}$ to be white. The natural complement of this operation is to assign all pairs in $\mathcal{O}$ to be black. Therefore, we branch on the following two options:

- *Branch 1*: Select any pair in $\mathcal{O}$ and assign it the color white.
- *Branch 2*: Assign all pairs in $\mathcal{O}$ the color black.

A visual representation of this branching process is presented in Figure 1(a).

An important part of this strategy is to select an appropriate orbit. The selection should attempt to maximize the size of the orbit (in order to exploit the number of pairs assigned in the second branch) while preserving as much symmetry as possible (in order to maintain large orbits in deeper stages of the search). It is difficult to determine the appropriate branching rule *a priori*, so it is beneficial to implement and compare the performance of several branching rules.

This use of orbital branching suffices to create a complete search of all uniquely K_r-saturated graphs, but is not very efficient. One significant drawback to this technique is the fact that the constraints (C1) and (C2) rely on black pairs forming cliques. In the next section, we create a custom augmentation step that is aimed at making these constraints trigger more frequently and thereby reducing the number of generated trigraphs.

3.2 Custom augmentations

We search for uniquely K_r-saturated graphs by enforcing at each step that every white pair has a unique K_r-completion. We place the following constraints on a trigraph:

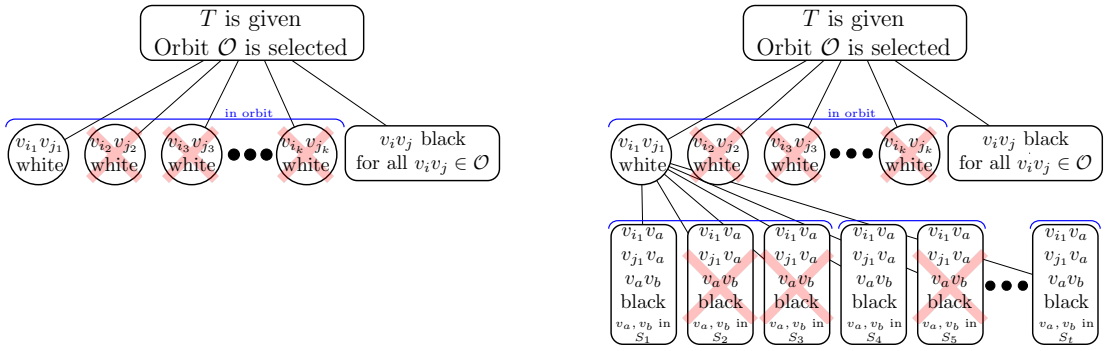
(C3) If v_iv_j is a white edge, then there exists a unique K_r-completion $S \subseteq \{v_1, \ldots, v_n\}$ for v_iv_j.

To enforce the constraint (C3), whenever we assign a white pair we shall also select a set of $r-2$ vertices to be the K_r-completion and assign the appropriate pairs to be black. The orbital branching procedure was built to assign only one white pair in a given step, so we can attempt all possible K_r-completions for that pair. However, if we perform an automorphism calculation and only augment for one representative set from every orbit of these sets, we can reduce the number of isomorphic duplicates.

We follow a two-stage orbital branching procedure. In the first stage, we select an orbit $\mathcal{O}$ of gray pairs. Either we select a representative pair $v_{i'}v_{j'} \in \mathcal{O}$ to set to white or assign v_iv_j to be black for all pairs $v_iv_j \in \mathcal{O}$. In order to guarantee constraint (C3), the white pair must have a K_r-completion. We perform a second automorphism computation to find $\mathrm{Stab}_{\{v_{i'},v_{j'}\}}(T)$, the set of automorphisms that set-wise stabilize the pair $v_{i'}v_{j'}$. Then, we compute all orbits of $(r-2)$-subsets S in $\{v_1, \ldots, v_n\} \setminus \{v_i, v_j\}$ under the action of $\mathrm{Stab}_{\{v_{i'},v_{j'}\}}(T)$. The second stage branches on each set-orbit $\mathcal{A}$, selects a single representative $S' \in \mathcal{A}$ and adds all necessary black pairs to make S' be a K_r-completion for $v_{i'}v_{j'}$. If at any point we attempt to assign a white pair to be black, that branch fails and we continue with the next set-orbit.

This branching process on a trigraph T is:

- *Branch 1*: Select any pair $v_{i_1}v_{j_1} \in \mathcal{O}$ to be white.
  - *Sub-Branch*: For *every* orbit $\mathcal{A}$ of $(r-2)$-subsets of $V(T) \setminus \{v_{i_1}, v_{i_2}\}$ under the action of $\mathrm{Stab}_{\{v_{i_1}, v_{j_1}\}}(T)$, select any set $S \in \mathcal{A}$, assign $v_{i_1}v_a$, $v_{j_1}v_a$, and v_av_b to be black for all $v_a, v_b \in S$.
- *Branch 2*: Set v_iv_j to be black for all pairs $v_iv_j \in \mathcal{O}$.



(a) Standard orbital branching.

(b) Custom augmentations.

Figure 1: Visual description of the branching process.

The full algorithm to output all uniquely K_r-saturated graphs on n vertices is given as the recursive method SaturatedSearch(n, r, T) in Algorithm 1, while the branching procedure is represented in Figure 1(b). The algorithm is initialized using the trigraph corresponding to a single white pair with a K_r-completion. The first step of every recursive call to SaturatedSearch(n, r, T) is to verify the constraints (C1) and (C2). If either constraint fails, no realization of the current trigraph can be uniquely K_r-saturated, so we return. After verifying the constraints, we perform a simple propagation step: If a gray pair $\{i, j\}$ has a K_r-completion we assign that pair to be white. We can assume that this pair is a white edge in order to avoid violation of (C1), and this assignment satisfies (C3).

The missing component of this algorithm is the *branching rule*: the algorithm that selects the orbit of unassigned pairs to use in the first stage of the branch. Based on experimentation, the most efficient branching rule we implemented only considers pairs where both vertices are contained in assigned pairs (if they exist) or pairs where one vertex is contained in an assigned pair (which must exist, otherwise), and selects from these pairs the orbit of largest size. This choice would guarantee the branching orbit has maximum interaction with currently assigned edges while maximizing the effect of assigning all representatives to be edges in the second branch.

Algorithm 1 SaturatedSearch(n, r, T)

if T contains a black r-clique **then**
  Constraint (C1) fails.
  return
else if there exists a pair v_iv_j with two K_r-completions in T **then**
  Constraint (C2) fails.
  return
else if there are no gray pairs **then**
  The trigraph T is uniquely K_r-saturated.
  Output T.
  return
end if
Propagate under constraint (C1).
for all gray pairs v_iv_j **do**
  if v_iv_j has a K_r-completion in T **then**
    Assign v_iv_j to be white.
  end if
end for
Compute pair orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots,$ of gray pairs $\{i, j\}$.
Select an orbit $\mathcal{O}_k$ using the branching rule.
Branch 1.
Let $v_{i'}v_{j'}$ be a representative of $\mathcal{O}_k$.
Compute orbits $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\ell$ of $(r-2)$-vertex sets in $\{v_1, \ldots, v_n\} \setminus \{v_{i'}, v_{j'}\}$.
for $t \in \{1, \ldots, \ell\}$ **do**
  Let S be a representative of $\mathcal{A}_t$.
  if $v_{i'}v_a, v_{j'}v_a, v_av_b$ not white for all $a, b \in S$ **then**
    Sub-Branch: Create T' from T by assigning $v_{i'}v_a, v_{j'}v_a, v_av_b$ to be black for all $a, b \in S$.
    call SaturatedSearch(n, r, T')
  end if
end for
Branch 2: Create T'' from T by assigning v_iv_j to be black for all $v_iv_j \in \mathcal{O}_k$.
call SaturatedSearch(n, r, T'')
return

n	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
10	0.10 s	0.37 s	0.13 s	0.01 s	0.01 s
11	0.68 s	5.25 s	1.91 s	0.28 s	0.09 s
12	4.58 s	1.60 m	25.39 s	1.97 s	1.12 s
13	34.66 s	34.54 m	6.53 m	59.94 s	20.03 s
14	4.93 m	10.39 h	5.13 h	20.66 m	2.71 m
15	40.59 m	23.49 d	10.08 d	12.28 h	1.22 h
16	6.34 h	1.58 y	1.74 y	34.53 d	1.88 d
17	3.44 d			8.76 y	115.69 d
18	53.01 d				
19	2.01 y				
20	45.11 y				

Table 3: CPU times to search for uniquely K_r-saturated graphs of order n. Execution times from the Open Science Grid [23] using the University of Nebraska Campus Grid [25]. The nodes available on the University of Nebraska Campus Grid consist of Xeon and Opteron processors with a range of speed between 2.0 and 2.8 GHz.

3.3 Implementation, Timing, and Results

The full implementation is available as the *Saturation* project in the *SearchLib* software library[4]. More information for the implementation is given in the *Saturation* User Guide, available with the software. In particular, the user guide details the methods for verifying the constraints (C1), (C2), and (C3). When $r \in \{4, 5\}$, we monitored clique growth using a custom data structure, but when $r \geqslant 6$ an implementation using Niskanen and Östergård's *cliquer* library [19] was more efficient.

Our computational method is implemented using the *TreeSearch* library [24], which abstracts the search structure to allow for parallelization to a cluster or grid. Table 3 lists the CPU time taken by the search for each $r \in \{4, 5, 6, 7, 8\}$ and $10 \leqslant n \leqslant N_r$ (where $N_4 = 20$, $N_5 = N_6 = 16$, and $N_7 = N_8 = 17$) until the search became intractable for $n = N_r + 1$. Table 1 lists the r-primitive graphs of these sizes. Constructions for the graphs are given in Section 5.

4 Infinite families of r-primitive Cayley graphs

In this section, we prove Theorems 2 and 3, which provide our two new infinite families of r-primitive graphs. We begin with some definitions that are common to both proofs.

Fix an integer n, a generator set $S \subseteq \mathbb{Z}_n$, and a Cayley complement $G = \overline{C}(\mathbb{Z}_n, S)$. For a set $X \subseteq \mathbb{Z}_n$ with $r = |X|$, list the elements of X as $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_{r-1} < n$. We shall assume that X is a clique in G (or in $G + e$ for some nonedge $e \in E(\overline{G})$).

[4] *SearchLib* is available online at http://www.math.uiuc.edu/~stolee/SearchLib/

Considering X as a subset of $\mathbb{Z}_n$, we let the kth *block* B_k be the elements of $\mathbb{Z}_n$ increasing from x_k (inclusive) to x_{k+1} (exclusive): $B_k = \{x_k, x_k + 1, \ldots, x_{k+1} - 1\}$. Note that $|B_k| = x_{k+1} - x_k$; we call a block of size s an *s-block*. For an integer $t \geqslant 1$ and $j \in \{0, \ldots, r-1\}$, the jth *frame* F_j is the collection of t consecutive blocks in increasing order starting from B_j: $F_j = \{B_j, B_{j+1}, \ldots, B_{j+\ell-1}\}$. A *frame family* is a collection $\mathcal{F}$ of frames.

If F is a frame (or any set of blocks), define $\sigma(F) = \sum_{B_j \in F} |B_j|$, the number of elements covered by the blocks in F.

Observation 4. *If X is a clique in $\overline{C}(\mathbb{Z}_n, S)$ and F is a set of consecutive blocks in X, then $\sigma(F) \notin S$.*

4.1 Two Generators

Theorem 2. *Let $t \geqslant 1$, and set $n = 4t^2 + 1$, $r = 2t^2 - t + 1$. Then, $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.*

Proof. Let $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$. Note that G is regular of degree $n - 5$. If $t = 1$, then $n = 5$, G is an empty graph, and $r = 2$, and empty graphs are 2-primitive. Therefore, we consider $t \geqslant 2$.

Claim 5. *For a clique X, every frame F_j has at least one block of size at least three, and $\sigma(F_j) \geqslant 2t + 1$.*

All blocks B_j have at least two elements, since no pair of elements in X may be consecutive in $\mathbb{Z}_n$, so $\sigma(F_j) \geqslant 2t$. If for all $B_k \in F_j$ the block length $|B_k|$ is exactly two, then $\sigma(F_j) = 2t \in S$. Hence, there is some $B_k \in F_j$ so that $|B_k| \geqslant 3$ and $\sigma(F_j) \geqslant 2t + 1$.

We now prove there is no r-clique in G.

Claim 6. $\omega(G) < r$.

Suppose $X \subseteq \mathbb{Z}_n$ is a clique of order r in G. Let $\mathcal{F}$ be the frame family of all frames ($\mathcal{F} = \{F_j : j \in \{0, \ldots, r-1\}\}$) and consider the sum $\sum_{j=0}^{r-1} \sigma(F_j)$. Using the bound $\sigma(F_j) \geqslant 2t + 1$, we have that this sum is at least $(2t + 1)r$. Each block length $|B_k|$ is counted in t evaluations of $\sigma(F_j)$ (for $j \in \{k-t+1, k-t+2, \ldots, k\}$). This sum counts each element of $\mathbb{Z}_n$ exactly t times, giving value tn. This gives $tn = \sum_{j=0}^{r-1} \sigma(F_j) \geqslant (2t + 1)r$, but $tn = 4t^3 + t < 4t^3 + t + 1 = (2t + 1)r$, a contradiction. Hence, X does not exist, proving the claim.

To prove unique saturation, we consider only the non-edge $\{0, 1\}$ since G is vertex-transitive and the map $x \mapsto -2tx$ is an automorphism of G mapping the edge $\{0, 2t\}$ to $\{0, -4t^2\} \equiv \{0, 1\} \pmod{n}$.

Claim 7. *There is a unique r-clique in $G + \{0, 1\}$.*

We may assume $X = \{0, 1, x_2, \ldots, x_{r-1}\}$ is an r-clique in $G + \{0, 1\}$. We use the frame family $\mathcal{F}$ defined as

$$\mathcal{F} = \{F_{jt+1} : j \in \{0, \ldots, 2t-2\}\}.$$

Note that $\mathcal{F}$ contains $2t-1$ *disjoint* frames containing disjoint blocks, and the block $B_0 = \{x_0\}$ is not contained in any frame within $\mathcal{F}$. Hence, $n-1 = \sum_{F\in\mathcal{F}} \sigma(F)$. By Claim 5, we know that every frame $F \in \mathcal{F}$ has $\sigma(F) \geqslant 2t+1$. This lower bound gives $\sum_{F\in\mathcal{F}} \sigma(F) \geqslant (2t+1)(2t-1) = n-2$. Thus, considering $\sigma(F)$ as an integer variable for each $F \in \mathcal{F}$, all solutions to the integer program with constraints $\sigma(F) \geqslant 2t+1$ and $\sum_{F\in\mathcal{F}} \sigma(F) = n-1$ have $\sigma(F) = 2t+1$ for all $F \in \mathcal{F}$ except a unique $F' \in \mathcal{F}$ with $\sigma(F') = 2t+2$.

The frame F' has two possible ways to attain $\sigma(F') = 2t+2$: (a) have two blocks of size three, or (b) have one block of size four. However, if F' has a block of size four, then there is a 2-block $B_j \in F'$ on one end of F' where $\sigma(F' \setminus \{B_j\}) = 2t \in S$, a contradiction. Thus, F' has two blocks of size three. In addition, if F' has fewer than $t-2$ blocks of size two between the two blocks of size three, then there is a pair $x, y \in X$ with $y = x + 2t$. Therefore, F' has two blocks of size three and they are the first and last blocks of F'.

This frame family demonstrates the following properties of X. First, there are exactly $2t$ blocks of size three ($2t-2$ frames have exactly one and F' has exactly two). Second, there is no set of t consecutive blocks of size two. Finally, no two blocks of size three have fewer than $t-2$ blocks of size two between them.

Consider the position of a 3-block in the first frame, F_1. If there are two 3-blocks in F_1, they appear as the first and last blocks in F_1, but then the distance from x_0 to x_{t-1} is $2t$, a contradiction. Since there is exactly one 3-block, B_k, in F_1, suppose $k < t$. Then the distance from x_0 to x_{t-1} is $2t$. Hence, B_t is the 3-block in F_1. By symmetry, there must be $t-1$ 2-blocks between the 3-block in $F_{(2t-2)t+1}$ and x_0.

Let $B_{k_1}, B_{k_2}, \ldots, B_{k_{2t}}$ be the 3-blocks in X with $k_1 < k_2 < \cdots < k_{2t}$. By the position of the 3-block in F_1, we have $k_1 = t$. By the position of the 3-block in $F_{(2t-2)t+1}$, we have $k_{2t} = (2t-2)t+1$. Since 3-blocks must be separated by at least $t-1$ 2-blocks, $k_{j+1} - k_j \geqslant t-1$ but since $k_{2t} = (2t-1)(t-1)+k_1$ we must have equality: $k_{j+1} - k_j = t-1$. Assuming X is an r-clique, it is uniquely defined by these properties. Indeed, all vertices of this set are adjacent. □

4.2 Three Generators

Theorem 3. *Let $t \geqslant 1$ and set $n = 9t^2-3t+1, r = 3t^2-2t+1$. Then, $\overline{C}(\mathbb{Z}_n, \{1, 3t-1, 3t\})$ is r-primitive.*

Proof. Let $G = \overline{C}(\mathbb{Z}_n, \{1, 3t-1, 3t\})$. Observe that G is vertex-transitive and there are automorphisms mapping $\{0, 3t-1\}$ to $\{0, 1\}$ or $\{0, 3t\}$ to $\{0, 1\}$. Thus, we only need to verify that G has no r-clique and $G + \{0, 1\}$ has a unique r-clique.

We prove that G is r-primitive in three steps. First, we show that there is no r-clique in G in Claim 11 using discharging. Second, assuming there are no 2-blocks in an r-clique of $G + \{0, 1\}$, we prove in Claim 12 that there is a unique such clique. This proof uses a counting method similar to the proof of Claim 7. Finally, we show that any r-clique in $G + \{0, 1\}$ cannot contain any 2-blocks. This step is broken into Claims 13 and 14, both of which slightly modify the discharging method from Claim 11 to handle the 1-block. Claim 14 requires a detailed case analysis.

We use several figures to aid the proof. Figure 2 shows examples of common features from these figures.

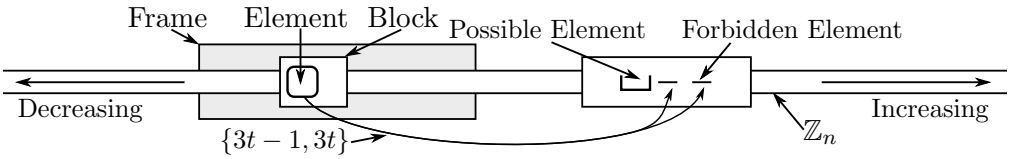


Figure 2: Key to later figures

We begin by showing some basic observations that are used frequently in the rest of the proof. These observations focus on interactions among blocks that are forced by the generators $3t-1$ and $3t$. In the observations below, we define functions φ_s and ψ_s that map s-blocks of X to other blocks of X. Always, φ_s maps blocks *forward* ($\varphi_s(B_k)$ has higher index than B_k) while ψ_s maps blocks *backward* ($\psi_s(B_k)$ has lower index than B_k).

It is intuitive that a maximum size clique uses as many small blocks as possible, to increase the density of the clique within G. However, Observation 8 shows that every 2-block induces a block of size at least five *in both directions*.

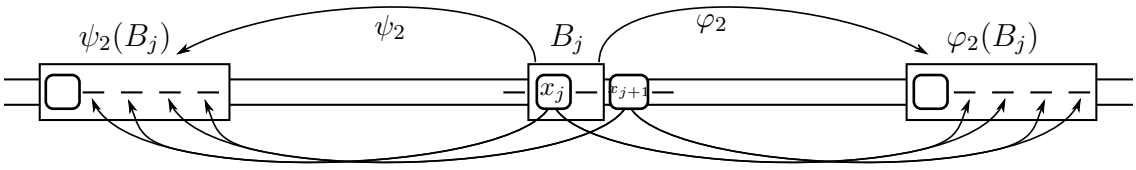


Figure 3: Observation 8 and a 2-block B_j.

Observation 8 (2-blocks)**.** *Let B_j be a 2-block, so $x_{j+1} = x_j + 2$. The elements x_j and x_{j+1} along with generators $3t-1$ and $3t$ guarantee that the sets $\{x_j + 3t - 1, x_j + 3t, x_j + 3t + 1, x_j + 3t + 2\}$ and $\{x_j - 3t, x_j - 3t + 1, x_j - 3t + 2, x_j - 3t + 3\}$ do not intersect X. Since these sets contain consecutive elements, each set is contained within a single block of X. We will use $\varphi_2(B_j)$ to denote the block containing $x_j + 3t$ and $\psi_2(B_j)$ to denote the block containing $x_j - 3t$. Both $\varphi_2(B_j)$ and $\psi_2(B_j)$ have size at least five.*

If in fact multiple 2-blocks induce the same big block, Observation 9 implies the big block has even larger size.

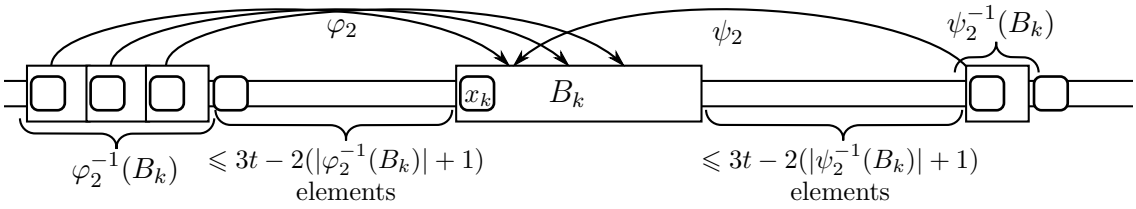


Figure 4: Observation 9 and a block B_k.

Observation 9 (Big blocks)**.** *Let B_k be a block of size at least five. The set $\varphi_2^{-1}(B_k)$ is the set of 2-blocks B_j so that $\varphi_2(B_j) = B_k$. Similarly, $\psi_2^{-1}(B_k)$ is the set of 2-blocks B_j so that $\psi_2(B_j) = B_k$. Note that when $s = |\varphi_2^{-1}(B_k)|$, there are at least $s+1$ elements of X (s from the 2-blocks in $\varphi_2^{-1}(B_k)$ and one following the last 2-block in $\varphi_2^{-1}(B_k)$) that block $2(s+1)$ elements from containment in X using the generators $3t-1$ and $3t$. Therefore,*

$$|B_k| \geqslant 2|\varphi_2^{-1}(B_k)| + 3, \quad \text{and} \quad |B_k| \geqslant 2|\psi_2^{-1}(B_k)| + 3.$$

Further, there are at most $3t - 2(|\varphi_2^{-1}(B_k)| + 1)$ elements between B_k and the last block of $\varphi_2^{-1}(B_k)$. Similarly, there are at most $3t - 2(|\psi_2^{-1}(B_k)| + 1)$ elements between B_k and the first block of $\psi_2^{-1}(B_k)$.

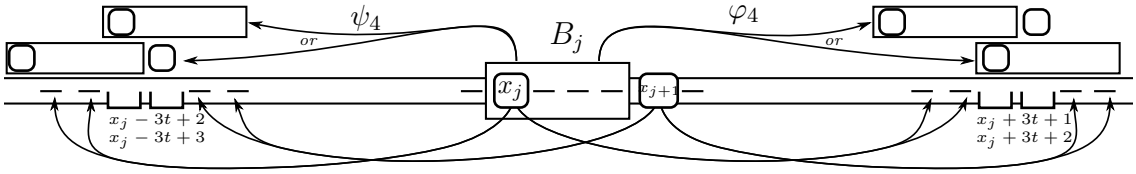


Figure 5: Observation 10 and a 4-block B_j.

Observation 10 (4-blocks)**.** *Let B_j be a 4-block, so $x_{j+1} = x_j + 4$. The elements $\{x_j+3t-1, x_j+3t, x_j+3t+3, x_j+3t+4\}$ are not contained in X, so $X \cap \{x_j+3t-1, \ldots, x_j+3t+4\} \subseteq \{x_j+3t+1, x_j+3t+2\}$. In G, no two elements of X are consecutive elements of $\mathbb{Z}_n$, so there is at most one element in this range. If there is no element of X in $\{x_j+3t+1, x_j+3t+2\}$, then there is a block of size at least seven that contains $x_j + 3t + 1$. Otherwise, there is a single element in $X \cap \{x_j + 3t + 1, x_j + 3t + 2\}$ and one of the adjacent blocks has size at least four. We use $\varphi_4(B_j)$ to denote one of these blocks of size at least four. By symmetry, we use $\psi_4(B_j)$ to denote a block of size at least four that contains or is adjacent to the block containing $x_j - 3t + 2$. In $G + \{0, 1\}$, the only elements of X that can be consecutive are 0 and 1, let $B_0 = \{0\}$ denote the first block of X. Thus, let $\varphi_4(B_j) = B_0$ if $x_j + 3t + 1 = 0$ and $\psi_4(B_j) = B_0$ if $x_j - 3t + 2 = 0$.*

We now use a two-stage discharging method to prove that there is no r-clique X in G. In Stage 1, we assign charge to the blocks of X and discharge so that all blocks have non-negative charge. In Stage 2, we assign charge to the frames of X using the new charges on the blocks and then discharge among the frames.

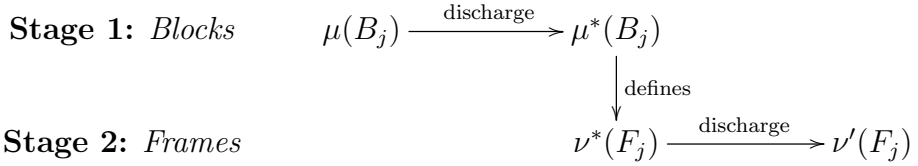


Figure 6: The two-stage discharging method.

We will use this framework three times, in Claims 11, 13, and 14, but we use a different set of rules for Stage 1 each time. Stage 2 will always use the same discharging rule.

Claim 11. $\omega(G) < r$.

Proof of Claim 11. Suppose X is an r-clique in G.

Let μ be a charge function on the blocks of X defined by $\mu(B_j) = |B_j| - 3$. All 2-blocks have charge -1, 3-blocks have charge 0, and all other blocks have positive charge. Moreover, the total charge on all blocks is

$$\sum_{j=0}^{r-1} \mu(B_j) = n - 3r = 3t - 2.$$

We shall discharge among the blocks to form a new charge function μ^*.

Stage 1α: Discharge by shifting one charge from $\varphi_2(B_j)$ to B_j for every 2-block B_j.

After Stage 1α, $\mu^*(B_j) = 0$ when $|B_j| \in \{2, 3\}$, $\mu^*(B_j) = 1$ when $|B_j| = 4$, and

$$\mu^*(B_j) = |B_j| - 3 - |\varphi_2^{-1}(B_j)| \geqslant |\varphi_2^{-1}(B_j)|$$

when $|B_j| \geqslant 5$. Note that if $|\varphi_2^{-1}(B_j)| = 0$ for a block B_j of size at least five, then $\mu^*(B_j) \geqslant 2$.

Now, μ^* is a non-negative function and $\sum_{j=0}^{r-1} \mu^*(B_j) = 3t - 2$.

For every frame F_j, define $\nu^*(F_j)$ as $\nu^*(F_j) = \sum_{B_{j+i} \in F_j} \mu^*(B_{j+i})$. Since every block is contained in exactly t frames, the total charge on all frames is

$$\sum_{j=0}^{r-1} \nu^*(F_j) = t \sum_{j=0}^{r-1} \mu^*(B_j) = t(3t - 2) = r - 1.$$

There must exist a frame with $\nu^*(F_j) = 0$; such a frame contains only 2- and 3-blocks. If this frame contained only blocks of length three and at most one block of length two, then $\sigma(F_j) \in \{3t - 1, 3t\}$, contradicting that X is a clique. Thus, any frame with $\nu^*(F_j) = 0$ must contain at least two 2-blocks where all blocks between are 3-blocks.

For each pair $B_k, B_{k'}$ of 2-blocks that are separated only by 3-blocks, define $L_{k,k'}$ to be the set of frames containing both B_k and $B_{k'}$, and $R_{k,k'}$ to be the set of frames containing both $\varphi_2(B_k)$ and $\varphi_2(B_{k'})$. If $\varphi_2(B_k) = \varphi_2(B_{k'})$, then $|R_{k,k'}| = t \geqslant |L_{k,k'}|$. Otherwise, there are fewer elements between $\varphi_2(B_k)$ and $\varphi_2(B_{k'})$ than between B_k and $B_{k'}$, and every block between $\varphi_2(B_k)$ and $\varphi_2(B_{k'})$ has size at least three (a 2-block B_j between $\varphi_2(B_k)$ and $\varphi_2(B_{k'})$ would induce a large block $\psi_2(B_j)$ between B_k and B'_k). Hence, there are at least as many blocks between B_k and B'_k as there are between $\varphi_2(B_k)$ and $\varphi_2(B_{k'})$ and so $|L_{k,k'}| \leqslant |R_{k,k'}|$. Let $f_{k,k'} : L_{k,k'} \to R_{k,k'}$ be any injection where $f_{k,k'}(F_j) = F_j$ for all $F_j \in L_{k,k'} \cap R_{k,k'}$.

Using these injections, we discharge among the frames to form a new charge function ν'.

Stage 2: For every frame F_j and every pair $B_k, B_{k'}$ of 2-blocks in F_j separated by only 3-blocks, F_j pulls one charge from $f_{k,k'}(F_j)$.

Since every frame F_j with $\nu^*(F_j) = 0$ has at least one such pair $B_k, B_{k'}$ and does not contain $\varphi_2(B_i)$ for any 2-block B_i, F_j pulls at least one charge but does not have any charge removed. Thus, $\nu'(F_j) \geqslant 1$.

We will show that frames F_j with $\nu^*(F_j) \geqslant 1$ have strictly less than $\nu^*(F_j)$ charge pulled during the second stage. Let $\{(B_{k_i}, B_{k'_i}; F_{j_i}) : i \in \{1, \ldots, \ell\}\}$ be the set of pairs $B_{k_i}, B_{k'_i}$ of 2-blocks and a common frame F_{j_i} where $f_{k_i,k'_i}(F_{j_i}) = F_j$. Since each map f_{k_i,k'_i} is an injection, the blocks B_{k_i} are distinct for all $i \in \{1, \ldots, \ell\}$, and exactly ℓ charge was pulled from F_j. While $B_{k'_i}$ and $B_{k_{i+1}}$ may be the same block, $B_{k_1}, \ldots, B_{k_\ell}, B_{k'_\ell}$ are $\ell + 1$ distinct 2-blocks. Every block B_{k_i} has $\varphi_2(B_{k_i}) \in F_j$ and $\varphi_2(B_{k'_\ell}) \in F_j$. Thus, $\nu^*(F_j) \geqslant \sum_{B_i \in F_j} |\varphi_2^{-1}(B_i)| \geqslant \ell + 1$, which implies $\nu'(F_j) \geqslant 1$.

Therefore, $\nu'(F_j) \geqslant 1$ for all frames F_j, and $r - 1 = \sum_{j=0}^{r-1} \nu'(F_j) \geqslant r$, a contradiction. Hence, there is no clique of size r in G, proving Claim 11. □

For the remaining claims, we assume X is an r-clique in $G + \{0, 1\}$ where X contains both 0 and 1. Then, B_0 is the block containing exactly $\{0\}$, and all other blocks from X have size at least two. Since 0 and 1 are in X, the sets $\{3t - 1, 3t, 3t + 1\}$ and $\{-3t - 1, -3t, -3t + 1\}$ of consecutive elements do not intersect X. Thus, there are two blocks B_{k_1} and B_{k_2} so that $\{3t - 1, 3t, 3t + 1\} \subset B_{k_1}$ and $\{-3t - 1, -3t, -3t + 1\} \subset B_{k_2}$. When B_{k_1} and B_{k_2} are 4-blocks, then $B_0 = \psi_4(B_{k_1}) = \varphi_4(B_{k_2})$ as in Observation 10.

With the assumption that there are no 2-blocks in X, uniqueness follows through an enumerative proof similar to Claim 7, given as Claim 12. After this claim, Claims 13 and 14 show that X has no 2-blocks, completing the proof.

Claim 12. *There is a unique r-clique in $G + \{0, 1\}$ with no 2-blocks.*

Proof of Claim 12. Consider the frame family $\mathcal{F} = \{F_{jt+1} : j \in \{0, \ldots, 3t - 2\}\}$ of $3t - 1$ disjoint frames. Note that the block B_0 is not contained in any of these frames. Since there are no 2-blocks, $\sigma(F_{jt+1}) \geqslant 3t$, but $\sigma(F_{jt+1}) \neq 3t$ so $\sigma(F_{jt+1}) \geqslant 3t + 1$. Thus,

$$n - 1 = \sum_{F_{jt+1} \in \mathcal{F}} \sigma(F_{jt+1}) \geqslant (3t - 1)(3t + 1) = n - 3.$$

From this inequality we have $\sigma(F_{jt+1}) = 3t + 1$ for all frames except either one frame F_k with $\sigma(F_k) = 3t + 3$ or two frames $F_k, F_{k'}$ with $\sigma(F_k) = \sigma(F_{k'}) = 3t + 2$.

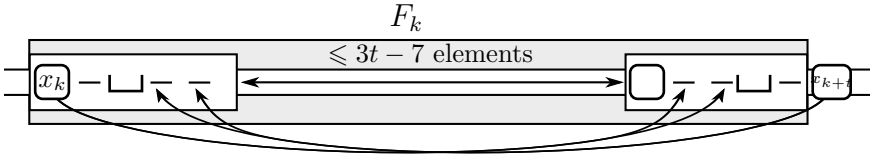


Figure 7: Claim 12, $\sigma(F_k) = 3t + 3$.

Suppose there is a frame F_k with $\sigma(F_k) = 3t+3$. Since $x_{k+t} = x_k + 3t + 3$, the elements

$$x_{k+t} - 3t = x_k + 3, \quad x_{k+t} - (3t - 1) = x_k + 4,$$

$$x_k + 3t - 1 = x_{k+t} - 4, \quad \text{and} \quad x_k + 3t = x_{k+t} - 3,$$

are not contained in X. Since we have no 2-blocks, the elements $x_k + 2$ and $x_{k+t} - 2$ are not in X. Thus, there are two blocks of size at least five in F_k. This means there are $t-2$ blocks for the remaining $3t-7$ elements, but $t-2$ blocks of size at least three cover at least $3t-6$ elements. Hence, no frame has $\sigma(F_k) = 3t+3$.

Suppose we have exactly two frames $F_k, F_{k'} \in \mathcal{F}$ with $\sigma(F_k) = \sigma(F_{k'}) = 3t+2$. If a frame F_j contains a block of size at least six, then $\sigma(F_j) \geqslant 3t+3$, so F_k and $F_{k'}$ each contain either one 5-block or two 4-blocks. However, if the first or last block (denoted by B_j) of F_k (or $F_{k'}$) has size three, then $\sigma(F_k \setminus \{B_j\}) = 3t-1$, a contradiction. Thus, the first and last blocks of F_k and $F_{k'}$ are not 3-blocks and hence are both 4-blocks. Therefore, there are exactly two frames in $\mathcal{F}$ containing exactly two 4-blocks and the rest contain exactly one 4-block, for a total of $3t$ 4-blocks in X.

Let $\ell_1, \ell_2, \ldots, \ell_{3t}$ be the indices of the 4-blocks. Since each frame F_i has at least one 4-block, $\ell_j \leqslant \ell_{j-1} + t$. Also, if a frame F_i has exactly two 4-blocks, then the blocks appear as the first and last blocks in F_j, giving $\ell_j \geqslant \ell_{j-1} + t - 1$.

Consider the position of B_{ℓ_1}. If B_{ℓ_1} is strictly between B_0 and B_{k_1}, then the frame F_1 contains two 4-blocks B_{ℓ_1} and B_{k_1}, and so $B_{\ell_1} = B_1$ and $B_{k_1} = B_t$. However, there are $3t-3$ elements between B_0 and B_{k_1}, but at least $3t-2$ elements between B_0 and B_t. Therefore, $B_{\ell_1} = B_{k_1}$ and there are $t-1$ 3-blocks between B_0 and B_{ℓ_1}, so $\ell_1 = t-1$. Similarly, $B_{\ell_{3t}} = B_{k_2}$ and there are $t-1$ 3-blocks between $B_{\ell_{3t}}$ and B_0, so $\ell_{3t} = (r-1) - (t-1) = 3t^2 - 3t + 1$.

There is exactly one solution to the constraints $\ell_j \in \{\ell_{j-1} + t - 1, \ell_{j-1} + t\}$ and $\ell_{3t} - \ell_1 = 3t^2 - 2t + 1 = (3t-1)(t-1)$ given by $\ell_j = \ell_{j-1} + t - 1$. This uniquely describes X as a clique in $G + \{0, 1\}$. □

We now aim to show that there are no 2-blocks in an r-clique X of G. This property can be quickly checked computationally for $t \leqslant 4$, so we now assume that $t \geqslant 5$.

The problem with applying the discharging method from Claim 11 is that B_0 starts with charge $\mu(B_0) = -2$ and there is no clear place from which to pull charge to make $\mu^*(B_0)$ positive. We define three values, a, b, and c, which quantify the *excess charge* from Stage 1α that can be redirected to B_0 while still guaranteeing that all frames end with positive charge. In Claim 13, we assume $a + b + c \geqslant 3$ and place all of this excess charge on B_0 in Stage 1β, giving $\mu^*(B_0) \geqslant 1$; an identical Stage 2 discharging leads to positive charge on all frames. In Claim 14, Stage 1γ pulls charge from B_{k_1} and B_{k_2} to result in $\mu^*(B_0) = 0$ and possibly $\mu^*(B_{k_1}) = 0$ or $\mu^*(B_{k_2}) = 0$. After Stage 1γ and Stage 2, there may be some frames with ν'-charge zero, but they must contain B_0, B_{k_1}, or B_{k_2}. By carefully analyzing this situation, we find a contradiction in that either X is not a clique or $a + b + c \geqslant 3$.

We now define the quantities a, b, and c.

If a block B_j has size at least five and $\varphi_2^{-1}(B_j)$ is empty, then no charge is removed from B_j in Stage 1α. If charge is pulled from frames containing B_j in Stage 2, there are other blocks that supply the charge required to stay positive. Therefore, we define a to

be the excess μ-charge that can be removed and maintain positive μ^*-charge:

$$a = \sum_{B_j \in \mathcal{A}} \left[|B_j| - 4 \right], \text{ where } \mathcal{A} \text{ is the set of blocks } B_j \text{ with } |B_j| \geqslant 5 \text{ and } \varphi_2^{-1}(B_j) = \emptyset.$$

If a block B_j has size at least five and $\varphi_2^{-1}(B_j)$ is not empty, charge is pulled from B_j in Stage 1α. However, if $|B_j| > 2|\varphi_2^{-1}(B_j)| + 3$, there is more charge left after Stage 1α than is required in Stage 2 to maintain a positive charge on frames containing B_j. We define b to be the excess charge left in this situation:

$$b = \sum_{B_j \in \mathcal{B}} \left[|B_j| - (2|\varphi_2^{-1}(B_j)| + 3) \right],$$

where $\mathcal{B}$ is the set of blocks B_j with $|B_j| \geqslant 5$ and $\varphi_2^{-1}(B_j) \neq \emptyset$.

If there is a frame F_j with three blocks $B_{\ell_0}, B_{\ell_1}, B_{\ell_2}$ where $|B_{\ell_i}| \geqslant 4$ for all $i \in \{0,1,2\}$ and $\varphi_2^{-1}(B_{\ell_1}) = \emptyset$, then let $c = 1$; otherwise $c = 0$. Since every frame containing B_{ℓ_1} also contains B_{ℓ_0} or B_{ℓ_2}, these frames are guaranteed a positive ν'-charge from B_{ℓ_0} or B_{ℓ_2}, so the single charge on B_{ℓ_1} that was not pulled from previous rules is free to pass to B_0.

Claim 13. *Suppose X is a set in $G + \{0,1\}$ with $|X| = r$. If $a + b + c \geqslant 3$, then X is not a clique.*

Proof of Claim 13. We proceed by contradiction, assuming that $a + b + c \geqslant 3$ and X is an r-clique. We shall modify the two-stage discharging from Claim 11 with a more complicated discharging rule to handle B_0 so that the result is the same contradiction: that all r frames have positive charge, but the amount of charge over all the frames is $r - 1$.

Let μ be the charge function on the blocks of X defined by $\mu(B_j) = |B_j| - 3$. We discharge using Stage 1β to form the charge function μ^*.

Stage 1β: There are four discharging rules:

1. If $|B_k| = 2$, B_k pulls one charge from $\varphi_2(B_k)$.
2. B_0 pulls $|B_k| - 4$ charge from every block B_k with $|B_k| \geqslant 5$ and $\varphi_2^{-1}(B_k) = \emptyset$. (The total charge pulled by B_0 in this rule is a.)
3. B_0 pulls $|B_k| - (2|\varphi_2^{-1}(B_k)| + 3)$ charge from every block B_k with $|B_k| \geqslant 5$ and $\varphi_2^{-1}(B_k) \neq \emptyset$. (The total charge pulled by B_0 in this rule is b.)
4. If there is a frame F_j with three blocks $B_{\ell_0}, B_{\ell_1}, B_{\ell_2}$ where $|B_{\ell_i}| \geqslant 4$ for all $i \in \{0,1,2\}$ and $\varphi_2^{-1}(B_{\ell_1}) = \emptyset$, then B_0 pulls one charge from B_{ℓ_1}. (The amount of charge pulled by B_0 in this rule is c.)

Since $a + b + c \geqslant 3$, B_0 pulls at least 3 charge, so $\mu^*(B_0) \geqslant 1$. Blocks of size two and three have μ^*-charge zero. If a block B_k has size four or has size at least five and $\varphi_2^{-1}(B_k) = \emptyset$, then $\mu^*(B_k) = 1$ except B_{ℓ_1} where $\mu^*(B_{\ell_1}) = 0$. Similarly, a block B_k of size at least five with $\varphi_2^{-1}(B_k) \neq \emptyset$ has charge $\mu^*(B_k) = |\varphi_2^{-1}(B_k)|$.

For every frame F_j, define $\nu^*(F_j) = \sum_{B_{j+i} \in F_j} \mu^*(B_{j+i})$. Note that if the charge $\nu^*(F_j)$ is zero, every block in F_j has zero charge since $\mu^*(B_k) \geqslant 0$ for all blocks.

Stage 2: For every frame F_j and every pair $B_k, B_{k'}$ of 2-blocks in F_j separated by only 3-blocks, F_j pulls one charge from $f_{k,k'}(F_j)$.

If $\nu^*(F_j) = 0$, then F_j contains only blocks B_k with $\mu^*(B_k) = 0$. These blocks are 2-blocks, 3-blocks, and B_{ℓ_1}. However, any frame that contains B_{ℓ_1} also contains B_{ℓ_0} or B_{ℓ_2}, which have positive charge. Thus, frames F_j with $\nu^*(F_j) = 0$ contain only 2- and 3-blocks. Since $\sigma(F_j) \notin \{3t, 3t-1\}$, F_j must contain at least two 2-blocks $B_k, B_{k'}$, so F_j pulls at least one charge in the second stage and loses no charge, so $\nu'(F_j) \geqslant 1$.

If $\nu^*(F_j) \geqslant 1$, the amount of charge pulled from F_j in Stage 2 is the number of 2-block pairs $B_k, B_{k'}$ separated by 3-blocks so that $\varphi_2(B_k), \varphi_2(B_{k'}) \in F_j$. Observe $\mu^*(B_i) = |\varphi_2^{-1}(B_i)|$ for all blocks B_i with $\varphi_2^{-1}(B_i) \neq \emptyset$, so $\nu^*(F_j) = \sum_{B_i \in F_j} \mu^*(B_i) \geqslant \sum_{B_i \in F_j} |\varphi_2^{-1}(B_i)|$. If there are ℓ pairs $B_k, B_{k'}$ that pull one charge from F_j in Stage 2, then there are at least $\ell + 1$ 2-blocks in $\cup_{B_i \in F_j} \varphi_2^{-1}(B_i)$, and $\nu^*(F_j) \geqslant \ell + 1$.

Therefore, $\nu'(F_j) \geqslant 1$ for all $j \in \{0, \ldots, r-1\}$, but since

$$r \leqslant \sum_{j=0}^{r-1} \nu'(F_j) = \sum_{j=0}^{r-1} \nu^*(F_j) = t \sum_{j=0}^{r-1} \mu^*(B_j) = t \sum_{j=0}^{r-1} \mu(B_j) = t(n-3r) = r-1,$$

we have a contradiction, and so X is not a clique. □

Claim 14. *If X is an r-clique in $G + \{0,1\}$ that contains a 2-block, then $a + b + c \geqslant 3$.*

Proof of Claim 14. We shall repeat the two-stage discharging from Claim 11 with a simpler rule for discharging to B_0 than in Claim 13. After this discharging is complete, we will investigate the configuration of blocks surrounding one of the 2-blocks and show that the sum $a + b + c$ has value at least three.

Let μ be the charge function on the blocks of X defined by $\mu(B_j) = |B_j| - 3$. We use Stage 1γ to discharge among the blocks and form a charge function μ^*.

Stage 1γ: We have two discharging rules:

1. If $|B_j| = 2$, B_j pulls one charge from $\varphi_2(B_j)$.
2. B_0 pulls one charge from B_{k_1} and one charge from B_{k_2}.

After the first rule within Stage 1γ there is at least one charge on all blocks of size at least four. Thus, removing one more charge from each of B_{k_1} and B_{k_2} in the second rule of Stage 1γ maintains that $\mu^*(B_{k_1})$ and $\mu^*(B_{k_2})$ are non-negative. Since B_0 receives two charge and every 2-block receives one charge, $\mu^*(B_j)$ is non-negative after Stage 1γ for all blocks B_j.

Define the charge function $\nu^*(F_j) = \sum_{B_i \in F_j} \mu^*(B_i)$.

Stage 2: For every frame F_j and every pair $B_k, B_{k'}$ of 2-blocks in F_j separated by only 3-blocks, F_j pulls one charge from $f_{k,k'}(F_j)$.

Again, $\sum_{j=0}^{r-1} \nu'(F_j) = r - 1$. Also, $\nu'(F_j) > 0$ whenever F_j contains a block of order at least four that is not B_{k_1} or B_{k_2}, or F_j contains two 2-blocks separated only by 3-blocks. Since one charge was removed from B_{k_1} and B_{k_2} in Stage 1γ, the frames containing B_{k_1} or

B_{k_2} are no longer guaranteed to have positive charge, but still have non-negative charge. In order to complete the proof of Claim 14, we must more closely analyze the charge function ν'.

Definition 15 (Pull sets)**.** A *pull set* is a set of blocks, $\mathcal{P} = \{B_{i_1}, \ldots, B_{i_p}\}$, where $|B_{i_j}| \geqslant 5$ for all $j \in \{1, \ldots, p\}$ and all blocks between B_{i_j} and $B_{i_{j+1}}$ are 3-blocks. Let $\varphi_2^{-1}(\mathcal{P}) = \cup_{B_i \in \mathcal{P}} \varphi_2^{-1}(B_i)$. A pull set $\mathcal{P}$ is *perfect* if all blocks $B_i \in \mathcal{P}$ have $|B_i| = 2|\varphi_2^{-1}(B_i)| + 3$. Otherwise, a pull set $\mathcal{P}$ contains a block $B_i \in \mathcal{P}$ with $|B_i| \geqslant 2|\varphi_2^{-1}(B_i)| + 4$ and $\mathcal{P}$ is *imperfect*. Given a pull set $\mathcal{P}$, the *defect* of $\mathcal{P}$ is $\delta(\mathcal{P}) = \sum_{B_i \in \mathcal{P}} \left[\mu^*(B_i) - |\varphi_2^{-1}(B_i)|\right] - 1$.

The defect $\delta(\mathcal{P})$ measures the amount of excess charge (more than one charge) the pull set $\mathcal{P}$ contributes to the ν'-charge of any frame containing $\mathcal{P}$. Note that pull sets $\mathcal{P}$ with $B_{k_1}, B_{k_2} \notin \mathcal{P}$ have defect $\delta(\mathcal{P}) \geqslant 0$, with equality if and only if $\mathcal{P}$ is perfect. Perfect pull sets $\mathcal{P}$ containing B_{k_1} or B_{k_2} have defect $\delta(\mathcal{P}) = -1$. For a block $B_i \in \mathcal{P}$, if $d \leqslant \mu^*(B_i) - |\varphi_2^{-1}(B_i)|$ then we say B_i *contributes* d to the defect of $\mathcal{P}$.

Consider a pull set $\mathcal{P} = \{B_{i_1}, \ldots, B_{i_p}\}$. Since there are at most $3t-4$ elements between $\varphi_2^{-1}(B_{i_p})$ and B_{i_p} and all blocks from B_{i_1} to B_{i_p} have order at least three, there exists a frame that contains all blocks of $\mathcal{P}$. Therefore, every pull set is contained within *some* frame.

If B_i is a block with $|B_i| \geqslant 5$, then $\mathcal{P} = \{B_i\}$ is a (not necessarily maximal) pull set, and $\{B_i\}$ is a subset of each frame containing B_i. For every frame F_j and block $B_i \in F_j$ with $|B_i| \geqslant 5$ there is a unique maximal pull set $\mathcal{P} \subseteq F_j$ containing B_i. Thus, if there are multiple maximal pull sets within a frame F_j, then they are disjoint.

Observation 16. *Let X be an r-clique and ν' be the charge function on frames of X after Stage 1γ and Stage 2. Then, for a frame F_j, $\nu'(F_j)$ is at least the sum of*

1. *the number of distinct pairs $B_k, B_{k'}$ of 2-blocks in F_j separated only by 3-blocks,*
2. *the number of 4-blocks in F_j not equal to B_{k_1}, B_{k_2},*
3. *$1 + \delta(\mathcal{P})$ for every maximal pull set $\mathcal{P} \subseteq F_j$.*

In Claim 14.4, we prove there exists a special block B_* in a frame F_z with $\nu'(F_z) = 0$. The proof of Claim 14.4 reduces to three special cases that are handled in Claims 14.1-14.3.

Recall $\sum_{j=0}^{r-1} \nu'(F_j) = r - 1$. Let Z be the number of frames F with $\nu'(F) = 0$. Then,

$$\sum_{j:\nu'(F_j)>0} [\nu'(F_j) - 1] = \sum_{j=0}^{r-1} [\nu'(F_j) - 1] + Z = (r-1) - r + Z = Z - 1.$$

Therefore, if there are at most $t+1$ frames with ν'-charge zero ($\nu'(F_j) = 0$), then the sum $\sum_{j:\nu'(F_j)>0}[\nu'(F_j) - 1]$ is bounded above by t. The proof of Claim 14.4 frequently reduces to a contradiction with this bound. Claims 14.1-14.3 provide some situations that guarantee this sum has value at least $t + 1$.

Claim 14.1. *Let $\mathcal{P}$ be a pull set containing a block B_j. If $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$ and $x_{k_1} + 6t^2 \leqslant x_j \leqslant x_{k_2}$, then there is a set $\mathcal{H}$ of frames with $\sum_{F_j \in \mathcal{H}} (\nu'(F_j) - 1) \geqslant t + 1$.*

Proof of Claim 14.1. Starting with $\mathcal{P}^{(0)} = \mathcal{P}$, we construct a sequence $\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\ell)}$ of pull sets with $\ell \leqslant \lceil \frac{t+1}{2} \rceil + 1$. We build $\mathcal{P}^{(k)}$ by following the map ψ_2 from $\varphi_2^{-1}(\mathcal{P}^{(k-1)})$. This process will continue until one of the sets is not a pull set, one of the sets is an imperfect pull set, or we reach $\lceil \frac{t+1}{2} \rceil$ pull sets. In either case, we find a set $\mathcal{H}$ of frames that satisfies the claim.

We initialize $\mathcal{P}^{(0)}$ to be $\mathcal{P}$, which contains B_j. Note that it is possible that $B_j = B_{k_2}$, but otherwise B_j precedes B_{k_2}. There will be at most $6t$ elements covered by the blocks starting at $\mathcal{P}^{(k)}$ to the blocks preceding $\mathcal{P}^{(k-1)}$. Note that since $x_j - x_{k_1} \geqslant 6t^2$, $\mathcal{P}^{(k)}$ will not contain B_{k_1} or B_{k_2} for any $k \in \{1, \ldots, \lceil \frac{t+2}{2} \rceil\}$.

Let $k \geqslant 1$ be so that $\mathcal{P}^{(k-1)}$ is a perfect pull set with $|\varphi_2^{-1}(\mathcal{P}^{(k-1)})| \geqslant 2$. For every block $B_i \in \mathcal{P}^{(k-1)}$, let B_ℓ be a 2-block in $\varphi_2^{-1}(B_i)$ and place $\psi_2(B_\ell)$ in $\mathcal{P}^{(k)}$. Then, place any block of size at least five that is positioned between to blocks of $\mathcal{P}^{(k)}$ into $\mathcal{P}^{(k)}$.

If $\mathcal{P}^{(k)}$ is always perfect for all $k \leqslant \lceil \frac{t+1}{2} \rceil$, then we have pull sets $\mathcal{P}^{(0)}$, $\ldots$, $\mathcal{P}^{(k)}$ and frames F_{j_0}, $F_{j'_0}$, $\ldots$, $F_{j_{k-1}}$, $F_{j'_{k-1}}$, where $k = \lceil \frac{t+1}{2} \rceil$. Thus, let $\mathcal{H} = \{F_{j_\ell}, F_{j'_\ell} : \ell \in \{1, \ldots, k\}\}$ and $\sum_{F \in \mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$, proving the claim. It remains to show that such a set $\mathcal{H}$ exists if some $\mathcal{P}^{(k)}$ is imperfect.

If $\mathcal{P}^{(k)}$ is a perfect pull set with $|\varphi_2^{-1}(\mathcal{P}^{(k)})| \geqslant 2$, then let F_{j_k} be the frame that starts at the last block of $\mathcal{P}^{(k)}$ and $F_{j'_k}$ be the frame that ends at the first block of $\mathcal{P}^{(k)}$. We claim that F_{j_k} and $F_{j'_k}$ have ν'-charge at least two. There are at most $3t - 4$ elements between the last block in $\mathcal{P}^{(k)}$ and the last 2-block in $\psi_2^{-1}(\mathcal{P}^{(k)})$. If there is at most one 2-block in F_{j_k}, then $\sigma(F_{j_k}) \geqslant 2 + 3(t-2) + 5 = 3t + 3$ and F_{j_k} contains all 2-blocks in $\psi_2^{-1}(\mathcal{P}^{(k)})$, a contradiction. Therefore, the frame F_{j_k} contains at least two 2-blocks. If those 2-blocks are separated by three blocks, they pull at least one charge in Stage 2. If those 2-blocks are not separated by three blocks, then either they are separated by a 4-block (which contributes at least one charge) or a second maximal pull set (which contributes at least one charge). Thus, $\nu'(F_{j_k}) \geqslant 2$. By a symmetric argument, $F_{j'_k}$ contains two 2-blocks and has $\nu'(F_{j'_k}) \geqslant 2$. Figure 8 shows how the frames F_{j_k} and $F_{j'_k}$ are placed among the pull sets $\mathcal{P}^{(k-1)}$ and $\mathcal{P}^{(k)}$.

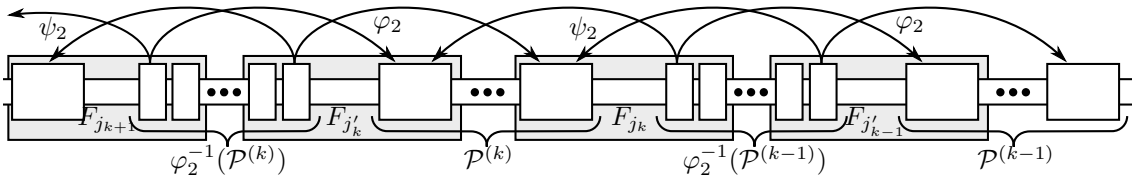


Figure 8: Claim 14.1, building $\mathcal{P}^{(k)}$ and frames $F_{j_k}, F_{j'_k}$.

If $\mathcal{P}^{(k)}$ is not a perfect pull set or $|\varphi_2^{-1}(\mathcal{P}^{(k)})| < 2$, either $\mathcal{P}^{(k)}$ is not a pull set or $\mathcal{P}^{(k)}$ is an imperfect pull set.

Case 1: $\mathcal{P}^{(k)}$ is not a pull set. In this case, there is a non-3-block B_j not in $\mathcal{P}^{(k)}$ that is between two blocks B_{ℓ_1}, B_{ℓ_2} of $\mathcal{P}^{(k)}$. If $|B_j| \geqslant 5$, then B_j would be added to $\mathcal{P}^{(k)}$. Therefore, $|B_j| \in \{2, 4\}$.

Case 1.i: $|B_j| = 4$. Every frame containing B_j also contains either B_{ℓ_1} or B_{ℓ_2}. Therefore, these t frames contain a 4-block and at least one pull set with non-negative defect so they have ν'-charge at least two. The frame starting at B_{ℓ_1} also contains B_j and B_{ℓ_2}, so this frame has two disjoint maximal pull sets and a 4-block and has ν'-charge at least three. Therefore, if $\mathcal{H}$ is the family of frames containing B_j, $\sum_{F\in\mathcal{H}}[\nu'(F)-1] \geqslant t+1$.

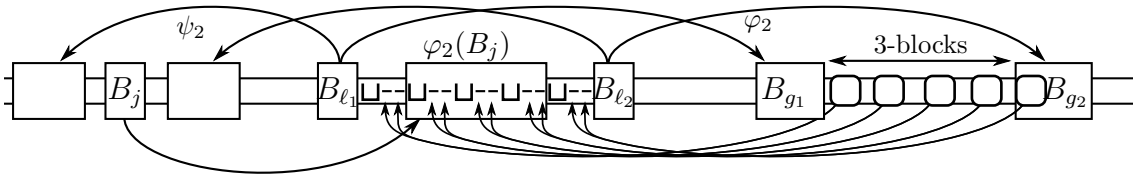


Figure 9: Claim 14.1, Case 1.ii.

Case 1.ii: $|B_j| = 2$. Let B_{ℓ_1} be the last 2-block preceding $\varphi_2(B_j)$ and B_{ℓ_2} be the first 2-block following $\varphi_2(B_j)$. Note that B_j is between $\psi_2(B_{\ell_1})$ and $\psi_2(B_{\ell_2})$, which must be in $\mathcal{P}^{(k)}$.

(a) Suppose $\{\varphi_2(B_j)\}$ is an imperfect pull set. Then $\varphi_2(B_j)$ contributes one to the defect of any pull set containing $\varphi_2(B_j)$. Place all frames containing $\varphi_2(B_j)$ into $\mathcal{H}$, as they have ν'-charge at least two. Also place the frame F starting at $\psi_2(B_{\ell_1})$ into $\mathcal{H}$. If F also contains $\psi_2(B_{\ell_2})$, it contains two disjoint maximal pull sets and thus has ν'-charge at least two. Otherwise, F must contain at least two 2-blocks that either pull a charge in Stage 2 or are separated by a block of size at least four; $\nu'(F) \geqslant 2$ in any case. This frame family $\mathcal{H}$ satisfies the claim.

(b) Suppose $\{\varphi_2(B_j)\}$ is a perfect pull set. Therefore, $|\varphi_2(B_j)| = 3+2h$ for some integer $h \geqslant 1$ and hence is odd. Let $B_{g_1} = \varphi_2(B_{\ell_1})$ and $B_{g_2} = \varphi_2(B_{\ell_2})$. Since B_{g_1} and B_{g_2} are in $\mathcal{P}^{(k-1)}$ and $\mathcal{P}^{(k-1)}$ is a pull set, there are only 3-blocks between B_{g_1} and B_{g_2}. Therefore, the elements $x_{g_1+1}, x_{g_1+2}, \ldots, x_{g_2}$ have $x_{g_1+i+1} = x_{g_1+i}+3$ for all $i \in \{1,\ldots,g_2-g_1-1\}$. The generators $3t-1$ and $3t$ guarantee that the elements of X strictly between x_{ℓ_1} and x_{ℓ_2} are a subset of $\{x_{\ell_1}+2+3i : i \in \{0,1,\ldots,g_2-g_1\}\}$. Therefore, all blocks between B_{ℓ_1} and B_{ℓ_2} (including $\varphi_2(B_j)$) have size divisible by three. So, $|\varphi_2(B_j)|$ is an odd multiple of three, but strictly larger than three; $|\varphi_2(B_j)| \geqslant 9$ and $|\varphi_2^{-1}(\varphi_2(B_j))| \geqslant 3$.

There are $t-2$ frames containing the first three 2-blocks in $\varphi_2^{-1}(\varphi_2(B_j))$. Since these 2-blocks are consecutive, each frame pulls two charge in Stage 2. Also, let F' be the frame whose last two blocks are the first two 2-blocks in $\varphi_2^{-1}(\varphi_2(B_j))$ and let F'' be the frame whose first two blocks are the last two 2-blocks in $\varphi_2^{-1}(\varphi_2(B_j))$. Either F' contains $\psi_2(B_{\ell_1})$ or contains another 2-block preceding $\varphi_2^{-1}(\varphi_2(B_j))$, and thus $\nu'(F') \geqslant 2$; by symmetric argument, $\nu'(F'') \geqslant 2$. Let $\mathcal{H}$ contain these frames, and note that $\sum_{F\in\mathcal{H}}[\nu'(F)-1] \geqslant t$. Also, add the frame F_i whose last block is $\varphi_2(B_j)$ to $\mathcal{H}$. If this frame is already included in $\mathcal{H}$, then the charge contributed by $\varphi_2(B_j)$ was not counted in the previous bound and $\sum_{F\in\mathcal{H}}[\nu'(F)-1] \geqslant t+1$. Otherwise, F_i does not contain two 2-blocks from $\varphi_2^{-1}(\varphi_2(B_j))$ and so F_i spans fewer

than $3t-8$ elements preceding $\varphi_2(B_j)$. Thus, F_i contains at least two 2-blocks that are separated either by only 3-blocks (where F_i pulls a charge in Stage 2) or by a block of size at least four (which contributes at least an additional charge to F_i), and so $\nu'(F_i) \geqslant 2$ and $\sum_{F\in\mathcal{H}}[\nu'(F)-1] \geqslant t+1$.

Case 2: $\mathcal{P}^{(k)}$ is an imperfect pull set. There is a block $B_\ell \in \mathcal{P}^{(k)}$ so that $|B_\ell| \geqslant 2|\varphi_2^{-1}(B_\ell)|+4$. Since B_ℓ contributes at least one to the defect of every pull set that contains B_ℓ, every frame containing B_ℓ has ν'-charge at least two. Let F_{j_k} be the frame that starts at the last block of $\mathcal{P}^{(k)}$, and note that F_{j_k} contains at least two 2-blocks. Therefore, F_{j_k} either contains a pull set and two 2-blocks separated by only 3-blocks, two disjoint maximal pull sets, or a pull set and a 4-block and in any case has ν'-charge at least two. If F_{j_k} contains B_ℓ, then one of the pull sets in F_{j_k} is imperfect and $\nu'(F_{j_k}) \geqslant 3$. Therefore, let $\mathcal{H}$ contain F_{j_k} and the frames containing B_ℓ, and $\mathcal{H}$ satisfies the claim. □

Claim 14.2. *Let B_i be a 5-block with $x_{k_2}-9t \leqslant x_i \leqslant x_{k_2}$. If every pull set $\mathcal{P}$ containing B_i has $|\varphi_2^{-1}(\mathcal{P})| = |\varphi_2^{-1}(B_i)| = 1$, then there is a set $\mathcal{H}$ of frames with $\sum_{F_j\in\mathcal{H}}(\nu'(F_j)-1) \geqslant t+1$.*

Proof of Claim 14.2. Let $B_j = \psi_2(\varphi_2^{-1}(B_i))$. If there is a pull set $\mathcal{P}$ containing B_j where $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then Claim 14.1 applies to $\mathcal{P}$ and we can set $\mathcal{H}$ to be the $t+1$ frames with ν'-charge at least two. Therefore, we assume no such pull set exists. This implies $|\varphi_2^{-1}(B_j)| \in \{0,1\}$.

We shall construct two disjoint sets $\mathcal{H}_1$ and $\mathcal{H}_2$ so that $\sum_{F\in\mathcal{H}_1}[\nu'(F_j)-1] \geqslant t$ and $\sum_{F\in\mathcal{H}_2}[\nu'(F)-1] \geqslant 1$ so $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ satisfies $\sum_{F_j\in\mathcal{H}}(\nu'(F_j)-1) \geqslant t+1$. To guarantee disjointness, there are blocks that must be contained in frames of $\mathcal{H}_2$ that cannot be contained in frames of $\mathcal{H}_1$. For instance, a frame in $\mathcal{H}_2$ may contain B_j, but no frames in $\mathcal{H}_1$ may contain B_j.

If $\varphi_2^{-1}(B_j) = \emptyset$ or if $|B_j| \geqslant 6$, then B_j contributes one to the defect of every pull set containing B_j, and hence every frame containing B_j has charge at least two. Place all of these frames in $\mathcal{H}_2$ and $\sum_{F\in\mathcal{H}_2}[\nu'(F)-1] \geqslant t$.

Therefore, we may assume that $|\varphi_2^{-1}(B_j)| = 1$ and $|B_j| = 5$. Hence, there are exactly $3t-4$ elements between $\varphi_2^{-1}(B_j)$ and B_j. Similarly, there are exactly $3t-4$ elements between B_j and $\psi_2^{-1}(B_j)$. In either of these regions, not all blocks may be 3-blocks. Let B_{g_1} be the last non-3-block preceding B_j and B_{g_2} be the first non-3-block following B_j. We shall guarantee that all frames in $\mathcal{H}_2$ contain at least one of B_j, B_{g_1}, or B_{g_2}.

There are exactly $3t-4$ elements between $\varphi_2^{-1}(B_i)$ and B_i. Since $3t-4 \equiv 2 \pmod 3$, this range contains at least one 2-block, two 4-blocks, or one block of order at least five. Let B_{ℓ_1} be the first non-3-block following $\varphi_2^{-1}(B_i)$ and B_{ℓ_2} be the first non-3-block preceding B_i.

Figure 10 demonstrates the arrangement of the blocks $B_i, B_j, B_{g_1}, B_{g_2}, B_{\ell_1}$, and B_{ℓ_2}, as well as two blocks B_{h_1} and B_{h_2} that will be selected later in a certain case based on the sizes of B_{g_1} and B_{g_2}.

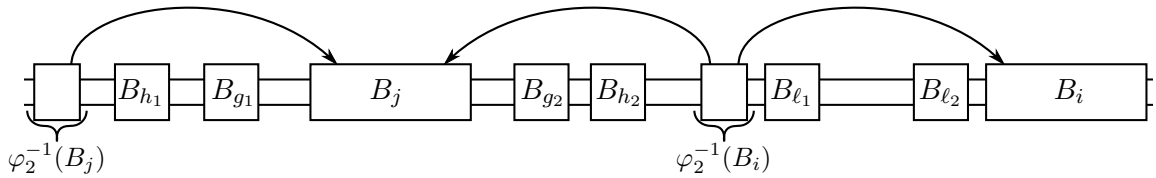


Figure 10: The blocks involved in the proof of Claim 14.2.

We consider cases depending on $|B_{\ell_1}|$ and $|B_{\ell_2}|$ and either find a contradiction or find at least one frame F to place in $\mathcal{H}_1$ so that F does not contain B_j or B_{g_2} and $[\nu'(F)-1] \geqslant 1$.

Case 1: $|B_{\ell_1}| = 2$. The block $\varphi_2(B_{\ell_1})$ follows B_i. If all blocks between B_i and $\varphi_2(B_{\ell_1})$ are 3-blocks, then B_i and $\varphi_2(B_{\ell_1})$ are contained in a common pull set $\mathcal{P}$ with $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, which we assumed does not happen. Therefore, there is a block B_k between B_i and $\varphi_2(B_{\ell_1})$ that is not a 3-block. If B_k is a 2-block, then $\psi_2(B_k)$ would be a large block between $\varphi_2^{-1}(B_i)$ and B_{ℓ_1}, a contradiction. If B_k is a 4-block, then $\psi_4(B_k)$ would be a large block between $\varphi_2^{-1}(B_i)$ and B_{ℓ_1}, another contradiction. Therefore, $|B_k| \geqslant 5$, but $\varphi_2^{-1}(B_k) = \emptyset$, since otherwise a 2-block from $\varphi_2^{-1}(B_k)$ would be strictly between $\varphi_2^{-1}(B_i)$ and B_{ℓ_1}. Then, every frame containing B_k has ν'-charge at least two. The frame F_k does not contain B_j, B_{g_1}, or B_{g_2}, so place F_k in $\mathcal{H}_1$.

Case 2: $|B_{\ell_2}| \geqslant 5$. If $\varphi_2^{-1}(B_{\ell_2}) \neq \emptyset$, B_{ℓ_2} and B_i are in a common pull set $\mathcal{P}$ with $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, but we assumed this did not happen. Therefore, $\varphi_2^{-1}(B_{\ell_2}) = \emptyset$ and every frame containing B_{ℓ_2} has ν'-charge at least two. The frame F_{ℓ_2} does not contain B_j, B_{g_1}, or B_{g_2}, so place F_{ℓ_2} in $\mathcal{H}_1$.

Case 3: $|B_{\ell_1}| \geqslant 5$. Since B_{ℓ_1} and B_i cannot be in a pull set, there is a non-3-block between B_{ℓ_1} and B_i, so $B_{\ell_1} \neq B_{\ell_2}$.

Case 3.i: $|B_{\ell_2}| = 2$. The frame F starting at $\psi_2(B_{\ell_2})$ also contains B_{ℓ_1} but does not contain B_j or B_{g_2}. Since $\varphi_2^{-1}(B_i)$ is between $\psi_2(B_{\ell_2})$ and B_{ℓ_1}, these blocks are in different pull sets and so $\nu'(F) \geqslant 2$. Place F in $\mathcal{H}_1$.

Case 3.ii: $|B_{\ell_2}| = 4$. The frame F starting at B_{ℓ_1} also contains B_{ℓ_2} but not B_j or B_{g_2}. Since F contains two 4-blocks, $\nu'(F) \geqslant 2$. Place F in $\mathcal{H}_1$.

Case 4: $|B_{\ell_1}| = 4$. Since $3t-4 \not\equiv 4 \pmod 3$, B_{ℓ_1} cannot be the only non-3-block between $\varphi_2^{-1}(B_i)$ and B_i, so $B_{\ell_1} \neq B_{\ell_2}$. Consider F_{ℓ_1}, the frame starting at B_{ℓ_1}.

If F_{ℓ_1} does not contain two 2-blocks, $\sigma(F_{\ell_1}) \geqslant 3t-4$ and F_{ℓ_1} contains B_i (and B_{ℓ_2}). If $|B_{\ell_2}| = 2$, then since $4+2 \not\equiv 3t-4 \pmod 3$ there is another block B_k between B_{ℓ_1} and B_i that is not a 3-block. Since F_{ℓ_1} does not contain two 2-blocks, $|B_k| \geqslant 4$ and therefore $\nu'(F_{\ell_1}) \geqslant 2$. Place F_{ℓ_1} in $\mathcal{H}_1$, and note that F_{ℓ_1} does not contain B_j, B_{g_1}, or B_{g_2}.

If F_{ℓ_1} does contain two 2-blocks, then either those two 2-blocks pull an extra charge in Stage 2, or they are separated by a block of size at least four. In either case, $\nu'(F_{\ell_1}) \geqslant 2$ so place F_{ℓ_1} in $\mathcal{H}_1$.

We now turn our attention to placing frames in $\mathcal{H}_2$ based on the sizes of B_{g_1} and B_{g_2}. Note that $\varphi_2^{-1}(B_{g_1}) = \varphi_2^{-1}(B_{g_2}) = \emptyset$, or else Claim 14.1 applies. If $|B_{g_1}| \geqslant 5$, then every frame containing B_{g_1} has ν'-charge at least two, so add these t frames to $\mathcal{H}_2$ to result in $\sum_{F\in\mathcal{H}}[\nu'(F) - 1] \geqslant t + 1$. Similarly, if $|B_{g_2}| \geqslant 5$, then every frame containing B_{g_2} has ν'-charge at least two, so add these frames to $\mathcal{H}_2$. Therefore, we may assume that $|B_{g_1}|, |B_{g_2}| \in \{2, 4\}$, which provides four cases.

Case 1: $|B_{g_1}| = |B_{g_2}| = 2$. There are at most $3t - 4$ elements between B_{g_1} and $\varphi_2(B_{g_1})$ or between $\psi_2(B_{g_2})$ and B_{g_2}. Let B_{h_1} be the last non-3-block preceding B_{g_1} and B_{h_2} be the first non-3-block following B_{g_2}. If B_{h_1} is a 2-block, let $\mathcal{P}_1 = \{\varphi_2(B_{h_1}), \varphi_2(B_{g_1})\}$. There cannot be a 4-block B_k or 2-block $B_{k'}$ between $\varphi_2(B_{h_1})$ and $\varphi_2(B_{g_1})$ or else $\psi_4(B_k)$ or $\psi_2(B_{k'})$ would be between B_{h_1} and B_{g_1}. Therefore, adding any non-3-block between $\varphi_2(B_{h_1})$ and $\varphi_2(B_{g_1})$ to $\mathcal{P}_1$ makes $\mathcal{P}_1$ be a pull set where $|\varphi_2^{-1}(\mathcal{P}_1)| \geqslant 2$, and by Claim 14.1 we are done. Similarly, if B_{ℓ_2}, the first non-3-block following B_{g_2}, is a 2-block, then let $\mathcal{P}_2 = \{\varphi_2(B_{\ell_2}), \varphi_2(B_{g_2})\}$ and we can expand $\mathcal{P}_2$ to a pull set where $|\varphi_2^{-1}(\mathcal{P}_2)| \geqslant 2$, and by Claim 14.1 we are done. Since we assumed this is not the case, B_{h_1} and B_{h_2} have size at least four. Either $\psi_2(B_{g_2}) = B_{h_1}$ or B_{h_1} follows $\psi_2(B_{g_2})$. Either $\varphi_2(B_{g_1}) = B_{h_2}$ or B_{h_2} precedes $\psi_2(B_{g_2})$. Thus, every frame containing B_j also contains B_{h_1} or B_{h_2} and thus contains at least a pull set and a 4-block or two maximal pull sets, which implies that the frame has ν'-charge at least two. Place these frames in $\mathcal{H}_2$.

Case 2: $|B_{g_1}| = |B_{g_2}| = 4$. There are at most $3t - 3$ elements between B_{g_1} and $\varphi_4(B_{g_1})$ or between $\psi_4(B_{g_2})$ and B_{g_2}. Since B_{g_1} is the last non-3-block preceding B_j, either $\psi_4(B_{g_2}) = B_{g_1}$ or $\psi_4(B_{g_2})$ precedes B_{g_1}. Similarly, either $\varphi_4(B_{g_1}) = B_{g_2}$ or $\varphi_4(B_{g_1})$ follows B_{g_1}. Therefore, every frame containing B_j also contains B_{g_1} or B_{g_2} and thus contains a pull set and a 4-block, which implies that the frame has ν'-charge at least two. Place these frames in $\mathcal{H}_2$.

Case 3: $|B_{g_1}| = 2$ *and* $|B_{g_2}| = 4$. There are at most $3t - 4$ elements between B_{g_1} and $\varphi_2(B_{g_1})$ and at most $3t-3$ elements between $\psi_4(B_{g_2})$ and B_{g_2}. Let B_{h_1} be the last non-3-block preceding B_{g_1}. If B_{h_1} is a 2-block, then there is a pull set $\mathcal{P}_1 = \{\varphi_2(B_{h_1}), \varphi_2(B_{g_1})\}$ where $|\varphi_2^{-1}(\mathcal{P}_1)| \geqslant 2$. We assumed this is not the case, so $|B_{h_1}| \geqslant 4$. Either $B_{h_1} = \psi_4(B_{g_2})$ or B_{h_1} follows $\psi_4(B_{g_2})$. Therefore, every frame containing B_j also contains B_{h_1} or B_{g_2} and thus contains a pull set and a 4-block or two maximal pull sets, which implies that the frame has ν'-charge at least two. Place these frames in $\mathcal{H}_2$.

Case 4: $|B_{g_1}| = 4$ *and* $|B_{g_2}| = 2$. This case is symmetric to Case 3.

Thus, $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ has been selected from $\mathcal{H}_1$ and $\mathcal{H}_2$ so that $\sum_{F\in\mathcal{H}}[\nu'(F) - 1] \geqslant t + 1$. □

Claim 14.3. *If there is a block B_ℓ with $|B_\ell| = 4$, $x_{k_2} - 12t \leqslant x_\ell \leqslant x_{k_2}$, and there is a block B_i between $\psi_4(B_\ell)$ and B_ℓ with $|B_i| \neq 3$, then there is a set $\mathcal{H}$ of frames so that* $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$.

Proof of Claim 14.3. Note that it may be the case that $B_\ell = B_{k_2}$. For the remainder of the proof, B_ℓ will not be used to bound the ν'-charge of frames in $\mathcal{H}$ and all other blocks will contain elements between $x_\ell - 12t$ and x_ℓ, so these blocks will not be one of B_0, B_{k_1}, or B_{k_2}.

Let $\psi_4^{(d)}$ denote the dth self-composition of the map ψ_4. Let $D \geqslant 1$ be the first integer so that $|\psi_4^{(D)}(B_\ell)| \neq 4$, if it exists. We will select blocks $B_{\ell_1}, B_{\ell_2}, B_{\ell_3}$, and B_{ℓ_4} based on the value of D. For all $d \leqslant D$, let $B_{\ell_d} = \psi_4^{(d)}(B_\ell)$.

If $D < 4$, then we must use different methods to find the remaining blocks B_{ℓ_d}. Note that $|B_{\ell_D}| \geqslant 5$. If $|\varphi_2^{-1}(B_{\ell_D})| \geqslant 2$, then by Claim 14.1 we are done. If $|\varphi_2^{-1}(B_{\ell_D})| = 1$ and $|B_{\ell_D}| = 5$, then either there is a pull set $\mathcal{P}$ containing B_{ℓ_D} with $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, and by Claim 14.1 we are done or every pull set $\mathcal{P}$ containing B_{ℓ_D} has $|\varphi_2^{-1}(\mathcal{P})| = 1$, and by Claim 14.2 we are done. Therefore, there are two remaining cases for B_{ℓ_D}: either (a) $\varphi_2^{-1}(B_{\ell_D}) = \emptyset$, or (b) $|\varphi_2^{-1}(B_{\ell_D})| = 1$ and $|B_{\ell_D}| \geqslant 6$.

We consider cases based on $|B_i|$.

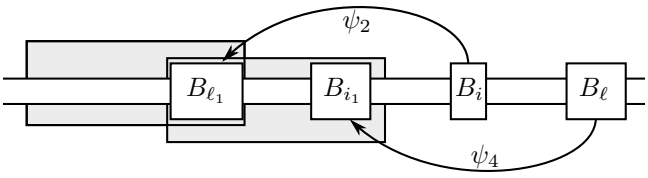


Figure 11: Claim 14.3, Case 1: $|B_\ell| = 4$ and $|B_i| = 2$, shown with $D \geqslant 4$.

Case 1: $|B_i| = 2$. Let $B_{i_1} = \psi_2(B_i)$. B_{i_1} is a block of size at least five preceding B_{ℓ_1}. If there exists a pull set $\mathcal{P}$ containing B_{i_1} so that $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then by Claim 14.1 we are done. Therefore, $|\varphi_2^{-1}(B_{i_1})| \in \{0, 1\}$.

Case 1.i: Suppose $|\varphi_2^{-1}(B_{i_1})| = 1$. If $|B_{i_1}| = 5$, then by Claim 14.2 we are done. Therefore, $|B_{i_1}| \geqslant 6$ and B_{i_1} contributes at least one to the defect of every pull set containing B_{i_1}, so every frame containing B_{i_1} has ν'-charge at least two. Place these frames in $\mathcal{H}$.

There are at most $3t - 4$ elements between B_{i_1} and B_i, so if does not contain B_{ℓ_1}, then F_{i_1} contains at least two 2-blocks. If these 2-blocks are separated only by 3-blocks, then $\nu'(F_{i_1}) \geqslant 3$ because the imperfect pull set containing B_{i_1} contributes two charge and these 2-blocks pull one charge in Stage 2. Otherwise, these 2-blocks are separated by some block of order at least four. Therefore, $\nu'(F_{i_1}) \geqslant 3$ since the imperfect pull set containing B_{i_1} contributes two charge and either the 4-blocks between the 2-blocks contributes one charge or the block of size at least five between the 2-blocks is contained in a pull set that contributes at least one charge. Thus, if F_{i_1} does not contain B_{ℓ_1}, we are done. We now assume that $B_{\ell_1} \in F_{i_1}$.

If $D \geqslant 2$, then $|B_{\ell_1}| = 4$. Then $\nu'(F_{i_1}) \geqslant 3$ because the imperfect pull set containing B_{i_1} contributes two charge and B_{ℓ_1} contributes one charge.

If $D = 1$, then $|B_{\ell_1}| \geqslant 5$. If $\varphi_2^{-1}(B_{\ell_1}) = \emptyset$, then B_{ℓ_1} contributes two charge to F_{i_1} and $\nu'(F_{i_1}) \geqslant 4$. Otherwise $|\varphi_2^{-1}(B_{\ell_1})| = 1$ and $|B_{\ell_1}| \geqslant 6$, so B_{ℓ_1} contributes at least one to the defect of any pull set containing B_{ℓ_1}, and thus $\nu'(F_{i_1}) \geqslant 3$.

Since $\mathcal{H}$ contains t frames of ν'-charge at least two and at least one frame (F_{i_1}) with ν'-charge at least three, $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$.

Case 1.ii: Suppose $|\varphi_2^{-1}(B_{i_1})| = 0$. B_{i_1} contributes at least two to the ν'-charge for every frame containing B_{i_1}. Place these t frames in $\mathcal{H}$. As in Case 1.i, the frame F_{i_1} must have charge $\nu'(F_{i_1}) \geqslant 3$ and $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$.

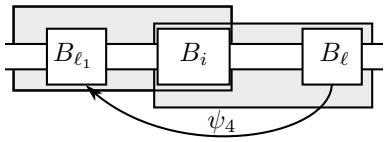


Figure 12: Claim 14.3, Case 2: $|B_\ell| = 4$ and $|B_i| = 2$, shown with $D \geqslant 4$.

Case 2: $|B_i| \geqslant 5$. Let $\mathcal{H}$ be the frames containing B_i. If there exists a pull set $\mathcal{P}$ containing B_i with $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then by Claim 14.1, we are done. If $|B_i| = 5$ and $|\varphi_2^{-1}(B_i)| = 1$, then by Claim 14.2, we are done. Therefore, either $\varphi_2^{-1}(B_i) = \emptyset$ and $|B_i| \geqslant 5$, or $|\varphi_2^{-1}(B_i)| = 1$ and $|B_i| \geqslant 6$. In either case, B_i contributes at least two charge to every frame in $\mathcal{H}$.

Consider the frame $F_{i-t+1} \in \mathcal{H}$ where B_i is the last block of F_{i-t+1}.

If F_{i-t+1} has fewer than two 2-blocks, then $\sigma(F_{i-t+1}) \geqslant 2 + 3(t-2) + |B_i| \geqslant 3t + 1$. Since there are at most $3t - 3$ elements between B_{ℓ_1} and B_ℓ, then $B_{\ell_1} \in F_{i-t+1}$ when F_{i-t+1} has fewer than two 2-blocks. If $|B_{\ell_1}| = 4$, then B_{ℓ_1} contributes another charge to F_{i-t+1} and $\nu'(F_{i-t+1}) \geqslant 3$. If $|B_{\ell_1}| \geqslant 5$ and $\varphi_2^{-1}(B_{\ell_1}) = \emptyset$ and B_{ℓ_1} contributes at least two charge to F_{i-t+1} and $\nu'(F_{i-t+1}) \geqslant 4$. Otherwise, $|B_{\ell_1}| \geqslant 5$ and $\varphi_2^{-1}(B_{\ell_1}) \neq \emptyset$. Since B_i is not contained within any pull set $\mathcal{P}$ with $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then either $\varphi_2^{-1}(B_i) = \emptyset$ or B_i and B_{ℓ_1} are not contained in a common pull set. In either case, B_{ℓ_1} contributes at least one more charge to F_{i-t+1} and $\nu'(F_{i-t+1}) \geqslant 3$.

If F_{i-t+1} has two or more 2-blocks, then either two 2-blocks are separated only by 3-blocks and contribute an extra charge to F_{i-t+1} or they are separated by a block of size at least four that is not in a pull set with B_i and contributes an extra charge to F_{i-t+1}.

Therefore, $\nu'(F_{i-t+1}) \geqslant 3$ and $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$.

Case 3: $|B_i| = 4$. Let $K \geqslant 1$ be the least positive integer so that $|\psi_4^{(K)}(B_i)| \neq 4$. For $d \in \{1, \ldots, K\}$, define $B_{i_d} = \psi_4^{(d)}(B_i)$.

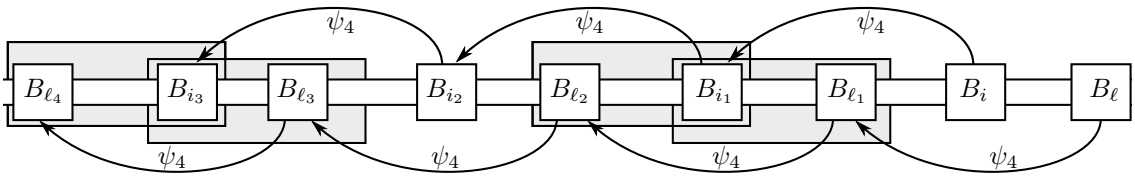


Figure 13: Claim 14.3, Case 3: $|B_\ell| = 4$ and $|B_i| = 4$, shown with $D \geqslant 4, D' \geqslant 3$.

Case 3.i: $D \geqslant 4$ and $K \geqslant 3$. Note that for $j \in \{1, 2, 3\}$, B_{i_j} is between $B_{\ell_{j+1}}$ and B_{ℓ_j}. There are at most $3t - 3$ elements between $B_{\ell_{j+1}}$ and B_{ℓ_j}, so every frame F containing B_{i_j} either contains one of $B_{\ell_{j+1}}$ or B_{ℓ_j} or has $\sigma(F) \leqslant 3t-4$. If F contains B_{i_j} and one of $B_{\ell_{j+1}}$ or B_{ℓ_j}, then either $\nu'(F) \geqslant 2$ or B_{i_j} is contained in a perfect pull set $\mathcal{P}$ with the other block and $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$ so by Claim 14.1 we are done. If $\sigma(F) \leqslant 3t - 3$, then there are at least three 2-blocks in F. At least two of these 2-blocks are on a common side of B_{i_j}, and either they are separated only by 3-blocks (and pull an extra charge to F) or they are separated by a block of size at least four (which contributes an extra charge to F). Therefore, every frame containing B_{i_j} has ν'-charge at least two. Build $\mathcal{H}$ from the frames containing B_{i_1} and the frames containing B_{i_3}. Then $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant 2t$.

Case 3.ii: $K < D < 4$. By definition, $|B_{i_K}| \geqslant 5$. Let $\mathcal{H}$ be the set of frames containing B_{i_K}.

If there exists a pull set $\mathcal{P}$ containing B_{i_K} so that $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$ then by Claim 14.1 we are done. If $|\varphi_2^{-1}(B_{i_K})| = 1$ and $|B_{i_K}| = 5$, then by Claim 14.2 we are done. Therefore, B_{i_K} contributes at least one to the defect of every pull set containing B_{i_K}, and hence every frame containing B_{i_K} has ν'-charge at least two.

The block B_{i_K} is between $B_{\ell_{K+1}}$ and B_{ℓ_K} and there are at most $3t - 3$ elements between $B_{\ell_{K+1}}$ and B_{ℓ_K}. Consider the frame F_{i_K}, which has B_{i_K} as the first block. If F_{i_K} contains B_{ℓ_K}, then $\nu'(F_{i_K}) \geqslant 3$ since B_{ℓ_K} is a 4-block and B_{i_K} contributed two charge to F_{i_K}. Otherwise, $\sigma(F_{i_K}) \leqslant 3t - 3$ and F_{i_K} contains at least two 2-blocks. Either these 2-blocks are separated by 3-blocks and pull a charge in Stage 2, or there is a block of size at least four between these blocks and contributes at least one more charge to F_{i_K}. Therefore, $\nu'(F_{i_K}) \geqslant 3$ and $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$.

Case 3.iii: $D \leqslant K < 4$. By definition, $|B_{\ell_D}| \geqslant 5$. Let $\mathcal{H}$ be the set of frames containing B_{ℓ_D}.

If there exists a pull set $\mathcal{P}$ containing B_{ℓ_D} so that $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$ then by Claim 14.1 we are done. If $|\varphi_2^{-1}(B_{\ell_D})| = 1$ and $|B_{\ell_D}| = 5$, then by Claim 14.2 we are done. Therefore, B_{ℓ_D} contributes at least one to the defect of every pull set containing B_{ℓ_D}, and hence every frame containing B_{ℓ_D} has ν'-charge at least two.

The block B_{ℓ_D} is between B_{i_D} and $B_{i_{D-1}}$ and there are at most $3t - 3$ elements between B_{i_D} and $B_{i_{D-1}}$. Consider the frame F_{ℓ_D}, which has B_{ℓ_D} as the first block. If F_{ℓ_D} contains $B_{i_{D-1}}$, then $\nu'(F_{\ell_D}) \geqslant 3$ since $B_{i_{D-1}}$ is a 4-block and B_{ℓ_D} contributed two charge. Otherwise, $\sigma(F_{\ell_D}) \leqslant 3t - 3$ and F_{ℓ_D} contains at least two 2-blocks.

Either these 2-blocks are separated by 3-blocks and pull a charge in Stage 2, or there is a block of size at least four between these blocks and contributes at least one more charge to F_{ℓ_D}. Therefore, $\nu'(F_{\ell_D}) \geqslant 3$ and $\sum_{F\in\mathcal{H}} [\nu'(F) - 1] \geqslant t + 1$. □

Since $\sum_{j=1}^{r} \nu'(F_j) = r - 1$, there is some frame F_z with $\nu'(F_z) = 0$. Also, the only frames where $\nu'(F_j)$ may be zero are those containing B_0, B_{k_1}, or B_{k_2}.

Claim 14.4. *There exists a block B_* and a frame F_z so that $B_* \in F_z$, $\nu'(F_z) = 0$, and for all 2-blocks B_j, B_* does not appear between $\psi_2(B_j)$ and $\varphi_2(B_j)$, inclusive.*

Proof of Claim 14.4. Using any frame F_z with $\nu'(F_z) = 0$, we will show that there is a block $B_* \in \{B_0, B_{k_1}, B_{k_2}\} \cap F_z$ so that for all 2-blocks B_j, B_* does not appear between $\psi_2(B_j)$ and $\varphi_2(B_j)$.

Consider five cases based on which blocks (B_0, B_{k_1}, or B_{k_2}) are within F_z and if there are other frames with zero charge.

Case 1: For some $i \in \{1, 2\}$, $B_{k_i} \in F_z$ and $|B_{k_i}| = 4$. Since $\nu'(F_z) = 0$, we must have that either $\nu^*(F_z) = 0$ or $\nu^*(F_z) > 0$ and charge was pulled from F_z in Stage 2.

If $\nu^*(F_z) = 0$, then F_z contains no block of size at least four other than B_{k_i}. If there are no 2-blocks, then every block of $F_z \setminus \{B_{k_i}\}$ is a 3-block and $\sigma(F_z) = 3t + 1$. All 2-blocks B_j have at most $3t - 4$ elements between B_j and $\varphi_2(B_j)$ or between $\psi_2(B_j)$ and B_j, so there are not enough elements to fit F_z in these ranges and hence $B_* = B_{k_i}$ suffices.

If there is exactly one 2-block in F_z, then $\sigma(F_z) = 3t$, a contradiction. Similarly, if there are exactly two 2-blocks in F_z, then $\sigma(F_z) = 3t-1$, a contradiction. Hence, there are at least three 2-blocks in F_z and some pair of 2-blocks is separated by only 3-blocks, so Stage 2 pulled at least one charge from another frame, contradicting $\nu'(F_z) = 0$.

If $\nu^*(F_z) > 0$, then there must be at least one block of order four or more other than B_{k_i}. If any of these blocks are 4-blocks, then the positive charge contributed cannot be removed by Stage 2. If any of these blocks have size at least five, the associated maximal pull set in F_z does not contain B_{k_1} or B_{k_2} so the defect is non-negative and Stage 2 leaves at least one charge, so $\nu'(F_z) > 0$.

Case 2: $B_{k_1} \in F_z$ and $|B_{k_1}| \geqslant 5$. Since $x_0 + 3t \in B_{k_1}$ and B_0 is not included in $\varphi_2^{-1}(B_{k_1})$, we have $|B_{k_1}| \geqslant 2|\varphi_2^{-1}(B_{k_1})| + 4$. Thus the maximal pull set in F_z containing B_{k_1} is imperfect and $\nu'(F_z) > 0$, a contradiction.

Case 3: $B_0 \in F_z$, there are no 2-blocks in F_z, and F_z does not contain B_{k_1} or B_{k_2}. Since $\nu'(F_z) = 0$, there is no block in F_z with size at least four, hence F_z contains $t - 1$ 3-blocks and B_0, so $\sigma(F_z) = 3t-2$. For a 2-block B_j, there are at most $3t-4$ elements contained in the blocks strictly between B_j and $\varphi_2(B_j)$ or the blocks strictly between B_j and $\psi_2(B_j)$. Then, if B_0 appears between $\psi_2(B_j)$ and $\varphi_2(B_j)$, then one of $\psi_2(B_j)$, B_j, or $\varphi_2(B_j)$ must be within F_z, a contradiction. Thus, $B_* = B_j$ suffices.

Case 4: $B_0 \in F_z$, F_z contains at least one 2-block, F_z does not contain B_{k_1} or B_{k_2}. Since F_z does not contain B_{k_1} or B_{k_2}, any block of size at least four implies $\nu'(F_z) \geqslant 1$, a contradiction. Further, if there are at least three 2-blocks in F_z, then two 2-blocks are separated by only 3-blocks and F_z pulls a charge in Stage 2, a contradiction. Therefore, F_z contains either one or two 2-blocks. If there are two 2-blocks, there must be one 2-block (call it B_{i_1}) preceding B_0 and another (call it B_{i_2}) following B_0. In either case, $\sigma(F_z) \in \{3t-4, 3t-3\}$.

Let B_{ℓ_1} be the block immediately following F_z and B_{ℓ_2} be the block immediately preceding F_z. If $\sigma(F_z) = 3t-3$ and B_{ℓ_j} has size two or three (for some $j \in \{1,2\}$), then $\sigma(F_z \cup \{B_{\ell_j}\}) \in \{3t-1, 3t\}$, a contradiction. If $\sigma(F_z) = 3t-4$ and $|B_{\ell_j}| \in \{3,4\}$ (for some $j \in \{1,2\}$), then $\sigma(F_z \cup \{B_{\ell_i}\}) \in \{3t-1, 3t\}$, a contradiction. Hence, $|B_{\ell_1}|, |B_{\ell_2}| \geqslant 4$ when exactly one 2-block exists, or $|B_{\ell_j}| = 2$ and the 2-block B_{i_j} is between B_0 and B_{ℓ_j} (and every frame containing both B_{i_j} and B_{ℓ_j} pulls a charge in Stage 2). Since all other frames containing B_0 contain either B_{ℓ_1} or B_{ℓ_2}, they have positive ν'-charge. Therefore, F_z is the *only* frame with zero charge and $\sum_{j:\nu'(F_j)>0} [\nu'(F_j) - 1] = 0$. Hence, if there exists any frame with ν'-charge at least two, we have a contradiction.

We consider if B_{i_1} and B_{i_2} both exist and whether or not $\psi_2(B_{i_j})$ is equal to B_{k_2} for some j.

Case 4.i: $\psi_2(B_{i_j}) = B_{k_2}$ for some $j \in \{1,2\}$. Since $|B_{k_2}| \geqslant 2|\psi_2^{-1}(B_{k_2})| + 4$, $|B_{k_2}| \geqslant 6$. If $\varphi_2^{-1}(B_{k_2}) = \emptyset$, then $\mu^*(B_{k_2}) \geqslant 2$ and every frame containing B_{k_2} has ν'-charge at least two, a contradiction. If $|\varphi_2^{-1}(B_{k_2})| \geqslant 2$, Claim 14.1 implies $\sum_{j:\nu'(F_j)>0} [\nu'(F_j)-1] \geqslant t+1$, a contradiction. Thus, $|\varphi_2^{-1}(B_{k_2})| = 1$. Let B_g be the unique 2-block in $\varphi_2^{-1}(B_{k_2})$. Note that $|\psi_2(B_g)| \geqslant 5$. If $|\psi_2(B_g)| \geqslant 2|\varphi_2^{-1}(\psi_2(B_g))| + 4$, then $\psi_2(B_g)$ contributes one to the defect of every pull set containing $\psi_2(B_g)$ and every frame containing $\psi_2(B_g)$ has ν'-charge at least two, a contradiction. Thus, $|\psi_2(B_g)| = 2|\varphi_2^{-1}(\psi_2(B_g))|+3 \geqslant 5$, and every pull set $\mathcal{P}$ that contains $\psi_2(B_g)$ has $|\varphi_2^{-1}(\mathcal{P})| \geqslant 1$. If any such pull set has $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then Claim 14.1 implies $\sum_{j:\nu'(F_j)>0}[\nu'(F_j) - 1] \geqslant t+1$. Otherwise, every pull set containing $\psi_2(B_g)$ has $|\varphi_2^{-1}(\mathcal{P})| = 1$, and Claim 14.2 implies $\sum_{j:\nu'(F_j)>0}[\nu'(F_j) - 1] \geqslant t+1$.

Case 4.ii: $\psi_2(B_{i_j}) \neq B_{k_2}$ for both $j \in \{1,2\}$. Consider some $j \in \{1,2\}$ so that B_{i_j} exists. If $|\varphi_2^{-1}(\psi_2(B_{i_j}))| \geqslant 2$, then Claim 14.1 provides a contradiction. If $|\psi_2(B_{i_j})| \geqslant 2|\varphi_2^{-1}(\psi_2(B_{i_j}))|+4$, then $\psi_2(B_{i_j})$ contributes at least one to the defect of any pull set containing $\psi_2(B_{i_j})$, and every frame containing $\psi_2(B_{i_j})$ has ν'-charge at least two, a contradiction. Therefore, the size of $\varphi_2^{-1}(\psi_2(B_{i_j}))$ is 1 and $|\psi_2(B_{i_j})| = 5$. Every pull set $\mathcal{P}$ that contains $\psi_2(B_{i_j})$ has $|\varphi_2^{-1}(\mathcal{P})| \geqslant 1$. If any such pull set has $|\varphi_2^{-1}(\mathcal{P})| \geqslant 2$, then Claim 14.1 provides a contradiction. Otherwise, every pull set containing $\psi_2(B_{i_j})$ has $|\varphi_2^{-1}(\mathcal{P})| = 1$ and Claim 14.2 provides a contradiction.

Case 5: $B_{k_2} \in F_z$ and $|B_{k_2}| \geqslant 5$. If $|B_{k_2}| \geqslant 2|\varphi_2^{-1}(B_{k_2})|+4$, then every pull set containing

B_{k_2} is imperfect and contributes at least one charge to every frame containing B_{k_2}, including F_z, a contradiction. Hence, $|B_{k_2}| = 2|\varphi_2^{-1}(B_{k_2})|+3$. Since we are not in Case 1 or Case 2, every frame with ν'-charge zero must contain B_{k_2} or B_0.

Suppose there is a frame $F_{z'}$ containing B_0 and not containing B_{k_2} with $\nu'(F_{z'}) = 0$. Since we are not in Case 3, $F_{z'}$ contains at least one 2-block and the proof of Case 4 shows that $F_{z'}$ is the *only* frame with ν'-charge zero containing B_0 and not containing B_{k_2}.

Therefore, there are at most $t+1$ frames with ν'-charge zero, whether or not there is a frame $F_{z'}$ with $\nu'(F_{z'}) = 0$ containing B_0 and not B_{k_2} and hence $\sum_{j:\nu'(F_j)>0}[\nu'(F_j)-1] \leqslant t$.

If $|\varphi_2^{-1}(B_{k_2})| \geqslant 2$, then Claim 14.1 implies $\sum_{j:\nu'(F_j)>0}[\nu'(F_j)-1] \geqslant t+1$. If $|\varphi_2^{-1}(B_{k_2})| = 1$, then Claim 14.2 implies $\sum_{j:\nu'(F_j)>0}[\nu'(F_j) - 1] \geqslant t + 1$. In either case we have a contradiction.

This completes the proof of Claim 14.4. □

Thus, we have a block B_* and a frame F_z so that $B_* \in F_z$, $\nu'(F_z) = 0$, and every 2-block B_j has $B_*, \psi_2(B_j), B_j$, and $\varphi_2(B_j)$ appearing in the cyclic order of blocks of X. Fix B_j to be the first 2-block that appears after B_* in the cyclic order. We will now prove that $a + b + c \geqslant 3$.

Consider $\psi_2(B_j)$. Observe that $\varphi_2^{-1}(\psi_2(B_j)) = \emptyset$, by the choice of B_* and B_j. Hence, $a \geqslant |\psi_2(B_j)| - 4$. If $|\psi_2(B_j)| \geqslant 7$, then $a \geqslant 3$. Thus, $|\psi_2(B_j)| \in \{5, 6\}$ and $\psi_2^{-1}(\psi_2(B_j)) = \{B_j\}$.

Consider the frame F_{j-t+1}, whose last block is B_j. By the choice of B_j, all blocks in $F_{j-t+1} \setminus \{B_j\}$ have size at least three, so $\sigma(F_{j-t+1}) \geqslant 3t-1$. This implies $\psi_2(B_j) \in F_{j-t+1}$. Since $\psi_2(B_j) \ni x_j - 3t$ and $|\psi_2(B_j)| \leqslant 6$, there are at least $3t-4$ elements strictly between $\psi_2(B_j)$ and B_j; these elements must be covered by at most $t-2$ blocks. Therefore, there exists some block B_k strictly between $\psi_2(B_j)$ and B_j with $|B_k| \geqslant 4$. Select B_k to be the first such block appearing after $\psi_2(B_j)$.

Case 1: $|\psi_2(B_j)| = 6$. This implies $a \geqslant 2$. If $|B_k| \geqslant 5$, by the choice of B_j we have $\varphi_2^{-1}(B_k) = \emptyset$ and $a \geqslant 3$. Therefore, $|B_k| = 4$ and $\psi_4(B_k)$ is a block of order at least four. If $|\psi_4(B_k)| \geqslant 5$, then $\varphi_2^{-1}(\psi_4(B_k)) = \emptyset$ and $a \geqslant 3$. Otherwise, $|\psi_4(B_k)| = 4$, and the frame F_i starting at $B_i = \psi_4(B_k)$ also contains $\psi_2(B_j)$ and B_k. Thus, $c = 1$ and $a + c \geqslant 3$.

Case 2: $|\psi_2(B_j)| = 5$ *and* $|B_k| \geqslant 5$. Note that $\varphi_2^{-1}(B_k) = \emptyset$ by the choice of B_j, which implies that $a \geqslant 2$. If $|B_k| \geqslant 6$, then $a \geqslant 3$; hence $|B_k| = 5$. Let $B_i = \psi_2(B_j)$ and consider the set $N_k = \{x_k - 3t, x_k - 3t + 1, x_k - 3t + 5, x_k - 3t + 6\}$. The elements in N_k are non-neighbors with at least one of x_k and x_{k+1}. Since X is a clique, X is disjoint from N_k. We must consider which elements in $A_k = \{x_k - 3t + 2, x_k - 3t + 3, x_k - 3t + 4\}$ are contained in X. If B_* appears before A_k, then since B_j is the first 2-block after B_*, there is at most one element of X in A_k. If B_* appears after A_k and two elements of

A_k are in X, then they form a 2-block $B_{j'}$ with $\varphi_2(B_{j'}) = B_k$, contradicting the choice of B_*. Hence, $|X \cap A_k| \leqslant 1$ and the elements from X in A_k form either blocks of size at least five or two consecutive blocks of order at least four.

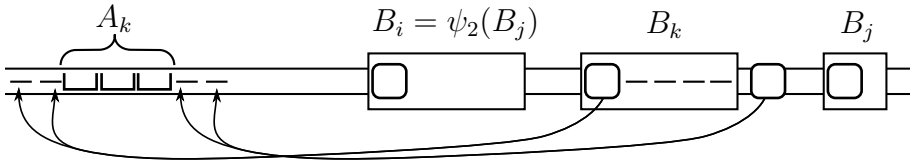


Figure 14: Claim 14, Case 2.

Case 2.i: $A_k \cap X = \emptyset$. Let B_ℓ be the block containing $x_k - 3t$. Note that $|B_\ell| \geqslant 8$. If $\varphi_2^{-1}(B_\ell) = \emptyset$, then $a \geqslant 4$. Otherwise $\varphi_2^{-1}(B_\ell) \neq \emptyset$, and B_* appears between B_ℓ and B_i. Then, there are at most $3t - 7$ elements between B_ℓ and B_k. Since $|B_*| \geqslant 1$, $|B_i| \geqslant 5$, and all other blocks have size at least three, the $t-2$ blocks after B_ℓ cover at least $3t - 6$ elements. Thus, every frame containing B_* (including F_z) must also contain B_ℓ or B_k. This implies that $\nu'(F_z) \neq 0$, a contradiction.

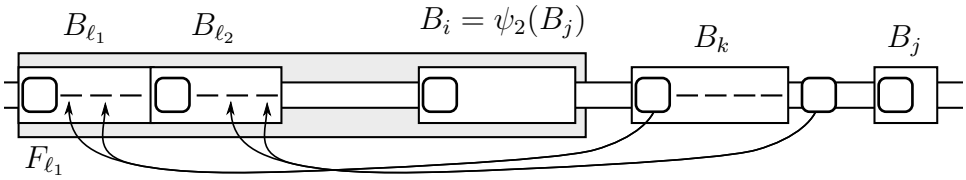


Figure 15: Claim 14, Case 2.ii.

Case 2.ii: $A_k \cap X = \{x_k - 3t + 3\}$. Then, the block starting at $x_k - 3t + 3$ and the block preceding it have size at least four. These two blocks (call them B_{ℓ_1} and B_{ℓ_2}) and $\psi_2(B_j)$ are contained in a single frame, F_{ℓ_1}, so $c = 1$ and $a + c \geqslant 3$.

Case 2.iii: $A_k \cap X \neq \{x_k - 3t + 3\}$ *and* B_* *appears before* A_k. Thus, the element in $A_k \cap X$ is either the first element in a block of size at least five or is the first element following a block of size at least five. In either case, this block, B_ℓ, has $\varphi_2^{-1}(B_\ell) = \emptyset$, by the choice of B_* and B_j. This implies $a \geqslant 3$.

Case 2.iv: $A_k \cap X \neq \{x_k - 3t + 3\}$ *and* B_* *appears between* A_k *and* B_i. Let B_ℓ be the block of size at least five that is guaranteed by the element in $A_k \cap X$. There are at most $3t - 3$ elements between B_ℓ and B_k. Since $|B_*| \geqslant 1$, $|B_i| = 5$, and all other blocks between B_ℓ and B_k have size at least three, the $t-1$ blocks following B_ℓ cover at least $3t - 3$ elements. Thus, any frame containing B_* also contains either B_ℓ or B_k and thus has positive charge. This includes F_z, but $\nu'(F_z) = 0$, a contradiction.

Case 3: $|\psi_2(B_j)| = 5$ *and all blocks between* $\psi_2(B_j)$ *and* B_j *have size at most four.* Since there are $3t - 4$ elements strictly between $\psi_2(B_j)$ and B_j that must be covered by at most $t - 2$ blocks of size at least three, there are at least two 4-blocks $B_k, B_{k'}$ between

$\psi_2(B_j)$ and B_j. Thus, the blocks $B_{\ell_0} = \psi_2(B_j), B_{\ell_1} = B_k$, and $B_{\ell_2} = B_{k'}$ are contained in a single frame and $c = 1$ giving $a + c \geqslant 3$.

This completes the proof of Claim 14. □

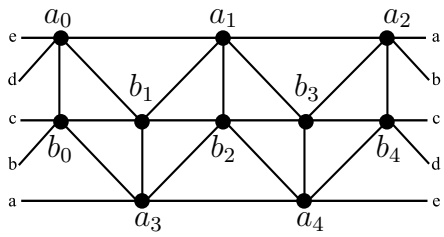
Claims 13 and 14 imply that an r-clique X in $G + \{0, 1\}$ has no 2-blocks. By Claim 12, $G + \{0, 1\}$ has a unique r-clique, and hence G is r-primitive. □

5 Constructions of Sporadic Graphs

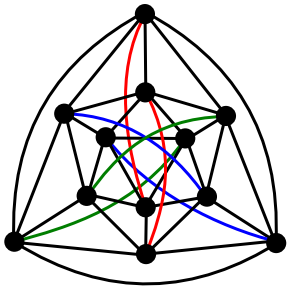
In this section, we give explicit constructions for all known r-primitive graphs, including those found in previous work. It is a simple computation to verify that every graph presented is uniquely K_r-saturated, so proofs are omitted. In addition to the descriptions given here, all graphs are available online[5].

5.1 Uniquely K_4-Saturated Graphs

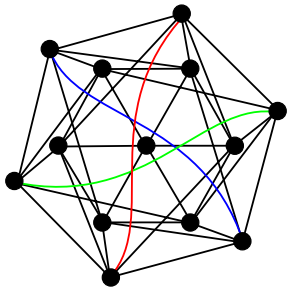
Construction 17 (Cooper [8], Figure 16(a))**.** G_{10} is the graph built from two 5-cycles a_0, a_1, a_2, a_3, a_4 and b_0, b_1, b_2, b_3, b_4 where a_i is adjacent to b_{2i-1}, b_{2i}, and b_{2i+1}.



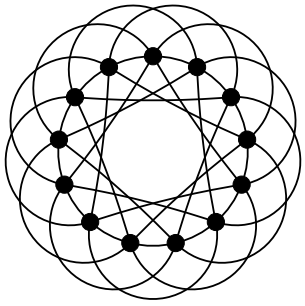
(a) Construction 17, G_{10}.



(b) Construction 18, G_{12}.



(c) Construction 19, G_{13}.



(d) Construction 20, Paley(13).

Figure 16: Uniquely K_4-saturated graphs on 10–13 vertices.

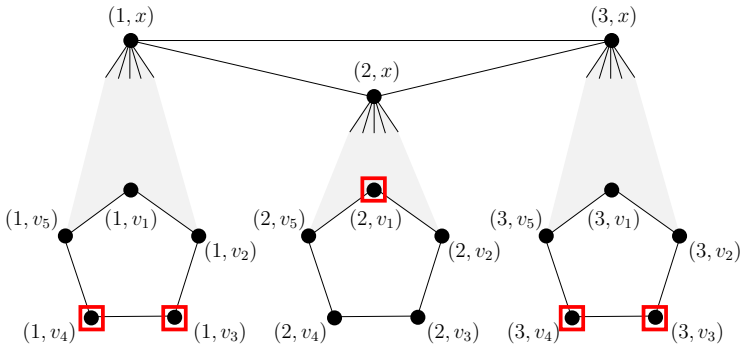
[5]Graphs available in graph6 format and as adjacency matrices at http://www.math.unl.edu/~shartke2/math/data/data.php.

Construction 18 (Collins [8], Figure 16(b))**.** The graph G_{12} is the vertex graph of the icosahedron with a perfect matching added between antipodal vertices. Another description takes vertices v_0, v_1 and two 5-cycles $u_{j,0}, \ldots, u_{j,4}$ ($j \in \{0,1\}$) with v_j adjacent to v_{j+1} and $u_{j,i}$ for all $i \in [5]$ and $u_{0,i}$ adjacent to $u_{1,i}$, $u_{1,i+1}$, and $u_{1,i+3}$ for all $i \in \mathbb{Z}_5$.

Construction 19 (Figure 16(c))**.** G_{13} is given by vertices $x, y_1, \ldots, y_6, z_1, \ldots, z_6$, where x is adjacent to every y_i, y_i and y_{i+1} are adjacent for all $i \in \{1, \ldots, 6\}$, and z_i and z_{i+1} are adjacent for all $i \in \{1, \ldots, 6\}$. Further, z_i is adjacent to z_{i+3}, y_i, y_{i-1}, and y_{i+2}.

Construction 20 (Figure 16(d))**.** The Paley graph [22] of order 13, Paley(13), is isomorphic to the Cayley complement $\overline{C}(\mathbb{Z}_{13}, \{1,3,4\})$.

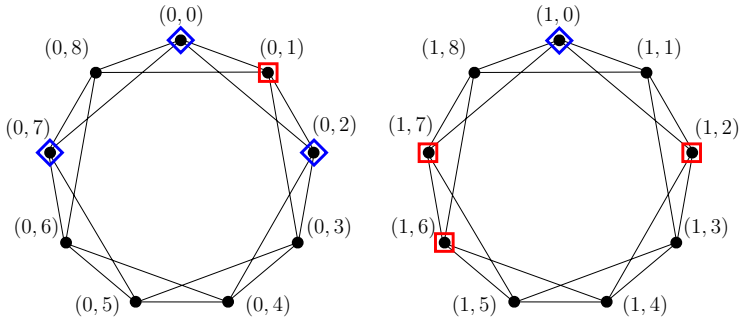
Construction 21 (Figure 17)**.** Let H be the graph on vertices $x, v_1, \ldots, v_5$ with x adjacent to every v_i and the vertices $v_1, \ldots, v_5$ form a 5-cycle. Note that H is uniquely K_4-saturated, as $v_1, \ldots, v_5$ induce C_5, which is 3-primitive. $G_{18}^{(A)}$ has vertex set $V = \mathbb{Z}_3 \times \{x, v_1, v_2, v_3, v_4, v_5\}$ where subscripts are taken modulo 5. The vertices (a, x) with $a \in \{1,2,3\}$ form a triangle. For each a, (a, x) is adjacent to (a, v_i) for each i but is not adjacent to $(a+1, v_i)$ or $(a+2, v_i)$ for any i. For each a and i, the vertex (a, v_i) is adjacent to (a, v_{i-1}) and (a, v_{i+1}) (within the copy of H) and also $(a+1, v_{i+2}), (a+1, v_{i-2}), (a-1, v_{i+2}), (a-1, v_{i-2})$ (outside the copy of H).



□ — $\{(2, v_1)\} \cup (N((2, v_1)) \cap \{(j, v_i) : j \in \{1,3\}, i \in \{1, \ldots, 5\}\})$.

Figure 17: Construction 21, $G_{18}^{(A)}$, is 4-primitive, 7-regular, on 18 vertices.

Construction 22 (Figure 18)**.** Let $G_{18}^{(B)}$ have vertex set $\mathbb{Z}_2 \times \mathbb{Z}_9$ where each coordinate is taken modulo two and nine, respectively. For fixed a, the vertices (a, i) and (a, j) are adjacent if and only if $|i-j| \leqslant 2$. For fixed i, the vertex $(0, i)$ is adjacent to $(1, 2i), (1, 2i+4)$ and $(1, 2i+5)$. Conversely, for fixed j the vertex $(1, j)$ is adjacent to $(0, 5j), (0, 5j+7)$ and $(0, 5j+2)$.



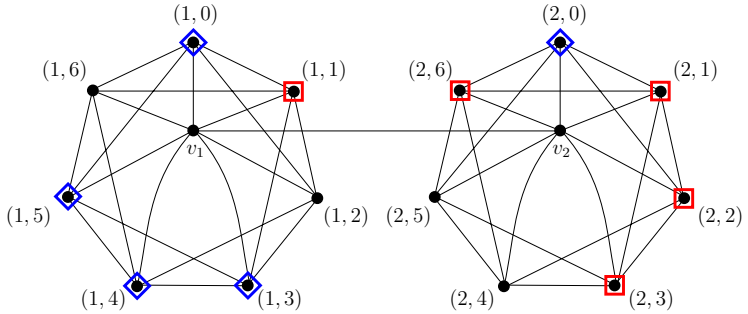
$\square$ — $\{(0,1)\} \cup (N((0,1)) \cap \{(1,i) : i \in \mathbb{Z}_9\})$.
$\diamond$ — $\{(1,0)\} \cup (N((1,0)) \cap \{(0,i) : i \in \mathbb{Z}_9\})$.

Figure 18: Construction 22, $G_{18}^{(B)}$, is 4-primitive, 7-regular, on 18 vertices.

5.2 Uniquely K_5-Saturated Graphs

Construction 23 (Figure 19)**.** Let $G_{16}^{(A)}$ have vertex set $\{v_1, v_2\} \bigcup (\{1,2\} \times \mathbb{Z}_7)$. The vertices v_1 and v_2 are adjacent. For each $j \in \{1,2\}$ and $i \in \mathbb{Z}_7$, v_j is adjacent to (j,i) and (j,i) is adjacent to $(j,i+1), (j,i+2), (j,i-1)$ and $(j,i-2)$. (Hence, the subgraph induced by (j,i) for fixed j and $i \in \mathbb{Z}_7$ is isomorphic to C_7^2.) For $i \in \mathbb{Z}_7$, the vertex $(1,i)$ is adjacent to $(2,2i), (2,2i+1), (2,2i-1)$, and $(2,2i-3)$. Conversely, for $i \in \mathbb{Z}_7$, the vertex $(2,i)$ is adjacent to $(1,4i), (1,4i-2), (1,4i+3)$, and $(1,4i-3)$.

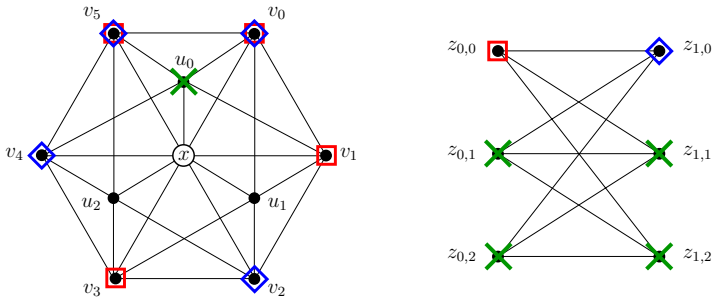
An interesting feature of $G_{16}^{(A)}$ is that it is not regular: v_1, and v_2 have degree 8 while the other vertices have degree 9. This dispels any temptation to conjecture that all uniquely K_r-saturated graphs with no dominating vertex were regular.



$\square$ — $\{(1,1)\} \cup (N((1,1)) \cap \{(2,i) : i \in \{0,1,\ldots,6\}\})$.
$\diamond$ — $\{(2,0)\} \cup (N((2,0)) \cap \{(1,i) : i \in \{0,1,\ldots,6\}\})$.

Figure 19: Construction 23, $G_{16}^{(A)}$, is 5-primitive *and irregular*, on 16 vertices.

Construction 24 (Figure 20)**.** The graph $G_{16}^{(B)}$ has vertex set $\{x\} \cup \{u_i : i \in \mathbb{Z}_3\} \cup \{v_j : j \in \mathbb{Z}_6\} \cup \{z_{k,i} : k \in \{0,1\}, i \in \mathbb{Z}_3\}$. The vertex x is adjacent to u_i for all $i \in \mathbb{Z}_3$ and v_j for all $j \in \mathbb{Z}_6$. There are no edges among the vertices u_i. The vertices v_j form a cycle, with an edge $v_j v_{j+1}$ for all $j \in \mathbb{Z}_6$. The vertices $z_{k,i}$ form a complete bipartite graph, with an edge $z_{0,i}z_{1,j}$ for all $i, j \in \mathbb{Z}_3$. For $i \in \{0,1,2\}$, the vertex u_i is adjacent to v_{2i-1}, v_{2i}, v_{2i+1}, and v_{2i+2}, and adjacent to $z_{k,i+1}$ and $z_{k,i-1}$ for $k \in \{0,1\}$. For $i \in \{0,1,2\}$, the vertex $z_{0,j}$ is adjacent to v_{2i}, v_{2i+1}, v_{2i+2}, and v_{2i+4}, while the vertex $z_{1,i}$ is adjacent to v_{2i-1}, v_{2i}, v_{2i+1}, and v_{2i+3}.



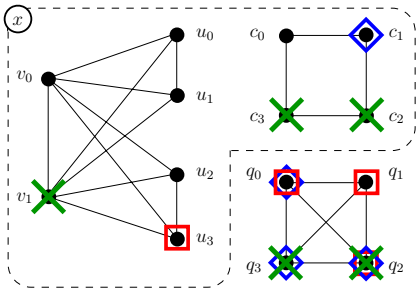
× — $\{u_0\} \cup (N(u_0) \cap \{z_{j,i} : j \in \{0,1\}, i \in \mathbb{Z}_3\})$.
□ — $\{z_{0,0}\} \cup (N(z_{0,0}) \cap \{v_i : i \in \mathbb{Z}_6\})$.
◇ — $\{z_{1,0}\} \cup (N(z_{1,0}) \cap \{v_i : i \in \mathbb{Z}_6\})$.

Figure 20: Construction 24, $G_{16}^{(B)}$, is 5-primitive, 9 regular, on 16 vertices.

5.3 Uniquely K_6-Saturated Graphs

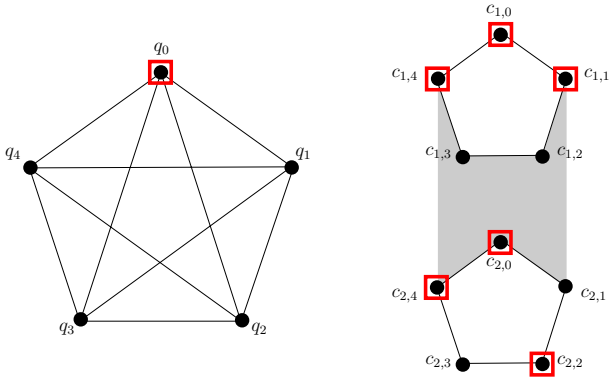
Construction 25 (Figure 21)**.** The graph $G_{15}^{(A)}$ has vertices x, v_0, v_1, u_1, ..., u_4, c_1, ..., c_4, q_1, ..., q_4. The vertex x dominates all but the q_i's. The vertices v_0, v_1 are adjacent and dominate the u_i's. Also, v_i dominates $c_{2i}, c_{2i+1}, q_{2i}, q_{2i+1}$ for each $i \in \mathbb{Z}_2$. The vertices u_0 and u_2 are adjacent as well as u_1 and u_3. The vertices u_i dominate the vertices c_j. Also, the vertex u_i is adjacent to q_j if and only if $i \neq j$. The vertices $c_1, \ldots, c_4$ form a cycle with edges $c_i c_{i+1}$. The vertices $q_1, \ldots, q_4$ form a clique. The vertices c_i and q_j are adjacent if and only if $i \neq j$.

Construction 26 (Figure 22)**.** The graph $G_{15}^{(B)}$ has vertices $q_i, c_{1,i}$, and $c_{2,i}$ for each $i \in \mathbb{Z}_5$. The subgraph induced by the vertices q_i is a 5-clique. For each $j \in \{1,2\}$, the subgraph induced by vertices $c_{j,i}$ for $i \in \mathbb{Z}_5$ is isomorphic to C_5 with edges $c_{j,i}c_{j,i+1}$ between consecutive elements. For each $i, i' \in \mathbb{Z}_5$, there is an edge between $c_{1,i}$ and $c_{2,i'}$. For each $i \in \mathbb{Z}_5$, the vertex q_i is adjacent to $c_{1,i}, c_{1,i-1}$, and $c_{1,i+1}$ as well as $c_{2,2i}$, $c_{2,2i-1}$, and $c_{2,2i+2}$.



$\square$ — $\{u_3\} \cup (N(u_3) \cap \{q_i : i \in [4]\})$.
$\diamond$ — $\{c_1\} \cup (N(c_1) \cap \{q_i : i \in [4]\})$.
$\times$ — $\{v_1\} \cup (N(v_1) \cap \{c_i, q_i : i \in [4]\})$.

Figure 21: Construction 25, $G_{15}^{(A)}$, is 6-primitive, 10 regular, on 15 vertices.



$\square$ — $\{q_0\} \cup (N(q_0) \cap \{c_{j,i} : j \in \{1,2\}, i \in \mathbb{Z}_5\})$.

Figure 22: Construction 26, $G_{15}^{(B)}$, is 6-primitive, 10 regular, on 15 vertices.

Construction 27 (Figure 23)**.** The graph $G_{16}^{(C)}$ is composed of three disjoint induced subgraphs isomorphic to K_4, K_4, and $\overline{C_8}$. Let the vertices $q_{0,0}, \ldots, q_{0,3}$, and $q_{1,0}, \ldots, q_{1,3}$ be the two copies of K_4 and vertices $c_0, \ldots, c_7$ be the $\overline{C_8}$, where the non-edges go between consecutive elements c_i and c_{i+1}. For $i \in \{0,1,2,3\}$, the vertex $q_{1,i}$ is adjacent to c_{2i+d} for all $d \in \{0,1,2,3,4,5\}$. For $i \in \{0,1,2,3\}$, the vertex $q_{2,i}$ is adjacent to c_{2i+d} for all $d \in \{0,1,3,4,5,6\}$. For $i \in \mathbb{Z}_4$, the vertex $q_{1,i}$ is adjacent to $q_{2,i+1}$ and $q_{2,i-1}$.

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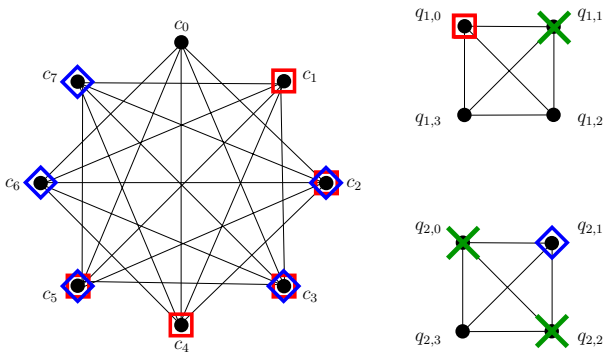

Figure 23: Construction 27, $G_{16}^{(C)}$, is 6-primitive, 10 regular, on 16 vertices.

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