Distance-regular graphs with a relatively small eigenvalue multiplicity

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Abstract

Godsil showed that if Γ is a distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$, and θ is an eigenvalue of Γ with multiplicity $m \geqslant 2$, then $k \leqslant \frac{(m+2)(m-1)}{2}$.

In this paper we will give a refined statement of this result. We show that if Γ is a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 2$ and an eigenvalue θ with multiplicity $m \geqslant 2$, such that k is close to $\frac{(m+2)(m-1)}{2}$, then θ must be a tail. We also characterize the distance-regular graphs with diameter $D \geqslant 3$, valency $k \geqslant 3$ and an eigenvalue θ with multiplicity $m \geqslant 2$ satisfying $k = \frac{(m+2)(m-1)}{2}$.

1 Introduction

For definitions and preliminaries, see Sections 2 and 3. In [6], Godsil showed

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Theorem 1. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Let θ be an eigenvalue of Γ with multiplicity $m \geqslant 2$. Then $k \leqslant \frac{(m+2)(m-1)}{2}$.*

In this paper we will give, in Theorem 13, a refined statement of this result. We show that if Γ is a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 2$ and an eigenvalue θ with multiplicity $m \geqslant 2$, such that k is close to $\frac{(m+2)(m-1)}{2}$, then θ must be a so-called tail. This, for example, implies that several Krein parameters vanish. Using the fact that θ is a (light) tail, we are also able to characterize in Theorem 14 the distance-regular graphs with diameter $D \geqslant 3$, valency $k \geqslant 3$ and an eigenvalue θ with multiplicity $m \geqslant 2$ satisfying $k = \frac{(m+2)(m-1)}{2}$.

In Section 2 we give the necessary definitions, and in Section 3 some preliminary results. In Section 4 we characterize the (non-bipartite) Taylor graphs as the non-bipartite distance-regular graphs with diameter at least three, having a light tail such that its accompanying eigenvalue equals -1 (Theorem 12). In Section 5 we state and prove Theorem 13 and Theorem 14.

2 Definitions

All the graphs considered in this paper are finite, undirected and simple (for unexplained terminology, examples and more details, see [4, 7]). Suppose that Γ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of two adjacent vertices. The *distance* $d_\Gamma(x, y)$ between any two vertices x and y in a graph Γ is the length of a shortest path connecting x and y. If the graph Γ is clear from the context, then we simply use $d(x, y)$. We define the *diameter* D of Γ as the maximum distance in Γ. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely i from x $(0 \leqslant i \leqslant D)$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$.

A connected graph Γ with diameter D is called *distance-regular* if there are integers b_i, c_i $(0 \leqslant i \leqslant D)$ such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. A graph Γ is said to be *strongly regular* with parameters (v, k, λ, μ) whenever Γ has v vertices and is regular with valency k, adjacent vertices of Γ have precisely λ common neighbors, and distinct non-adjacent vertices of Γ have precisely μ common neighbors. Note that distance-regular graphs of diameter two are strongly regular. We define $a_i := k - b_i - c_i$ for notational convenience. Note that $a_i =\mid \Gamma(y) \cap \Gamma_i(x) \mid$ holds for any two vertices x, y with $d(x, y) = i$ $(0 \leqslant i \leqslant D)$.

For a distance-regular graph Γ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p^h_{ij} := |\{w | w \in \Gamma_i(x) \cap \Gamma_j(y)\}|$ for any $y \in \Gamma_h(x)$. It is easy to see that $k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ and hence it does not depend on x. The numbers a_i, b_{i-1} and c_i $(1 \leqslant i \leqslant D)$ are called the *intersection numbers*, and the array $\{b_0, b_1, \cdots, b_{D-1}; c_1, c_2, \cdots, c_D\}$ is called the *intersection array* of Γ.

Suppose that Γ is a distance-regular graph with diameter $D \geqslant 2$ and valency $k \geqslant 2$, and let A_i be the matrix of Γ such that the rows and the columns of A_i are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are at distance i and 0 otherwise. We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. The eigenvalues of the graph Γ are the eigenvalues of A.

We find that $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [4, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ of the *primitive idempotents* of Γ, and A can be written as $A = \sum_{i=0}^{D} \theta_i E_i$, where θ_i is the *eigenvalue* of Γ associated with E_i $(0 \leqslant i \leqslant D)$. We denote by m_i the multiplicity of θ_i. For an eigenvalue $\theta = \theta_i$ we will also write E_θ instead of E_i.

For an eigenvalue θ of Γ, the sequence $(\omega_i)_{i=0,1,\ldots,D} = (\omega_i(\theta))_{i=0,1,\ldots,D}$ satisfying $\omega_0 = \omega_0(\theta) = 1$, $\omega_1 = \omega_1(\theta) = \theta/k$, and

$$c_i\omega_{i-1} + a_i\omega_i + b_i\omega_{i+1} = \theta\omega_i \quad (i = 1, 2, ..., D-1)$$

is called the *standard sequence* corresponding to the eigenvalue θ ([4, p.128]). A *sign change* of $(\omega_i)_{i=0,1,\ldots,D}$ is a pair (i, j) with $0 \leqslant i < j \leqslant D$ such that $\omega_i\omega_j < 0$ and $\omega_t = 0$ for $i < t < j$.

Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant D$, so M is closed under $\circ$. Thus there exist complex scalars q_{ij}^h $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |V(\Gamma)|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \leqslant i, j \leqslant D).$$

By [2, p. 170], q_{ij}^h is real and nonnegative for $0 \leqslant h, i, j \leqslant D$. The q_{ij}^h are called the *Krein parameters*. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two $(0 \leqslant h, i, j \leqslant D)$ [4, p. 59].

For each vertex $x \in V(\Gamma)$, we let $\Delta(x)$ denote the subgraph of Γ induced on $\Gamma(x)$. We refer to $\Delta(x)$ as the *local graph* at vertex x. We observe that $\Delta(x)$ has k vertices, and is regular with valency a_1.

A graph Γ is called *bipartite* if it has no odd cycle. (A distance-regular graph Γ with diameter D is bipartite if and only if $a_1 = a_2 = \ldots = a_D = 0$.) An *antipodal* graph is a connected graph Γ with diameter $D \geqslant 2$ for which being at distance 0 or D is an equivalence relation. If, moreover, all equivalence classes have the same size r, then Γ is also called an *antipodal r-cover*. A distance-regular graph Γ with intersection array $\{k, \mu, 1; 1, \mu, k\}$ is called a *Taylor graph*. These are precisely the distance-regular antipodal 2-covers with diameter 3.

We define tails as follows: An eigenvalue θ of a distance-regular graph Γ with valency k is called a *tail* if $\theta \neq k$ and $E_\theta \circ E_\theta = \alpha J + \beta E_\theta + \gamma E_{\theta'}$ for some eigenvalue $\theta' \neq k, \theta$ and some α, β and $\gamma \neq 0$. We call θ' the *accompanying* eigenvalue for the tail θ. We call θ a *light* tail if $\beta = 0$ and *heavy* otherwise. Note that $\alpha > 0$ and $\beta \geqslant 0$. (Note that in [13], [10], they also allow $\gamma = 0$ for a tail and a light tail, respectively. Note that for diameter $D \geqslant 3$ this case of $\gamma = 0$ only occurs if Γ is an antipodal distance-regular graph of diameter $D = 3$ and $\theta = -1$ ([10, Theorem 4.1(b)]).)

3 Preliminaries

In this section we will give some preliminary results.

The following lemma is a special case of the Absolute Bound and we state it for distance-regular graphs only.

Lemma 2. ([15]) *Let Γ be a distance-regular graph with diameter $D \geqslant 2$. Then* $\sum_{q_{ii}^j \neq 0} m_j \leqslant \frac{m_i(m_i+1)}{2}$ $(0 \leqslant j \leqslant D)$.

The next result relates the multiplicity of an eigenvalue and its number of vertices for a strongly regular graph. A graph Γ is called *coconnected* if its complement is connected.

Lemma 3. *Let Γ be a connected and coconnected strongly regular graph with v vertices and distinct eigenvalues $k > \sigma > \tau$ with corresponding multiplicities $1, f, g$. Then*
(i) $v \leqslant \min\{\frac{f(f+3)}{2}, \frac{g(g+3)}{2}\}$.
(ii) If $v > \frac{g(g+1)}{2}$, *then τ is a light tail, that is,* $\mu = \frac{-(\sigma+1)\tau(\tau+\sigma^2)}{\tau-\sigma(\sigma+2)}$.
(iii) If $v > \frac{f(f+1)}{2}$, *then σ is a light tail, that is,* $\mu = \frac{-(\tau+1)\sigma(\sigma+\tau^2)}{\sigma-\tau(\tau+2)}$.
(iv) If $v = \frac{g(g+3)}{2}$, *then*

$$\begin{aligned}
\mu &= \sigma^3(2\sigma+3),\\
k &= 2\mu,\\
\lambda &= \sigma(2\sigma^3+\sigma^2-3\sigma+1),\\
v &= (2\sigma+1)^2(2\sigma^2+2\sigma-1),\\
\tau &= -\sigma^2(2\sigma+3),
\end{aligned}$$

and $\sigma > 0$ and $\tau < -1$ are integers except for the case $\sigma = \frac{-1+\sqrt{5}}{2}$, $\tau = \frac{-1-\sqrt{5}}{2}$ *and Γ is the pentagon.*

(v) If $v = \frac{f(f+3)}{2}$, *then*

$$\begin{aligned}
\mu &= \tau^3(2\tau+3),\\
k &= 2\mu,\\
\lambda &= \tau(2\tau^3+\tau^2-3\tau+1),\\
v &= (2\tau+1)^2(2\tau^2+2\tau-1),\\
\sigma &= -\tau^2(2\tau+3),
\end{aligned}$$

and $\sigma > 0$ and $\tau < -1$ are integers except for the case $\sigma = \frac{-1+\sqrt{5}}{2}$, $\tau = \frac{-1-\sqrt{5}}{2}$ *and Γ is the pentagon.*

Proof: (i) This follows from the absolute bound, Lemma 2. See also [19, p.169].
(ii) It follows from [15, Theorem 2] and [5, Theorem 6.1].

(iii) If we take the complement of Γ then it is a strongly regular graph satisfying (ii), and the result follows easily.
(iv) See [14] (cf. [19, p.169–170]).
(v) If we take the complement of Γ then it is a strongly regular graph satisfying (iv), and the result follows easily. ∎

Next, we introduce the Fundamental Bound and tight distance-regular graphs.

Lemma 4. ([9, Theorem 6.2]) *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Then the following inequality holds.*

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_D + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2} \tag{1}$$

We refer to (1) as the *Fundamental Bound.* A distance-regular graph Γ is *tight* if Γ is not bipartite and equality holds in (1).

The next lemma gives some known results on tight distance-regular graphs.

Lemma 5. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Then*
(i) ([9, Theorem 12.6]) *Γ is tight if and only if for all $x \in V(\Gamma)$, the local graph $\Delta(x)$ is connected strongly regular with distinct eigenvalues a_1, $-1-\frac{b_1}{\theta_D+1}$, $-1-\frac{b_1}{\theta_1+1}$.*
(ii) ([9, Theorem 11.7]) *If Γ is tight, then the intersection number a_D satisfies $a_D = 0$.*
(iii) ([18, Lemma 3.5], cf.[17]) *If Γ is tight, then the Krein parameter q^i_{1D} satisfies $q^i_{1D} = 0$ unless $i = D-1$ $(0 \leqslant i \leqslant D)$.*

The next result is due to Terwilliger and concerns the eigenvalues of the local graph $\Delta(x)$ at a vertex x of a distance-regular graph Γ.

Proposition 6. ([4, Theorem 4.4.4]) *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. If θ_i has multiplicity m_i with $1 < m_i < k$, then $\theta_i \in \{\theta_1, \theta_D\}$. Putting $b = \frac{b_1}{\theta_i+1}$ we have that each local graph $\Delta(x)$ has eigenvalue $-1-b$ with multiplicity at least $k - m_i$; in case $-1-b = a_1$ its multiplicity is at least $k - m_i + 1$.*

The following lemma is a consequence of Proposition 6.

Lemma 7. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. Then $m_1 + m_D \geqslant k+1$.*

Proof: As the sum of the multiplicities of $-1-\frac{b_1}{\theta_1+1}$ and $-1-\frac{b_1}{\theta_D+1}$ as eigenvalues of the local graph at vertex x is at most $k-1$ if $-1-\frac{b_1}{\theta_D+1} \neq a_1$ and at most k if equals a_1, the result follows. ∎

In the next lemma we show that the accompanying eigenvalue of a light tail θ is the third-largest eigenvalue, if θ is the second-largest eigenvalue.

Lemma 8. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. If $\theta = \theta_1$ is a light tail, then the accompanying eigenvalue θ' satisfies $\theta' = \theta_2$.*

Proof: Let E_i be the primitive idempotent corresponding to θ_i. Now $E_1 \circ E_1 = \alpha E_0 + \beta E_i$, where α and β are positive numbers. As the standard sequence corresponding to θ_1 is strictly decreasing, this implies that the standard sequence corresponding to θ_i has at most two sign changes ([10, Theorem 4.1(iii)]). But as $i \neq 0, 1$ it follows that $i = 2$. ∎

4 Characterizations of Taylor graphs

In this section we will give some characterizations of the Taylor graphs. We start with the following result, due to Taylor.

Lemma 9. ([4, Proposition 1.5.1, Theorem 1.5.3])
(i) If Γ is a Taylor graph with valency k, then for every $x \in V(\Gamma)$, the local graph $\Delta(x)$ is strongly regular with parameters (v', k', λ', μ') and satisfies $a_1 = k' = 2\mu'$, and $v' = k$.
(ii) If Δ is a (non-complete) connected strongly regular graph with (v', k', λ', μ') such that $k' = 2\mu'$, then there exists a Taylor graph Γ and a vertex x of Γ such that the local graph $\Delta(x)$ of Γ is isomorphic to Δ.

Remark: We denote by Tay(Δ), the Taylor graph as in Lemma 9(ii), where Δ is a (non-complete) connected strongly regular graph with (v', k', λ', μ') satisfying $k' = 2\mu'$.

The next result gives some sufficient conditions for a distance-regular graph to be tight.

Lemma 10. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. Then the following hold.*
(i) If $m_1 + m_D = k + 1$, then Γ is an antipodal 2-cover, and Γ is tight or bipartite.
(ii) If for all vertices x the local graph $\Delta(x)$ is strongly regular and $m_1, m_D < k$, then Γ is tight.

Proof: (i) If $m_1 + m_D = k + 1$, then we need to consider two cases: $m_D = 1$ and $m_D \geqslant 2$. If $m_D = 1$, then Γ is bipartite and $\theta_D = -k$ by [4, Proposition 4.4.8(i)]. If $m_i = 1$ and $i \geqslant 1$, then $i = D$, $\theta_D = -k$ and Γ is bipartite. So from now we may assume $m_1 \geqslant 2$ and $m_D \geqslant 2$. Now let $m_D \geqslant 2$. Then $m_1 = k + 1 - m_D < k$. If $\theta_D = -1 - \frac{b_1}{\theta_1 + 1}$, then the local graph $\Delta(x)$ at vertex x has eigenvalues a_1 and $-1 - \frac{b_1}{\theta_1+1}$ with corresponding multiplicities $k - m_D + 1$ and $k - m_1$ by Proposition 6. So this means that $\Delta(x)$ is a disjoint union of cliques. Since $\theta_1 > 0$, we find that $-1 - \frac{b_1}{\theta_1+1} < -1$. But it is not possible. So we find that $\theta_D \neq -1 - \frac{b_1}{\theta_1+1}$. Then again by Proposition 6 we find that for all vertices x the local graph $\Delta(x)$ has eigenvalues a_1, $-1 - \frac{b_1}{\theta_D+1}$, $-1 - \frac{b_1}{\theta_1+1}$ with corresponding multiplicities 1,

$k - m_D$, $k - m_1$. So this means that $\Delta(x)$ is strongly regular by [7, Lemma 10.1.5], and hence by Lemma 5(i), we find Γ is tight. So we have shown that Γ is tight or bipartite. This means that $a_D = 0$ by Lemma 5(ii). By [8] it follows that $k_D = 1$ as otherwise $-1 - \frac{b_1}{\theta_1+1}$ has multiplicity at least $k + 1 - m_1$ in $\Delta(x)$ for any vertex x. This shows (i).
(ii) Let x be a vertex of Γ and consider the local graph $\Delta(x)$. Proposition 6 implies that $-1 - \frac{b_1}{\theta_1+1}$ and $-1 - \frac{b_1}{\theta_D+1}$ are both eigenvalues of $\Delta(x)$. Now $-1 - \frac{b_1}{\theta_1+1} \neq -1$, so that means $\Delta(x)$ is not the disjoint union of cliques, and hence is connected. But this shows that Γ is tight in similar fashion as in (i). ∎

Remark: (i) The bipartite distance-regular graphs with an eigenvalue having multiplicity k are determined by N. Yamazaki [21] and K. Nomura [16]. They found the following:
(a) $2d$-gons,
(b) complete bipartite graphs,
(c) complements of $2 \times (k+1)$-grids,
(d) Hadamard graphs,
(e) antipodal 2-covers with the intersection array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$ and $\gamma \geqslant 2$,
(f) hypercubes.

For the fifth case, if $\gamma = 2$, then the graph is 2-cover of Higman-Sims graph, and for $\gamma \geqslant 3$, no graph is known.

(ii) The Taylor graphs have $m_1 + m_3 = k + 1$. Besides them there are feasible intersection arrays known for diameter 4 with $m_1 + m_4 = k + 1$. These are

$$\{56, 45, 12, 1; 1, 12, 45, 56\},$$
$$\{115, 96, 20, 1; 1, 20, 96, 115\},$$
$$\{204, 175, 30, 1; 1, 30, 175, 204\} \text{ and,}$$
$$\{329, 288, 42, 1; 1, 42, 288, 329\}.$$

For the first intersection array, it is known that there are no distance-regular graphs with this intersection array([3, 11.4.6 Theorem]). There are no feasible intersection arrays known for larger diameter.

In Theorem 12 below, we show that the (non-bipartite) Taylor graphs are the distance-regular graphs with diameter $D \geqslant 3$, valency k and intersection number $a_1 \neq 0$ having a light tail such that its accompanying eigenvalue equals -1. To show this result we first need the following lemma.

Lemma 11. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency k, intersection number $a_1 \neq 0$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Let θ be a light tail of Γ with standard sequence $1 = \omega_0, \omega_1, \ldots, \omega_D$ and let θ' be the accompanying eigenvalue of θ. For all $x \in V(\Gamma)$, let the local graph $\Delta(x)$ be a (non-complete) strongly regular graph with parameters $(v' = k, k' = a_1, \lambda', \mu')$. Then the following statements are equivalent.*

(i) $\theta' = -1$.
(ii) θ *is a root of* $x^2 - (a_1 - b_1)x - k$.
(iii) $k' = 2\mu'$.
(iv) $\omega_2 = -\omega_1$.

Proof: The equivalence (i)⇔(ii) follows from [10, Theorem 4.1(a)].
The equivalence (ii)⇔(iii) follows from [10, Corollary 6.3].
The equivalence (ii)⇔(iv) is straightforward. ∎

In the next result we show that any of the 4 statements in Lemma 11 is equivalent with Γ be a Taylor graph.

Theorem 12. *Let* Γ *be a distance-regular graph with diameter* $D \geqslant 3$*, valency* k*, intersection number* $a_1 \neq 0$ *and distinct eigenvalues* $k = \theta_0 > \theta_1 > \ldots > \theta_D$*. Let* $\theta \neq \pm k$ *be an eigenvalue of* Γ*. Then the following statements are equivalent:*
(i) θ *is a light tail of* Γ *with standard sequence* $1 = \omega_0, \omega_1, \ldots, \omega_D$ *such that its accompanying eigenvalue* θ' *equals* -1.
(ii) Γ *is a Taylor graph and* $\theta \in \{\theta_1, \theta_3\}$.

Proof: (i)⇒(ii) As $a_1 \neq 0$ and θ is a light tail it follows that $\theta \in \{\theta_1, \theta_D\}$ by [10, Remarks 3.3(iii)]. If $\theta = \theta_1$, then $\theta_2 = \theta' = -1$ by Lemma 8. If $D \geqslant 4$, then $\theta_2 \geqslant \min\{0, a_2, a_4\} \geqslant 0$. This implies that $D = 3$. By [10, Theorem 5.1] and Lemma 11 we find $c_3 = k\frac{\omega_3(1-\omega_1)}{\omega_3-\omega_2} = k\frac{\omega_3(1-\omega_1)}{\omega_3+\omega_1}$. This implies $c_3 = k$ and $\omega_3 = -1$ as $\omega_1 > 0$ and hence Γ is an antipodal r-cover. By [4, p.142–143], $\omega_3 = -1/(r-1)$ and hence Γ is a Taylor graph. Let us assume that $\theta = \theta_D$, then we need to consider two cases: $D = 3$ and $D \geqslant 4$. If $D = 3$, then let α be the largest root of $x^2 - (a_1 - b_1)x - k$. Let Tay(Δ) be the Taylor graph corresponding to $\Delta = \Delta(x)$ as in Lemma 9(ii). Here note that as θ is a light tail the local graph $\Delta = \Delta(x)$ is a (non-complete) strongly regular graph with parameters $(v' = k, k' = a_1, \lambda', \mu')$ and it satisfies $k' = 2\mu'$ by Lemma 11. Now Δ has the smallest eigenvalue $-1 - \frac{b_1}{\alpha+1}$ as α is an eigenvalue of Tay(Δ) and Tay(Δ) is tight. This implies $\theta_1 \leqslant \alpha$ ([4, Theorem 4.4.3]). But then $a_1 + a_2 + a_3 = k + \theta_1 + \theta_2 + \theta_3 \leqslant k + \alpha + \theta_2 + \theta_3 = 2a_1$ as Tay(Δ) has eigenvalues k, α, θ_2, θ_3. Hence $a_1 \geqslant a_2 + a_3$. But $a_2 + a_3 \geqslant a_1$ by [11, Proposition 4]. So $a_2 + a_3 = a_1$ and this implies $a_3 = 0$ and $b_2 = 1$ and hence Γ is a Taylor graph. If $D \geqslant 4$, then by [12, Theorem 3.1(iii)], $\theta_1 \geqslant \frac{a_1+\sqrt{a_1^2+4k}}{2} > a_1 + 1$. But again from the proof of $D = 3$ we have $\theta_1 \leqslant \alpha$, where α is the largest root of $x^2 - (a_1 - b_1)x - k$. But if we evaluate the polynomial $x^2 - (a_1 - b_1)x - k$ in point $a_1 + 1$ we see that it is always non-negative. This means that $\alpha \leqslant a_1 + 1$ and $\alpha \geqslant \theta_1 > a_1 + 1$, a contradiction. So this case can not occur.
(ii)⇒(i) It is easily checked that if Γ is a Taylor graph then $\theta \in \{\theta_1, \theta_3\}$ is a light tail and its accompanying eigenvalue θ' equals -1. ∎

5 The refined bound

In this section, we will show the following refined version of Theorem 1.

Theorem 13. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 2$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots, \theta_D$. Let $\theta \neq \pm k$ be an eigenvalue of Γ with multiplicity $m \geqslant 2$. Then $k \leqslant \frac{(m+2)(m-1)}{2}$. More precisely, the following hold.*
(i) If $m = 2$, then $k = 2$.
(ii) If θ is not a tail and $m \geqslant 3$, then $k \leqslant \frac{(m-1)(m+4)}{4}$.
(iii) If θ is a heavy tail with $\theta' \notin \{\theta_1, \theta_D\}$ and $m \geqslant 3$, then $k \leqslant \frac{(m+1)(m-2)}{2}$.
(iv) If θ is a heavy tail with $\theta' \in \{\theta_1, \theta_D\}$ and $m \geqslant 3$, then $k \leqslant \frac{(m-2)(m+3)}{2}$.
(v) If θ is a light tail, then $k \leqslant \frac{(m+2)(m-1)}{2}$.

Proof: Let $\theta = \theta_i \neq \pm k$ be an eigenvalue of Γ with multiplicity $m = m_i$. [4, Proposition 4.4.8(ii)] shows $k = 2$ if and only if $m = 2$. This shows (i). So from now on we may assume $m \geqslant 3$ and $k \geqslant 3$. We will first consider the case $m < k$ and later we will consider $m \geqslant k$. Let us first assume $m < k$. Then $i \in \{1, D\}$ by Proposition 6, and $a_1 \neq 0$ by [10, Theorem 3.2]. If there are at least two distinct $j_1, j_2 \notin \{0, i\}$ satisfying $q_{ii}^{j_1} \neq 0 \neq q_{ii}^{j_2}$, then by Lemma 2 and Lemma 7 we have $\frac{m(m+1)}{2} \geqslant m_0 + m_{j_1} + m_{j_2} \geqslant 1 + k + k - m + 1$ and hence $k \leqslant \frac{(m-1)(m+4)}{4}$. If $q_{ii}^j = 0$ for all $j \notin \{0, i\}$, then by [10, Theorem 4.1(b)], Γ is antipodal with diameter 3 and $\theta = \theta_2 = -1$. But then $m = k$. This shows (ii) if $m < k$. Now let us assume θ is a tail and θ' its accompanying eigenvalue. Let m' be the multiplicity of θ'. If θ is a heavy tail with $\theta' \notin \{\theta_1, \theta_D\}$, then by Lemma 2 and Proposition 6, $\frac{m(m+1)}{2} \geqslant 1+m+k$ and this shows (iii) if $m < k$. If θ is a heavy tail with $\theta' \in \{\theta_1, \theta_D\}$, then by Lemma 2, Proposition 6, and Lemma 7, $\frac{m(m+1)}{2} \geqslant 1 + m + m' \geqslant 1 + m + k + 1 - m = k + 2$. But if $\frac{m(m+1)}{2} = k + 2$, then $m + m' = k + 1$ and it follows by Lemma 10 that Γ is tight. But $q_{1D}^j = 0$ if $j \neq D - 1$ by Lemma 5(iii), so this give a contradiction. This shows (iv) when $m < k$. Now if θ is a light tail, then for all vertices x the local graph $\Delta(x)$ is strongly regular by [10, Corollary 6.3]. If $m' < k$, then $\{\theta, \theta'\} = \{\theta_1, \theta_D\}$ and by Lemma 10(ii) Γ is tight. But this is not possible by Lemma 5(iii). This means $m' \geqslant k$. Now by Lemma 2, $\frac{m(m+1)}{2} \geqslant m_0 + m' \geqslant 1 + k$. This shows (v). So we have shown the theorem if $m < k$. As $m \leqslant \frac{(m-1)(m+4)}{4}$, $m \leqslant \frac{(m-2)(m+3)}{2}$, and $m \leqslant \frac{(m+2)(m-1)}{2}$ if $m \geqslant 3$, it follows that cases (ii), (iv), and (v) also hold if $m \geqslant k \geqslant 3$. For case (iii) and $m \geqslant k \geqslant 3$ we see that $m \leqslant \frac{(m+1)(m-2)}{2}$ unless $m = 3$. If $m = 3$ and $m \geqslant k \geqslant 3$, then we see that $k = 3$ and $a_1 = 0$ as $D \geqslant 3$. But then θ is a light tail, a contradiction with the assumption that θ is a heavy tail. ∎

In the following theorem, we characterize the distance-regular graphs with valency at least three which attain the bound in Theorem 13.

Theorem 14. *Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 3$ and an eigenvalue θ having multiplicity $m \geqslant 2$. Then the following statements are equivalent.*
(i) $k = \frac{(m+2)(m-1)}{2}$.
(ii) Γ *is a Taylor graph with intersection array* $\{(2\alpha+1)^2(2\alpha^2+2\alpha-1), 2\alpha^3(2\alpha+3), 1; 1, 2\alpha^3(2\alpha+3), (2\alpha+1)^2(2\alpha^2+2\alpha-1)\}$ *where α is an integer $\neq 0, -1$ or* $\alpha = \frac{-1\pm\sqrt{5}}{2}$, *(and* $m = 4\alpha^2+4\alpha-1$*).*

Proof: (i)⇒(ii) The only distance-regular graphs with an eigenvalue having multiplicity 2 are the polygons. So θ has multiplicity $m \geqslant 3$. As $m < \frac{(m+2)(m-1)}{2}$ if $m \geqslant 3$, we have $m < k$ and hence $a_1 \neq 0$. By Theorem 13, the eigenvalue θ is a light tail. To complete the proof, we will show that for any vertex x of Γ, the local graph $\Delta(x)$ at the vertex x is a strongly regular graph with parameters (v', k', λ', μ') satisfying $k' = 2\mu'$. Then by Lemma 11 the accompanying eigenvalue θ' of θ is equal to -1, and hence by Theorem 12 the graph Γ is a Taylor graph with the parameters as stated in the theorem.
Let x be a vertex of Γ. Then the local graph $\Delta(x)$ is a strongly regular graph. If $\Delta(x)$ is not connected, then $\Delta(x)$ is the disjoint union of $\frac{k}{a_1+1}$ complete graphs with a_1+1 vertices. Then by [10, Corollary 6.3], we have

$$\theta = \theta_D = -1 - \frac{b_1}{a_1+1} = \frac{-k}{a_1+1}$$

and also by [10, Theorem 3.2] we have

$$m = k - \frac{b_1}{a_1+1} \geqslant \frac{k}{2} + 1.$$

As $k = \frac{(m+2)(m-1)}{2} \geqslant 2m-1$ if $m \geqslant 3$, we find that $\Delta(x)$ must be connected. By [10, Corollary 6.3] we find that $\Delta(x)$ has an eigenvalue $\frac{a_1\theta}{\theta+k}$ with multiplicity $m-1$. Now by parts (iv) and (v) in Lemma 3, we find that the local graph $\Delta(x)$ at the vertex x is strongly regular with parameters $(v', k', \lambda', \mu') = ((2\alpha+1)^2(2\alpha^2+2\alpha-1), 2\alpha^3(2\alpha+3), \alpha(2\alpha^3+\alpha^2-3\alpha+1), \alpha^3(2\alpha+3))$ satisfying $k' = 2\mu'$, where α is an integer $\neq 0, -1$ or $\alpha = \frac{-1\pm\sqrt{5}}{2}$. This shows (i).
(ii)⇒(i) Trivial.
This finishes the proof. ∎

Remark: Note that the distance-2 graph of a graph $\Gamma = (V(\Gamma), E(\Gamma))$ has as vertex set $V(\Gamma)$ and two vertices are adjacent if they have distance 2 in Γ. Then the distance-2 graph of a Taylor graph with intersection array

$$\{(2\alpha+1)^2(2\alpha^2+2\alpha-1), 2\alpha^3(2\alpha+3), 1; 1, 2\alpha^3(2\alpha+3), (2\alpha+1)^2(2\alpha^2+2\alpha-1)\},$$

where α is an integer $\neq 0, 1$ or $\alpha = \frac{-1\pm\sqrt{5}}{2}$, is again a Taylor graph with intersection array

$$\{(2\beta+1)^2(2\beta^2+2\beta-1), 2\beta^3(2\beta+3), 1; 1, 2\beta^3(2\beta+3), (2\beta+1)^2(2\beta^2+2\beta-1)\},$$

where $\beta = -\alpha - 1$.

Also, the following hold:
(i) Γ is the Icosahedron if $\alpha = \frac{-1\pm\sqrt{5}}{2}$,
(ii) Γ is the Gosset graph if $\alpha = 1$,
(iii) Γ is the distance-2 graph of Gosset graph if $\alpha = -2$,
(iv) Γ is the Tay(McLaughlin graph) (see [20]) if $\alpha = -3$,
(v) Γ is the distance-2 graph of Tay(McLaughlin graph) if $\alpha = 2$,
(vi) For the other α nothing is known.

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