

On the Sprague-Grundy Values of the $\mathcal{F}$ -Wythoff Game

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Abstract

We examine the Sprague-Grundy values of $\mathcal{F}$ -Wythoff, a restriction of Wythoff's game introduced by Ho, where the integer ratio of the pile sizes must be preserved if the same number of tokens is removed from both piles. We answer two conjectures raised by Ho. First, we show that each column of Sprague-Grundy values is ultimately additively periodic. Second, we prove that every diagonal of Sprague-Grundy values contains all the nonnegative integers. We also investigate the asymptotic behavior of the sequence of positions attaining a given Sprague-Grundy value.

Keywords: Wythoff's game; $\mathcal{P}$ -positions; Sprague-Grundy function; combinatorial games

1 Introduction

Wythoff's game is played on two piles of tokens with two players alternating moves. A player may remove any nonzero number of tokens from one pile or may remove the same nonzero number of tokens from both piles. The last player to move wins. $\mathcal{F}$ -Wythoff [3] is a restriction of Wythoff's game in which a player may remove any nonzero number of tokens from one pile or may remove $1 \leq j \leq a-1$ tokens from both piles if $\left\lfloor \frac{b-j}{a-j} \right\rfloor = \left\lfloor \frac{b}{a} \right\rfloor$, where $b \geq a$ are the sizes of the piles. We say a position (c, d) is a *follower* of (a, b) if $(a, b) \rightarrow (c, d)$ is a move. Let $F(a, b)$ be the set of followers of (a, b) . We call positions of the form $(a-i, b-i) \in F(a, b)$ *slant followers* of (a, b) .

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The *Sprague-Grundy value* $\mathcal{G}(p)$ of a position p is defined recursively by $\mathcal{G}(p) = \text{mex}\{\mathcal{G}(q) : q \in F(p)\}$, where $\text{mex}(A) = \min(\mathbb{N}_0 \setminus A)$ denotes the *minimal excludant* of the set A . We call a position p a g -position if $\mathcal{G}(p) = g$. A $\mathcal{P}$ -position is one from which the previous player has a winning strategy. An $\mathcal{N}$ -position is one from which the next player has a winning strategy. It is a standard game theory result that a position is a $\mathcal{P}$ -position if and only if it is a 0-position.

Suppose the Sprague-Grundy values of Wythoff are written in a chart with $\mathcal{G}(i, j)$ in entry (i, j) (see Table 1). Define *row* b to be the sequence $\{\mathcal{G}(i, b)\}_{i \geq 0}$, *column* a to be the sequence $\{\mathcal{G}(a, j)\}_{j \geq 0}$, and *diagonal* d to be the sequence $\{\mathcal{G}(a, a + d)\}_{d \geq 0}$. We say that a sequence $\{g_n\}_{n \geq 0}$ is *ultimately additively periodic* if there exist N and $j > 0$ such that $g_{n+j} = g_n + j$ for all $n \geq N$, that is, if and only if $\{g_n - n\}_{n \geq 0}$ is ultimately periodic.

Table 1: Table of Sprague-Grundy Values

12	12	13	14	11	10	9	8	15	1	4	17	7	16
11	11	10	9	12	8	15	13	0	2	3	16	14	7
10	10	11	8	9	7	14	15	3	13	5	6	16	17
9	9	8	11	10	12	13	1	2	6	7	5	3	4
8	8	9	10	7	11	0	12	4	5	6	13	2	1
7	7	6	5	8	9	1	10	11	4	2	3	0	15
6	6	7	4	5	0	2	3	10	12	1	15	13	8
5	5	4	7	6	3	8	2	1	0	13	14	15	9
4	4	5	6	1	2	3	0	9	11	12	7	8	10
3	3	2	0	4	1	6	5	8	7	10	9	12	11
2	2	3	1	0	6	7	4	5	10	11	8	9	14
1	1	0	3	2	5	4	7	6	9	8	11	10	13
0	0	1	2	3	4	5	6	7	8	9	10	11	12
j/i	0	1	2	3	4	5	6	7	8	9	10	11	12

In Wythoff's game, it is known that the set of $\mathcal{P}$ -positions is $\{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) : n \geq 0\}$, where ϕ is the golden ratio, up to reordering of the coordinates [6]. Moreover, the Sprague-Grundy values of standard Wythoff have been studied extensively in [1], [4], and [5]. It is known that every column of Sprague-Grundy values is ultimately additively periodic [4]. Additionally, every row, column, and diagonal of Sprague-Grundy values contains each nonnegative integer exactly once [5, 1]. In [5], it is shown that for a fixed g , the positive difference between the n th g -position and the n th 0-position is bounded by a constant depending solely on g , where the sequences are in order of increasing first coordinates with the smaller pile size in the first coordinate.

In [3], Ho introduces the game $\mathcal{F}$ -Wythoff and shows that the $\mathcal{P}$ -positions of $\mathcal{F}$ -Wythoff, with the exception of $(0,0)$, are translations of the $\mathcal{P}$ -positions of standard Wythoff by 1 in each entry. Moreover, the 1-positions and 2-positions in $\mathcal{F}$ -Wythoff, with finitely many exceptions, are each translations of Wythoff's $\mathcal{P}$ -positions by 2 and 4, respectively [3]. Similar to Wythoff, every row and column of Sprague-Grundy values contains each nonnegative integer exactly once in $\mathcal{F}$ -Wythoff [3]. For $\mathcal{F}$ -Wythoff, Ho

conjectures that $\{\mathcal{G}(a, n)\}_{n \geq 0}$ is ultimately additively periodic. He also conjectures that each nonnegative integer appears exactly once along every diagonal of Sprague-Grundy values.

In the following sections, we will prove Ho's first conjecture and a slight variant of Ho's second conjecture, where we allow a Sprague-Grundy value to appear multiple times along each diagonal. We have a counterexample to Ho's original conjecture which posits that every Sprague-Grundy value appears exactly once along each diagonal. In Section 2, we will show the ultimate additive periodicity of each column of Sprague-Grundy values. Next, we give an algorithm to compute the sequence of g -positions and use the algorithm to prove that all the nonnegative integers appear in each diagonal of Sprague-Grundy values. In the subsequent section, we provide bounds on each coordinate of the n th g -position for a given g . In the last section, we conjecture that for each g , the ratio of the larger pile size to the smaller pile size of the n th g -position approaches the golden ratio as n tends to ∞ . We also conjecture that the set of g -positions of $\mathcal{F}$ -Wythoff are not eventually translations of the $\mathcal{P}$ -positions of Wythoff for $g > 2$.

2 Additive Periodicity of the Sprague-Grundy Values

In this section, we will show that every column of Sprague-Grundy values is ultimately additively periodic. We start by bounding $\mathcal{G}(a, b)$.

Lemma 1. $b - 2a \leq \mathcal{G}(a, b) \leq a + b$.

Proof. This claim is proved in [4] for Wythoff's game and the proof generalizes naturally to $\mathcal{F}$ -Wythoff. We reproduce the proof here for the convenience of the reader. We first prove the lower bound. Let $g = \mathcal{G}(a, b)$. Then $g \neq \mathcal{G}(a, k)$ for $0 \leq k \leq b - 1$, i.e. g does not appear as $\mathcal{G}(a, k)$ for exactly b values of k . If $g \neq \mathcal{G}(a, k)$, either $\mathcal{G}(a, k) < g$ or $\mathcal{G}(a, k) > g$. The former case occurs for at most g values of k with $0 \leq k \leq b - 1$ since $0 \leq \mathcal{G}(a, k) \leq g - 1$. In the latter case, either there is some $0 \leq j_k \leq a - 1$ such that $\mathcal{G}(j_k, k) = g$, or there is some $1 \leq i_k \leq \min\{a, k\}$ such that $\mathcal{G}(a - i_k, k - i_k) = g$, where taking i_k away from each of k and a preserves the integer ratio. Note there is no $0 \leq l_k \leq k - 1$ such that $\mathcal{G}(a, l_k) = g$ since $\mathcal{G}(a, k) \neq g$ for all $0 \leq k \leq b - 1$. Since each column can have only one occurrence of g and $1 \leq i_k \leq a$, we know that $\mathcal{G}(a - i_k, k - i_k) = g$ occurs for at most a values of k with $0 \leq k \leq b - 1$. For the same reason $\mathcal{G}(j_k, k) = g$ occurs for at most a values of k with $0 \leq k \leq b - 1$. Then the number of k 's such that $\mathcal{G}(a, k) \neq g$ with $0 \leq k \leq b - 1$ is at most $g + 2a$. So $b \leq g + 2a$ or $b - 2a \leq g$.

For the upper bound, we induct on $a + b$. The statement holds for $a + b = 0$ since $\mathcal{G}(0, 0) = 0$. Assume that for i, j such that $i + j \leq a + b - 1$, we have $\mathcal{G}(i, j) \leq i + j$. Then $\mathcal{G}(a, b) = \text{mex}\{\mathcal{G}(c, d) : (c, d) \in F(a, b)\} \leq a + b$ since $(c, d) \in F(a, b)$ implies $\mathcal{G}(c, d) \leq c + d < a + b$, i.e. $a + b \notin \{\mathcal{G}(c, d) : (c, d) \in F(a, b)\}$. $\square$

Next, we show that (a, b) has no slant followers if $b > a^2$.

Lemma 2. If $(a - i, b - i) \in F(a, b)$, then $b \leq a^2$.

Proof. Let $(a - i, b - i) \in F(a, b)$. If $b \leq a$, then $b \leq a^2$. Assume $b > a$. Since $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{b-i}{a-i} \rfloor$, we have

$$\begin{aligned} \left| \frac{b}{a} - \frac{b-i}{a-i} \right| &< 1 \\ \Rightarrow \frac{b-i}{a-i} - \frac{b}{a} &< 1 \\ \Rightarrow (b-i)a - b(a-i) &< (a-i)a \\ \Rightarrow i(b-a) &< a(a-i). \end{aligned}$$

Since $b - a \leq i(b - a) < a(a - i) \leq a(a - 1)$, we have $b < a^2$. $\square$

Remark 3. In fact, $b \leq (a - 1)a$ for $(b - i, a - i) \in F(a, b)$ (by a different argument). But this sharper bound is unnecessary for our purposes.

Using the two lemmas above and a method of Landman [4], we will prove the ultimate additive periodicity conjecture of Ho.

Theorem 4 (Conjecture 13 of [3]). For $a \geq 0$, there exist M and $j > 0$ such that $\mathcal{G}(a, b + j) = \mathcal{G}(a, b) + j$ for all $b \geq M$.

Proof. Define $\mathcal{H}(a, b) = \mathcal{G}(a, b) - b + 2a$. By Lemma 1, we know that $0 \leq \mathcal{H}(a, b) \leq 3a$ for all b . By definition, $\{\mathcal{H}(a, b)\}_{b \geq 0}$ is ultimately periodic if and only if $\{\mathcal{G}(a, b)\}_{b \geq 0}$ is ultimately additively periodic. We will show that we can compute the sequence $\{\mathcal{H}(a, b)\}_{b \geq 0}$ by a finite-state machine. For a given position, we will store the data regarding its followers in a finite amount of space independent of b . This enables us to calculate $\{\mathcal{H}(a, b)\}_{b \geq 0}$ with only a finite number of states, which will prove our result.

Let $L(a, b) = \{\mathcal{G}(a - k, b) : 1 \leq k \leq a\}$ and $D(a, b) = \{\mathcal{G}(a, b - k) : 1 \leq k \leq b\}$. To get bounds independent of b , let $L'(a, b) = \{b - 2a, \dots, a + b\} \setminus L(a, b)$. Similarly, define $D'(a, b) = \{b - 2a, \dots, a + b\} \setminus D(a, b)$. Then $|L'(a, b)|$ and $|D'(a, b)|$ are at most $3a + 1$. Represent $L'(a, b)$ and $D'(a, b)$ by a string of $3a + 1$ bits, with 1 in position j if $b - 2a + j$ is in the set and 0 otherwise. Since we only need to show $\{\mathcal{H}(a, b)\}_{b \geq 0}$ is eventually periodic, we may assume that the finite-state machine starts with $b > a^2$. By Lemma 2, (a, b) has no slant followers for $b > a^2$. So we do not need to compute $\lfloor \frac{b}{a} \rfloor$ to test for slant followers, i.e., it is not necessary to store b at any stage. Then $\mathcal{G}(a, b) = \min(L'(a, b) \cap D'(a, b))$ since $b - 2a \leq \mathcal{G}(a, b) \leq a + b$ by Lemma 1. For each stage in the finite-state machine, store the data of $L'(0, b), \dots, L'(a, b)$, $D'(0, b), \dots, D'(a, b)$, and $\mathcal{H}(a, b - 1)$. There are $2a + 3$ strings each having at most $3a + 1$ bits, which takes up $O(a^2)$ bits. Thus, it remains to show that we can compute $L'(0, b + 1), \dots, L'(a, b + 1)$, $D'(0, b + 1), \dots, D'(a, b + 1)$, and $\mathcal{H}(a, b)$ from $L'(0, b), \dots, L'(a, b)$, $D'(0, b), \dots, D'(a, b)$, and $\mathcal{H}(a, b - 1)$.

First, we compute and store $\mathcal{H}(a, b)$ from $L'(a, b)$ and $D'(a, b)$. Second, we will compute and store $D'(i, b + 1)$ for each $i \in \{0, \dots, a\}$. For $i \in \{0, \dots, a\}$, compute $\mathcal{H}(i, b)$ from $L'(i, b)$ and $D'(i, b)$, and then use $\mathcal{H}(i, b)$ and $D'(i, b)$ to get $D'(i, b + 1)$. Lastly, we compute and store $L'(i, b + 1)$ for $i \in \{0, \dots, a\}$ in order, starting with $i = 0$. We know

that $L'(0, b+1) = \{b+1\}$. Then $\mathcal{H}(0, b+1)$ can be computed from our stored $D'(0, b+1)$ and the known $L'(0, b+1)$. Next, $L'(1, b+1)$ can be found from $L'(0, b+1)$ and $\mathcal{H}(0, b+1)$. Now, we can compute $\mathcal{H}(1, b+1)$ from the stored $D'(1, b+1)$ and $L'(1, b+1)$. Continue in this manner to obtain $L'(i, b+1)$ for each $i \in \{0, \dots, a\}$. To make sure not to exceed column a , store the column number while building up the $L'(i, b+1)$ s from $i = 0$ to $i = a$. This requires no more than $(a+1)\log(a)$ bits.

Calculating each $\mathcal{H}(a, b)$ for $b > a^2$ takes $O(a^2)$ bits. So there are $2^{O(a^2)}$ possible states with each state dependent only on the previous one. Since a is constant and the sequence $\{\mathcal{H}(a, b)\}_{b \geq 0}$ is infinite, the procedure revisits some state after $2^{O(a^2)}$ steps. Hence for a fixed a , $\{\mathcal{H}(a, b)\}_{b \geq 0}$ is ultimately periodic, which means $\{\mathcal{G}(a, b)\}_{b \geq 0}$ is ultimately additively periodic. $\square$

Remark 5. In [2], Ho introduces another restriction of Wythoff's game called $\mathcal{R}$ -Wythoff, in which a player may either remove any number of tokens from the larger pile or remove an equal number of tokens from both piles. Ho conjectures that in $\mathcal{R}$ -Wythoff, $\{\mathcal{G}(a, b)\}_{b \geq 0}$ is ultimately additively periodic. This can be proved using a similar technique as above by letting $\mathcal{H}(a, b) = \mathcal{G}(a, b) - b + 2a - 1$. Define $D(a, b) = \{\mathcal{G}(a, b - k) : 1 \leq k \leq b\}$ and $D'(a, b) = \{b - 2a + 1, \dots, a + b - 1\} \setminus D(a, b)$. Similarly, define $S(a, b) = \{\mathcal{G}(a - k, b - k) : 1 \leq k \leq a\}$ and $S'(a, b) = \{b - 2a + 1, \dots, a + b - 1\} \setminus S(a, b)$. By Theorem 2.12 and 2.13 of [2], $0 \leq \mathcal{H}(a, b) \leq 3n - 2$ for $b \geq a \geq 4$. At each stage, store the bit arrays representing $D'(0, b), \dots, D'(a, b)$, $S'(0, b), \dots, S'(a, b)$, and $\mathcal{H}(a, b - 1)$. The cases that $a \in \{0, 1, 2, 3\}$ can be checked by induction.

3 Diagonal Sprague-Grundy Values

In this section, we modify an algorithm of Blass and Fraenkel's [1], which computes the positions that attain a given Sprague-Grundy value in standard Wythoff. Our algorithm will compute the corresponding positions in $\mathcal{F}$ -Wythoff. With the aid of this algorithm, we adapt Blass and Fraenkel's technique [1] to prove the second conjecture of Ho, that each diagonal of Sprague-Grundy values contains all the nonnegative integers.

We consider only positions (a, b) with $a \leq b$ unless otherwise specified. We will use Algorithm $\mathcal{FWSG}$ defined below to compute a sequence of positions called $T_j = \{(a_n^j, b_n^j)\}_{n \geq 0}$. Later, we will show that T_j is the sequence of j -positions in increasing order of the first coordinate.

To compute entry k in T_j , use the following algorithm to compute $T_0, \dots, T_{j-1}$ in order, up to some large number of entries, and then compute entries 0 through k of T_j in order.

Algorithm $\mathcal{FWSG}$

1. $p \leftarrow \text{mex}\{a_n^j, b_n^j : 0 \leq n < k\}$.
2. $q \leftarrow m$, smallest $m \geq 0$ such that

$$(a) \text{ for each } n \in \{0, \dots, k-1\} \text{ such that } m = b_n^j - a_n^j, \text{ we have } \left\lfloor \frac{b_n^j}{a_n^j} \right\rfloor \neq \left\lfloor \frac{p+m}{p} \right\rfloor,$$

- (b) $(p, p + m) \notin T_i$ for all $0 \leq i < j$,
- (c) $p + m \neq b_n^j$ for all $n \in \{0, \dots, k - 1\}$.

3. $(a_k^j, b_k^j) \leftarrow (p, p + q)$.

Let $A_j = \{a_n^j\}_{n \geq 0}$, $B_j = \{b_n^j\}_{n \geq 0}$, and $D_j = \{b_n^j - a_n^j\}_{n \geq 0}$.

We start by showing some properties of the sequences A_j and T_j to help us prove the validity of the algorithm.

Proposition 6. *The sequence $\{a_n^j\}_{n \geq 0}$ is strictly increasing in n .*

Proof. Consider the sequence $A_j = \{a_n^j\}_{n \geq 0}$. We induct on n . Assume that $\{a_k^j\}_{k \geq 0}$ is strictly increasing for $k < n$. The set $\{a_k^j, b_k^j : 0 \leq k \leq n - 2\}$ contains all the integers from 0 to $a_{n-1}^j - 1$ since $a_{n-1}^j = \text{mex}\{a_k^j, b_k^j : 0 \leq k < n - 1\}$. So $a_n^j = \text{mex}\{a_k^j, b_k^j : 0 \leq k < n\} = \text{mex}\{0, \dots, a_{n-1}^j\} \geq a_{n-1}^j + 1$, completing the induction. $\square$

Proposition 7. *If $(a, b) \in T_j$, then $(p, a) \notin T_j$ for $p \neq a$.*

Proof. Let $(a, b) \in T_j$. Suppose for contradiction that there is some $p \neq a$ such that $(p, a) \in T_j$. Then $p < a$ since a is the second coordinate. Because A_j is strictly increasing, (p, a) must appear before (a, b) in T_j . Then $a = \text{mex}\{p, a, \dots\} \neq a$, which is a contradiction. $\square$

Now we are ready to give a characterization of the sequence T_j . The following lemma is a slight variation of a result of Blass and Fraenkel [1].

Lemma 8. *Every T_j consists exactly of the positions having Sprague-Grundy j if and only if every T_j satisfies*

1. $T_i \cap T_j = \emptyset$ for $i \neq j$.
2. If $(a, b) \in T_j$, then (a, b) has no follower in T_j .
3. If $(s, t) \notin T_0 \cup \dots \cup T_j$, then (s, t) has a follower in T_j .

Proof. The “only if” direction follows by definition of the Sprague-Grundy function. We will show that the three conditions are sufficient to conclude that $T_j = \{(a, b) : \mathcal{G}(a, b) = j\}$. Assume the three conditions hold for every j . We induct on j . So assume T_i consists exactly of positions with Sprague-Grundy i for $i < j$.

First, we will show that if $\mathcal{G}(a, b) = j$, then $(a, b) \in T_j$. Suppose for contradiction that there is some $(a, b) \notin T_j$ with $\mathcal{G}(a, b) = j$. Then $(a, b) \notin T_i$ for $i \leq j$ by the induction hypothesis. So (a, b) has a follower $(a', b') \in T_j$ by condition 3. Then $\mathcal{G}(a', b') > j - 1$ by condition 1. Moreover, $\mathcal{G}(a', b') \neq j$ since $(a', b') \in F(a, b)$. Then (a', b') has a follower (a'', b'') such that $\mathcal{G}(a'', b'') = j$ (by definition of $\mathcal{G}$). Moreover, $(a'', b'') \notin T_j$ by condition 2 since $(a'', b'') \in F(a', b')$. Continuing similarly gives an infinite sequence of moves $(a, b) \rightarrow (a', b') \rightarrow (a'', b'') \rightarrow (a''', b''') \dots$ such that $\mathcal{G}(a, b) = j$, $\mathcal{G}(a', b') > j$, $\mathcal{G}(a'', b'') = j$,

$\mathcal{G}(a''', b''') > j, \dots$ This contradicts the fact that a game of $\mathcal{F}$ -Wythoff must terminate in a finite number of moves. Hence $\mathcal{G}(a, b) = j$ implies $(a, b) \in T_j$.

Second, we will show that if $(u, v) \in T_j$, then $\mathcal{G}(u, v) = j$. If $(u, v) \in T_j$ and $\mathcal{G}(u, v) \neq j$, then $\mathcal{G}(u, v) > j$ by the induction hypothesis and condition 1. Thus (u, v) has a follower (u', v') such that $\mathcal{G}(u', v') = j$ (by definition of $\mathcal{G}$), i.e. $(u', v') \in T_j$ by the above paragraph. This contradicts condition 2 since both (u, v) and (u', v') are in T_j . Hence $(u, v) \in T_j$ implies $\mathcal{G}(u, v) = j$. $\square$

Using the two propositions and the lemma above, we show that the algorithm produces the sequence of j -positions in increasing order of the smaller coordinate.

Theorem 9. *The sequence T_j determined by Algorithm $\mathcal{FWSG}$ consists of exactly all the positions having Sprague-Grundy j in increasing order of the first coordinate.*

Proof. We will show that T_j satisfies the three conditions of Lemma 8.

1. $T_i \cap T_j = \emptyset$ for $i \neq j$ by step 2b of the algorithm.
2. Let $(a, b) \in T_j$. We will show that none of the followers of (a, b) are in T_j . For $1 \leq k \leq a$, $(a - k, b) \notin T_j$ since B_j consists of distinct terms by step 2c. Let $1 \leq k \leq b$. If $b - k \geq a$, then $(a, b - k) \notin T_j$ since A_j has only distinct terms. If $b - k < a$, then $(b - k, a) \notin T_j$ by Proposition 7. Let $1 \leq k \leq a - 1$ be such that $\lfloor \frac{b-k}{a-k} \rfloor = \lfloor \frac{b}{a} \rfloor$. Suppose for contradiction that $(a - k, b - k) \in T_j$. When $b - a$ is considered as a candidate for q with $p = a$, the difference $b - a$ is rejected at step 2 since it violates condition 2a. So $q > b - a$, contradicting that $(a, b) \in T_j$. Hence (a, b) has no follower in T_j .
3. Assume $(s, t) \notin T_i$ for $0 \leq i \leq j$, with $s \leq t$. We will show that (s, t) has a follower in T_j . Since $A_j \cup B_j = \mathbb{N}_0$ by step 1 of the algorithm, $s = b_n^j$ or $s = a_n^j$ for some n . First, assume that $s = b_n^j$. Then $t - a_n^j \geq t - b_n^j = t - s \geq 0$. If $t = a_n^j$, then $(s, t) \in T_j$, contradicting the assumption. If $t > a_n^j$, then $(s, t) \rightarrow (a_n^j, b_n^j)$ is a move by taking $t - a_n^j$ away from t . Hence we may assume $s \neq b_n^j$.

Second, assume that $s = a_n^j$. If $t > b_n^j$, then $(s, t) \rightarrow (a_n^j, b_n^j)$ is a move by taking $t - b_n^j$ away from t . The case that $t = b_n^j$ cannot happen since $(s, t) \notin T_j$. Hence, we may assume that $t < b_n^j$. If (s, t) has a slant follower in T_j , then we are done. If (s, t) does not have a slant follower in T_j , then either (a) $t - s$ does not appear before $b_n^j - a_n^j$ in D_j or (b) all positions $(s - k, t - k)$ with $1 \leq k \leq s - 1$ appearing before (a_n^j, b_n^j) in D_j satisfy $\lfloor \frac{t-k}{t-s} \rfloor \neq \lfloor \frac{t}{s} \rfloor$.

- (a) Suppose that $t - s$ does not appear before $b_n^j - a_n^j$ in D_j , i.e., either $t - s \notin D_j$ or else for all (a_h^j, b_h^j) with $b_h^j - a_h^j = t - s$, we have $a_h^j > a_n^j = s$. Since $(s, t) \notin T_j$, the difference $t - s$ was considered as a candidate for q with $p = a_n^j$, and was rejected. So $t - s$ violates at least one condition of step 2 of the algorithm. Since $t - s$ either appears after $b_n^j - a_n^j$ or never appears in D_j , $t - s$ passes condition 2a. Moreover, we assumed that $(s, t) \notin T_i$ for all $0 \leq i \leq j$, so the

difference $t - s$ passes condition 2b. Then $t - s$ must have violated condition 2c, i.e., $t \in B_j$ appeared before b_n^j . Say $t = b_k^j$. Since b_k^j occurred before b_n^j and A_j is increasing, $a_k^j < a_n^j$. Then $(s, t) \rightarrow (a_k^j, b_k^j) \in T_j$ is a move.

- (b) Suppose that for all $(a_h^j, b_h^j) \in T_j$ with $a_h^j < a_n^j$ such that there is some $1 \leq k \leq s - 1$ with $a_h^j = s - k$ and $b_h^j = t - k$, the integer ratio $\left\lfloor \frac{b_h^j}{a_h^j} \right\rfloor \neq \left\lfloor \frac{t}{s} \right\rfloor$. In T_j , for $p = a_n^j$, the difference $t - s$ passes condition 2a as a candidate for q since the corresponding integer ratio is never $\left\lfloor \frac{t}{s} \right\rfloor$ for previous elements of T_j having the same difference. Moreover, $(s, t) \notin T_i$ for $0 \leq i < j$ from the assumption. If t does not appear in the B_j constructed so far, then $t - s$ passes all the conditions of step 2, i.e. $(s, t) \in T_j$. But $(s, t) \notin T_j$. So t must have appeared before b_n^j in T_j , i.e. $t = b_\ell^j$ for some $\ell < n$. So $(s, t) \rightarrow (a_\ell^j, b_\ell^j) \in T_j$ is a move.

□

Using our algorithm, we now bound the number of occurrences of a fixed Sprague-Grundy value along each diagonal, which will help us to prove the second conjecture of Ho.

Lemma 10. *For $d, g \in \mathbb{N}_0$, g can appear at most $d + 2$ times along the diagonal $\{\mathcal{G}(a, a + d)\}_{a \geq 0}$.*

Proof. Since $\left\lfloor \frac{a+d}{a} \right\rfloor = 1$ for all $a \geq d + 1$, there are at most $d + 2$ distinct integer ratios in the sequence $\left\{ \left\lfloor \frac{a+d}{a} \right\rfloor \right\}_{a \geq 0}$. For $a_1, a_2 \geq d + 1$, the position $(a_1, a_1 + d)$ must have a different Sprague-Grundy value from $(a_2, a_2 + d)$ because one is a follower of the other. Hence a Sprague-Grundy value can occur at most $d + 2$ times in $\{\mathcal{G}(a, a + d)\}_{a \geq 0}$. □

Corollary 11. *For j and $c \in \mathbb{N}_0$, there is some M such that $b_i^j - a_i^j > c$ for all $i \geq M$.*

Proof. For each $d \in \{0, \dots, c\}$, the difference d can occur at most finitely many times in D_j since there are at most finitely many values a such that $\mathcal{G}(a, a + d) = j$ by Lemma 10. Choose M past the last position where there is a $d \in \{0, \dots, c\}$ that occurs in D_j . Then $b_i^j - a_i^j > c$ for all $i \geq M$. □

In [3], Ho conjectures that every diagonal of Sprague-Grundy values contains each non-negative integer exactly once. However, the conjecture is not completely true because one Sprague-Grundy value may appear multiple times along the same diagonal, e.g. $9 = \mathcal{G}(1, 8) = \mathcal{G}(3, 10) = \mathcal{G}(5, 12)$. Therefore, we modify the conjecture so that the condition that each number appears only once is not required.

Theorem 12 (Modified Conjecture 15 of [3]). *For $d, g \in \mathbb{N}_0$, there is some $a \in \mathbb{N}_0$ (not necessarily unique) such that $\mathcal{G}(a, a + d) = g$.*

Proof. The idea is to show that if some $g \in \mathbb{N}_0$ is missing in some diagonal of Sprague-Grundy values, then the difference between $\max_{0 \leq \ell \leq i} \{b_\ell^j\}$ and a_i^j is bounded, which contradicts Corollary 11.

First, we know that $\mathcal{G}(\lfloor \phi d \rfloor + 1, \lfloor \phi d \rfloor + 1 + d) = 0$ for all $d \geq 0$ by Theorem 5 of Ho [3]. Second, the case of $g = 0$ is proved in [3]. Now suppose there is some $d > 0$ and $j > 0$ such that $\mathcal{G}(a, a + d) \neq j$ for all $a \in \mathbb{N}_0$. By Lemma 10, for $i < j$, the value i can appear at most finitely many times in $\{\mathcal{G}(a, a + d)\}_{a \geq 0}$. Choose some N_1 such that for $x \geq N_1$, $(x, x + d) \notin T_i$ for all $i < j$. Consider some $x \geq N_1$. Since $A_j \cup B_j = \mathbb{N}_0$, we have $x \in A_j$ or $x \in B_j$. If $x \in A_j$, then d was considered as a candidate for q with $p = x$ at step 1 of the algorithm, and therefore must have been rejected at step 2. Since $d \notin D_j$ (by the assumption) and $(x, x + d) \notin T_i$ for $i < j$, we have $x + d \in B_j$. If $x \notin A_j$, then $x \in B_j$. So for $x \geq N_1$, we have $x + d \in B_j$ or $x \in B_j$.

Since the claim is true for $d = 0$, we have $|A_j \cap B_j| \geq 1$. By Proposition 7, if $a \in A_j \cap B_j$, then a must appear in the same position in A_j as in B_j . Since $j > 0$, we must have $|A_j \cap B_j| = 1$.

Next, we will show that there is some $N \geq N_1$ such that for $x \geq N$, at most one of x and $x + d$ occurs in A_j . Recall that at least one of x and $x + d$ is in B_j . If both x and $x + d$ are in B_j , then A_j has at most one of them since $|A_j \cap B_j| = 1$. Say $A_j \cap B_j = \{p_0\}$. If B_j has only one of x and $x + d$, then choosing $N > p_0$ will ensure that at most one of x and $x + d$ is in A_j for $x \geq N$.

Choose N such that $N > p_0$ and $N > N_1$. Let $B'_j = \{b'_n{}^j\}_{n \geq 0}$ be the sequence of elements of B_j in increasing order. Fix $n \geq N$. Since A_j and B'_j are both increasing, $a_n^j \geq a_N^j \geq N$ and $b'_n{}^j \geq b'_N{}^j \geq N$ for $n \geq N$. For $k > 0$, let $U_k = \{a_n^j + \ell : 0 \leq \ell \leq 2kd - 1\}$ and $V_k = \{b'_n{}^j + \ell : 0 \leq \ell \leq 2kd - 1\}$. To use the fact that at least one of x and $x + d$ appears in B_j and that at most one of them appears in A_j , we pair off elements that are d away from each other in U_k and V_k . We may write $U_k = \cup_{t=0}^{k-1} \{a_n^j + i + 2td, a_n^j + i + 2td + d : 0 \leq i \leq d - 1\}$ and $V_k = \cup_{t=0}^{k-1} \{b'_n{}^j + i + 2td, b'_n{}^j + i + 2td + d : 0 \leq i \leq d - 1\}$. Since A_j has at most one of x and $x + d$ for $x \geq N$, we know that U_k has at most kd elements from A_j . Similarly, V_k has at least kd elements from B_j . If $a_{n+kd}^j \in U_k$, then $a_n^j, \dots, a_{n+kd}^j \in U_k$, contradicting that $|U_k \cap A_j| \leq kd$. So $a_{n+kd}^j \notin U_k$, which means $a_{n+kd}^j \geq a_n^j + 2kd$. If $b'_{n+kd-1}{}^j \notin V_k$, then $b'_\ell{}^j \notin V_k$ for $\ell \geq n + kd - 1$, contradicting that $|V_k \cap B_j| \geq kd$. So $b'_{n+kd-1}{}^j \leq b'_n{}^j + 2kd - 1$.

Next, we will bound $b'_i{}^j - a_i^j$ independent of i . Since at least one of $b'_{n+kd-1}{}^j + 1$ and $b'_{n+kd-1}{}^j + d + 1$ is in B'_j and $b'_{n+kd}{}^j$ is the smallest element in B'_j larger than $b'_{n+kd-1}{}^j$, we know that $b'_{n+kd}{}^j \leq b'_{n+kd-1}{}^j + d + 1$. Then

$$\begin{aligned} b'_{n+kd}{}^j - a_{n+kd}^j &\leq b'_{n+kd-1}{}^j + d + 1 - (a_n^j + 2kd) \\ &\leq (b'_n{}^j + 2kd - 1) + d + 1 - (a_n^j + 2kd) = b'_n{}^j - a_n^j + d. \end{aligned}$$

Substitute $n = N + \ell$, where $1 \leq \ell \leq d$. Then $b'_{N+\ell+kd}{}^j - a_{N+\ell+kd}^j \leq d + \max_{1 \leq i \leq d} (b'_{N+i}{}^j - a_{N+i}^j)$ for all $k > 0$. Let $c = d + \max_{1 \leq i \leq d} (b'_{N+i}{}^j - a_{N+i}^j)$. Then $b'_i{}^j - a_i^j \leq c$ for $i > N + d$.

By Corollary 11, there is an M such that $b'_i{}^j - a_i^j > c$ for $i \geq M$. Let $M > N + d$. Then $b'_i{}^j - a_i^j \leq c$ for $i \geq M$. Hence $b'_i{}^j - a_i^j < b'_i{}^j - a_i^j$, so $b'_i{}^j > b'_i{}^j$ for $i \geq M$. There are exactly $M + 1$ elements $b'_i{}^j$ bounded above by $b'_M{}^j$. On the other hand, there are at most M values $b'_i{}^j$ bounded above by $b'_M{}^j$ since $b'_i{}^j > b'_i{}^j \geq b'_M{}^j$ for $i \geq M$, i.e. at most

$b_0^j, \dots, b_{M-1}^j$ are bounded above by b_M^j , contradicting that B_j' has the same elements as B_j . So the supposition that $j \notin \{\mathcal{G}(a, a+d)\}_{a \geq 0}$ is not possible. Hence every diagonal of Sprague-Grundy values contains all nonnegative integers. $\square$

4 Bounds

In this section, we provide bounds on a_n^j , b_n^j , and d_n^j . We wish eventually to show that for a given j , the positive difference between the n th j -position and the n th 0-position is bounded independent of n .

First we bound a_n^j .

Proposition 13. *For all $j, n \in \mathbb{N}_0$, $n \leq a_n^j \leq 2n$.*

Proof. Since $a_{n+1}^j \geq a_n^j + 1$ and $a_0^j = 0$, we conclude that $a_n^j \geq n$. By step 1 of the algorithm, a_n^j is the minimal excludant of the $2n$ integers $a_0^j, \dots, a_{n-1}^j, b_0^j, \dots, b_{n-1}^j$. So $a_n^j \leq 2n$. $\square$

Next, in seeking a lower bound for d_n^j , we examine the last position that the difference d occurs in the sequence D_j , and we denote this position by L_d^j . So L_d^j is the largest integer n such that $d_n^j = d$ in D_j .

Proposition 14. *For all $j \in \mathbb{N}_0$, $L_0^j \leq j + 1$.*

Proof. In D_0 , the last occurrence of 0 is d_1^0 since $A_0 \cap B_0 = \{0, 1\}$, and $(0, 0)$ and $(1, 1)$ happen in positions 0 and 1 of the sequence T_0 . So $L_0^0 = 1$. To show that $L_0^j \leq j + 1$, we suppose for contradiction that $L_0^j > j + 1$ for some $j > 0$. Recall that 0 can occur only once in D_j because if there exist $n < m$ with $d_n^j = 0$ and $d_m^j = 0$, then $(a_n^j, b_n^j) \in F(a_m^j, b_m^j)$. So the first occurrence of 0 is also the last occurrence of 0 in D_j , i.e. each of $d_0^j, \dots, d_{j+1}^j$ is not 0. Consider the candidate 0 for d_i^j with $i \in \{0, \dots, j + 1\}$. Then 0 passes condition 2a of the algorithm. Since $a_i^j = \text{mex}\{a_n^j, b_n^j : 0 \leq n < i\} \neq b_k^j$ for all $k \in \{0, \dots, i - 1\}$, the candidate 0 passes condition 2c of the algorithm. So for each d_i^j , the candidate 0 must violate condition 2b, i.e., for each $i \in \{0, \dots, j + 1\}$, we have $(a_i^j, a_i^j) \in T_{k_i}$ for some $k_i \in \{0, \dots, j - 1\}$. From the proof of Theorem 12, we have $|A_k \cap B_k| = 1$ for $k \neq 0$ and $|A_0 \cap B_0| = 2$. There are $j + 2$ such positions (a_i, a_i) that must fit into the $j + 1$ total spots in $T_0, \dots, T_{j-1}$, which is impossible by the pigeonhole principle. Thus $L_0^j \leq j + 1$. $\square$

Remark 15. From computer experiments, we suspect that $L_d^j \leq j + 1 + d$. If this is true, then it will give a lower bound of $d_n^j \geq n - (j + 1)$.

We now give an upper bound for d_n^j .

Proposition 16. *For all $j, n \in \mathbb{N}_0$, $d_n^j \leq (n + 1)j + n$.*

Proof. Let $d_0^j < \dots < d_{n-1}^j$ be the differences $d_0^j, \dots, d_{n-1}^j$ arranged in order. Let $d > d_{n-1}^j$ be a candidate being considered as q for $p = a_n^j$ in the algorithm. Then d passes condition 2a. If d fails condition 2c, then $a_n^j + d = d_i^j + a_i^j$ for some $0 \leq i < n$, so $a_n^j + d \leq d_{n-1}^j + a_i^j <$

$d_{n-1}^j + a_n^j$, i.e. $d < d_{n-1}^j$, which contradicts our choice of $d > d_{n-1}^j$. So d passes condition 2c. If d fails condition 2b, then $(a_n^j, a_n^j + d) \in T_i$ for some $0 \leq i < j$. There are at most j distinct numbers that can fail condition 2b since a_n^j only appears once in each A_i . As the candidates $d_{n-1}^j + 1, \dots, d_{n-1}^j + j$ are considered in order, the worst case is that all of them fail condition 2b. So there must be some number in $\{0, \dots, d_{n-1}^j + j + 1\}$ that passes all three conditions for q in the algorithm. Therefore $d_n^j \leq d_{n-1}^j + j + 1$. Since $d_0^j = j$, induction yields $d_n^j \leq (n+1)j + n$. $\square$

As a result of the bounds for a_n^j and d_n^j , we get the following bound for b_n^j .

Corollary 17. *For all $j, n \in \mathbb{N}_0$, $n \leq b_n^j \leq 3n + (n+1)j$.*

5 Conjectures

In standard Wythoff, for each $j \in \mathbb{N}_0$, we know that $\lim_{n \rightarrow \infty} \frac{b_n^j}{a_n^j} = \phi = \frac{1+\sqrt{5}}{2}$ by Theorem 2.3 of [5]. Based on computer experiments, we conjecture that this asymptotic behavior holds in $\mathcal{F}$ -Wythoff.

Conjecture 18. For $j \in \mathbb{N}_0$, $\frac{b_n^j}{a_n^j} \rightarrow \phi$ as $n \rightarrow \infty$.

The following theorem of Nivasch [5] reduces the above conjecture to the statement that the positive difference between d_n^j and n is bounded by some constant c_j independent of n .

Theorem 19 (Nivasch [5]). *Let $A := \{a_n : n \geq 0\} \subset \mathbb{N}_0$ be a strictly increasing sequence and $B := \{b_n : n \geq 0\} \subset \mathbb{N}_0$ be a sequence of distinct elements. Suppose A and B satisfy:*

1. $|A \cap B| < \infty$,
2. $A \cup B = \mathbb{N}_0$, and
3. *there exists c such that $|(b_n - a_n) - n| \leq c$ for all n .*

Then there exists M_1 and M_2 such that $|\phi n - a_n| \leq M_1$ and $|\phi^2 n - b_n| \leq M_2$ for all n .

We know that for all j , the sequences A_j and B_j satisfy 1 and 2 in the above theorem. Hence, if there is some c_j such that $|d_n^j - n| \leq c_j$ for all n , then Conjecture 18 holds.

The next conjecture proposes an answer to Question 9 of Ho in [3], which asks whether there is a $j > 2$ such that, with finitely many exceptions, the sequence T_j in $\mathcal{F}$ -Wythoff is a translation of the $\mathcal{P}$ -positions in standard Wythoff. Based on computer experiments, we suspect that B_j is not ultimately increasing for all $j > 2$, which leads to the following conjecture.

Conjecture 20. For $j > 2$, there is no pair (s, t) and finite set S such that $T_j = S \cup (\{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) : n \in \mathbb{N}_0\} + (s, t))$.

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