Cross-intersecting families of labeled sets

Huajun Zhang*

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China

huajunzhang@zjnu.cn

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Abstract

For two positive integers n and p, let $\mathcal{L}_p$ be the family of labeled n-sets given by

$$\mathcal{L}_p = \big\{\{(1,\ell_1),(2,\ell_2),\dots,(n,\ell_n)\} : \ell_i \in [p], i = 1,2\dots,n\big\}.$$

Families $\mathcal{A}$ and $\mathcal{B}$ are said to be cross-intersecting if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In this paper, we will prove that for $p \geqslant 4$, if $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting subfamilies of $\mathcal{L}_{\mathfrak{p}}$, then $|\mathcal{A}||\mathcal{B}| \leqslant p^{2n-2}$, and equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are an identical largest intersecting subfamily of $\mathcal{L}_p$.

Keywords: EKR theorem; Intersecting family; cross-intersecting family; labeled set

1 Introduction

For a positive integer n, let $[n]$ denote the set $\{1,2,\dots,n\}$. Given a set X, by $\binom{X}{k}$ we denote the set of all k-subsets of X, and let 2^X denote the set of all subsets of X. A family $\mathcal{A}$ of sets is said to be t-intersecting if $|A \cap B| \geqslant t$ for every pair $A, B \in \mathcal{A}$. Usually, $\mathcal{A}$ is called intersecting if $t = 1$.

The Erdős-Ko-Rado Theorem [15] says that if $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{k}$ where $n \geqslant 2k$, then $|\mathcal{A}| \leqslant \binom{n-1}{k-1}$. This theorem is a central result in extremal set theory and inspires abundant fruits in this field, for an excellent introduction to this we recommend the survey paper [13].

This theorem has many generalizations, analogs and variations. First, finite sets are analogous to finite vector spaces ([17, 18, 20]), permutations ([11, 12, 27]) and labeled sets (signed sets [4, 6] or colored sets [22]), etc. Second, the intersection condition was

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generalized to t-intersection and cross-intersection. Here, families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be cross-intersecting if $A \cap B \neq \emptyset$ for any $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \neq j$. Many authors studied the bound of $\sum_{i=1}^{m} |\mathcal{A}_i|$ ([19, 5, 6, 7, 8, 9, 10, 29, 30]), and Pyber [25] first considered the bound of $|\mathcal{A}||\mathcal{B}|$ for cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$. His result was slightly refined by Matsumoto and Tokushige [24] and Bey [3] as follows.

Theorem 1. *If $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{\ell}$ are cross-intersecting with $n \geqslant \max\{2k, 2\ell\}$, then*

$$|\mathcal{A}||\mathcal{B}| \leqslant \binom{n-1}{k-1}\binom{n-1}{\ell-1}.$$

Moreover, the equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : i \in A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{\ell} : i \in B\}$ for some $i \in [n]$, unless $n = 2k = 2\ell$.

Tokushige [26] and Ellis, Friedgut and Pilpel [14] generalized the above result to cross-t-intersecting families of finite sets and cross-t-intersecting subfamilies of the symmetric group S_n, respectively. This paper provides an analogue of Theorem 1 for families whose sets we refer to as labeled sets, following [5].

For an n-tuple $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ such that $p_1, p_2, \ldots, p_n$ are positive integers with $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$, we define the family $\mathcal{L}_{\mathfrak{p}}$ of labeled sets by

$$\mathcal{L}_{\mathfrak{p}} = \big\{\{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n)\} : \ell_i \in [p_i], i = 1, 2 \ldots, n\big\}.$$

Berge [2] determined the maximum size of intersecting families of labeled n-sets, Livingston [23] characterized partial optimal intersecting families and Borg [5] completely solved it by using the shift operator in an inductive argument.

Theorem 2 (Berge, Livingston, Borg)**.** *If $\mathcal{A}$ is an intersecting subfamily of $\mathcal{L}_{\mathfrak{p}}$, then $|\mathcal{A}| \leqslant p_2 p_3 \cdots p_n$. When $p_1 \geqslant 3$, equality holds if and only if $\mathcal{A} = \big\{\{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n)\} : \ell_i = j\big\}$, where $p_i = p_1$ and $j \in [p_1]$.*

In [5], Borg also determined the upper bound of $\sum_{1 \leqslant i \leqslant m} |\mathcal{A}_i|$ for cross-intersecting subfamilies $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ of $\mathcal{L}_{\mathfrak{p}}$.

In this paper, we consider a special case: $p_1 = p_2 = \cdots = p_n = p$. In this case, we write $\mathcal{L}_{\mathfrak{p}}$ as $\mathcal{L}_p$. The main result in this paper is the following theorem.

Theorem 3. *Let n and p be two positive integers with $p \geqslant 4$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting families in $\mathcal{L}_p$, then*

$$|\mathcal{A}||\mathcal{B}| \leqslant p^{2n-2},$$

and equality holds if and only if $\mathcal{A} = \mathcal{B} = \big\{\{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n)\} : \ell_i = j\big\}$ for some $i \in [n]$ and $j \in [p]$.

We will present some preliminary results in the next section, and complete the proof of the above theorem in Section 3.

2 Preliminary Results

For the labeled set $\mathcal{L}_p$, we can construct a simple graph, whose vertex set is $\mathcal{L}_p$, and $A, B \in \mathcal{L}_p$ are adjacent if and only if $A \cap B = \emptyset$. For convenience, this graph is also denoted by $\mathcal{L}_p$. Set $\Gamma = S_n \wr S_p = \{(f, g_1, g_2, \ldots, g_n) : f \in S_n \text{ and } g_1, g_2, \ldots, g_n \in S_p\}$, the wreath product of the symmetric groups on $[n]$ and $[p]$. For $\sigma = (f, g_1, g_2, \ldots, g_n) \in \Gamma$ and $\{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n)\} \in \mathcal{L}_p$, define

$$\sigma(\{(1, \ell_1), \ldots, (n, \ell_n)\}) = \{(f(1), g_1(\ell_1)), \ldots, (f(n), g_n(\ell_n))\}.$$

Then Γ acts transitively on $\mathcal{L}_p$. In other words, the graph $\mathcal{L}_p$ is vertex-transitive. Moreover, every intersecting subfamily of the labeled set $\mathcal{L}_p$ corresponds to an independent set of the graph $\mathcal{L}_p$. In the sequel we shall alternatively use the terms "set" and "graph" when referring to $\mathcal{L}_p$.

For a graph G, let $\alpha(G)$ denote the independence number of G. Given a subset A of $V(G)$, we define

$$N_G(A) = \{b \in V(G) : \{a, b\} \in E(G) \text{ for some } a \in A\}$$

$$\overline{N}_G(A) = V(G) - N_G(A).$$

If G is clear from the context, for simplicity, we will omit the index G. For $B \subseteq V(G)$, by $G[B]$ we denote the induced subgraph of G. For short, we abbreviate $\alpha(G[B])$ to $\alpha(B)$.

For the labeled set $\mathcal{L}_p$ we construct another graph $\widehat{\mathcal{L}}_p$, whose vertex set is the set $\{(A, B) \in \mathcal{L}_p \times \mathcal{L}_p : A \cap B \neq \emptyset\}$, and (A_1, B_1) and (A_2, B_2) are non-adjacent if and only if $A_1 \cap B_2 \neq \emptyset$ and $B_1 \cap A_2 \neq \emptyset$. By definition it is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting subfamilies of $\mathcal{L}_p$, then $\mathcal{A} \times \mathcal{B}$ is an independent set of $\widehat{\mathcal{L}}_p$. Therefore, $|\mathcal{A}||\mathcal{B}| \leqslant \alpha(\widehat{\mathcal{L}}_p)$. To complete the proof of Theorem 3, it suffices to determine the size and structure of the maximum independent sets in $\widehat{\mathcal{L}}_p$.

Note that the action of Γ on $\mathcal{L}_p$ induces an action on the graph $\widehat{\mathcal{L}}_p$ defined by $\sigma(A, B) = (\sigma(A), \sigma(B))$ for $\sigma \in \Gamma$ and $(A, B) \in \widehat{\mathcal{L}}_p$. For $1 \leqslant i \leqslant n$, set $\widehat{\mathcal{L}}_{p,i} = \{(A, B) \in \mathcal{L}_p \times \mathcal{L}_p : |A \cap B| = i\}$. Clearly, $|A \cap B| = |\sigma(A) \cap \sigma(B)|$ holds for all $\sigma \in \Gamma$ and $A, B \in \mathcal{L}_p$, and it is easy to verify that $\widehat{\mathcal{L}}_{p,1}, \widehat{\mathcal{L}}_{p,2}, \ldots, \widehat{\mathcal{L}}_{p,n}$ are all orbits of Γ on $\widehat{\mathcal{L}}_p$. In other words, every induced subgraph $\widehat{\mathcal{L}}_{p,i}$ is vertex-transitive.

In the context of vertex-transitive graphs, the following result named the "no-homomorphism lemma" is useful to get bounds on the size of independent sets.

Lemma 4 (Albertson and Collins [1])**.** *Let G and G' be two graphs such that G is vertex-transitive and there exists a homomorphism $\phi : G' \mapsto G$. Then $\frac{\alpha(G)}{|V(G)|} \leqslant \frac{\alpha(G')}{|V(G')|}$, and the equality holds if and only if for any independent set I of cardinality $\alpha(G)$ in G, $\phi^{-1}(I)$ is an independent set of cardinality $\alpha(G')$ in G'.*

The following Lemma is a variation of the above.

Lemma 5. *(see [11, Theorem 3]) Let G be a vertex-transitive graph, and Ω a transitive subgroup of $Aut(G)$. Let I be an independent set of G, and let $B \subseteq V(G)$, then $\frac{|I|}{|V(G)|} \leqslant \frac{\alpha(B)}{|B|}$. Equality holds if and only if $|I \cap \sigma(B)| = \alpha(B)$ holds for all $\sigma \in \Omega$.*

Proof. Set $\mathcal{D} = \{\sigma(B) : \sigma \in \Omega\}$ and $\mathcal{D}_u = \{D \in \mathcal{D} : u \in D\}$ for $u \in V(G)$. Note that the action of Ω on $V(G)$ is transitive. The size of $\mathcal{D}_u$, denoted by r, is independent of the choice of u. Hence, $r|V(G)| = |B||\mathcal{D}|$. On the other hand, for each $D \in \mathcal{D}$, $I \cap D$ is also an independent set of D, and so $|D \cap I| \leqslant \alpha(G[B])$. Therefore, $r|I| \leqslant \alpha(G[B])|\mathcal{D}|$. Combining the above two inequalities gives $\frac{|I|}{|V(G)|} \leqslant \frac{\alpha(G[B])}{|B|}$, and equality holds if and only if $|D \cap I| = \alpha(G[B])$ for each $D \in \mathcal{D}$. □

Since all $\widehat{\mathcal{L}}_{p,i}$ are vertex-transitive, the above lemma can be applied to them. In more detail, let $\widehat{\mathcal{K}}$ be a subset of $\widehat{\mathcal{L}}_p$ such that $\widehat{\mathcal{K}} \cap \widehat{\mathcal{L}}_{p,i} \neq \emptyset$ for $1 \leqslant i \leqslant n$. Write $\widehat{\mathcal{K}}_i = \widehat{\mathcal{K}} \cap \widehat{\mathcal{L}}_{p,i}$ for $i \in [n]$. Then, for any independent set $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{L}}_p$ and $i \in [n]$, $|\widehat{\mathcal{I}} \cap \widehat{\mathcal{L}}_{p,i}| \leqslant \alpha(\widehat{\mathcal{L}}_{p,i})$, and by Lemma 5, $\alpha(\widehat{\mathcal{L}}_{p,i}) \leqslant |\widehat{\mathcal{L}}_{p,i}|\frac{\alpha(\widehat{\mathcal{K}}_i)}{|\widehat{\mathcal{K}}_i|}$. Therefore,

$$|\widehat{\mathcal{I}}| = \sum_{i=1}^{n} |\widehat{\mathcal{I}} \cap \widehat{\mathcal{L}}_{p,i}| \leqslant \sum_{i=1}^{k} |\widehat{\mathcal{L}}_{p,i}| \frac{\alpha(\widehat{\mathcal{K}}_i)}{|\widehat{\mathcal{K}}_i|},$$

and equality holds if and only if $|\widehat{\mathcal{I}} \cap \widehat{\mathcal{L}}_{p,i}| = \alpha(\widehat{\mathcal{L}}_{p,i})$ and $|\widehat{\mathcal{I}} \cap \widehat{\mathcal{L}}_{p,i} \cap \sigma(\widehat{\mathcal{K}})| = \alpha(\widehat{\mathcal{K}}_i)$ for all $i = 1, 2, \ldots, n$ and $\sigma \in \Gamma$. Equivalently, for each $\sigma \in \Gamma$,

$$|\widehat{\mathcal{I}} \cap \sigma(\widehat{\mathcal{K}})| = \sum_{i=1}^{n} |\sigma^{-1}(\widehat{\mathcal{I}}) \cap \widehat{\mathcal{K}} \cap \widehat{\mathcal{L}}_{p,i}| = \sum_{i=1}^{n} \alpha(\widehat{\mathcal{K}}_i) = \alpha(\widehat{\mathcal{K}}).$$

We state it as a lemma as follows.

Lemma 6. *Let $\widehat{\mathcal{K}}$ be a subset of $\widehat{\mathcal{L}}_p$ such that $\widehat{\mathcal{K}}_i \neq \emptyset$ for $1 \leqslant i \leqslant n$, where $\widehat{\mathcal{K}}_i = \widehat{\mathcal{K}} \cap \widehat{\mathcal{L}}_{p,i}$. If $\widehat{\mathcal{I}}$ is an independent set of $\widehat{\mathcal{L}}_p$, then*

$$|\widehat{\mathcal{I}}| \leqslant \sum_{i=1}^{n} |\widehat{\mathcal{L}}_{p,i}| \frac{\alpha(\widehat{\mathcal{K}}_i)}{|\widehat{\mathcal{K}}_i|},$$

and equality holds if and only if $|\widehat{\mathcal{I}} \cap \sigma(\widehat{\mathcal{K}})| = \sum_{i=1}^{n} \alpha(\widehat{\mathcal{K}}_i) = \alpha(\widehat{\mathcal{K}})$ for each $\sigma \in \Gamma$.

Arrange the elements

$$(1,1), (2,1), \ldots, (n,1), (1,2), (2,2), \ldots, (n,2), \ldots, (1,p), (2,p), \ldots, (n,p)$$

in a cycle. Let R_i denote the ith n-interval $\{(s,j), (s+1,j) \ldots, (n,j), (1,j+1), \ldots, (s-1, j+1)\}$ of this cycle, where $i = n(j-1)+s$ with $1 \leqslant s \leqslant n$. Set $\mathcal{R} = \{R_1, R_2, \ldots, R_{np}\}$ and $\widehat{\mathcal{R}} = \{(A,B) \in \mathcal{R} \times \mathcal{R} : A \cap B \neq \emptyset\}$. Then, $\widehat{\mathcal{R}} \subseteq \widehat{\mathcal{L}}_p$ and $\widehat{\mathcal{R}}_i = \widehat{\mathcal{R}} \cap \widehat{\mathcal{L}}_{p,i} \neq \emptyset$ for each $1 \leqslant i \leqslant n$.

Clearly, $R_i \cap R_j \neq \emptyset$ if and only if $|i-j| < n$ or $|i+np-j| < n$ for $R_i, R_j \in \mathcal{R}$, and the subgraph of $\mathcal{L}_p$ induced by $\mathcal{R}$, which will also be denoted by $\mathcal{R}$, is isomorphic to the well-known circular graph $\mathrm{Circ}(n, np)$. Here, the graph $\mathrm{Circ}(n, np)$ has the vertex set $[np]$, and i and j are not adjacent if and only if $|i-j| < n$ or $|np+i-j| < n$. Hence, $\alpha(\mathcal{R}) = n$, and by the well-known result of Katona [21], the maximum independent sets of $\mathcal{R}$ are stars. In the following we will prove that $\widehat{\mathcal{R}}$ is the desired subset.

Let $\mathcal{A}$ and $\mathcal{B}$ be cross-intersecting subfamilies of $\mathcal{R}$. Then, it is obvious that $\mathcal{B} \subseteq \overline{N}_{\mathcal{R}}(\mathcal{A})$. For every non-empty $A \subset V(\mathrm{Circ}(n, np))(p \geqslant 3)$, we have proved that if $|A| \geqslant 2n$, $\overline{N}(A) = \emptyset$; if $|A| < 2n$, $|\overline{N}(A)| + |A| \leqslant 2n$, and equality holds if and only if $A = \{i, i+1, \ldots, i+|A|-1\}$ for some i (see [16, Lemma 3.1] or [28, Lemma 2.3]). Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are both non-empty, then $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}| + |\overline{N}_{\mathcal{R}}(\mathcal{A})| \leqslant 2n$. Note that $|\mathcal{A}||\mathcal{B}| = 0$ if one of $\mathcal{A}$ and $\mathcal{B}$ is empty. So we have that $|\mathcal{A}||\mathcal{B}| \leqslant |\mathcal{A}|(2n - |\mathcal{A}|) \leqslant n^2$, and equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are some identical maximum independent set of $\mathcal{R}$. Therefore, $\alpha(\widehat{\mathcal{R}}) = n^2$. In the following, we give a stronger result.

Lemma 7. *Suppose $p \geqslant 4$. Then*

$$\alpha(\widehat{\mathcal{R}}) = n^2 = \sum_{i=1}^{n} \alpha(\widehat{\mathcal{R}}_i),$$

and $\widehat{\mathcal{I}}$ is a maximum independent set of $\widehat{\mathcal{R}}$ if and only if $\widehat{\mathcal{I}} = \mathcal{S} \times \mathcal{S}$ for some maximum independent set of $\mathcal{R}$.

Proof. For any subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{R}$ and $1 \leqslant i \leqslant n$, set $(\mathcal{A}, \mathcal{B})_i = |(\mathcal{A} \times \mathcal{B}) \cap \widehat{\mathcal{R}}_i|$. Let $\mathcal{S}$ be a fixed maximum independent set of $\mathcal{R}$ and write $(\mathcal{S}, \mathcal{S})_i = a_i$. Clearly, a_i does not depend on the choice of $\mathcal{S}$, and $\alpha(\widehat{\mathcal{R}}) = n^2 = \sum_{1 \leqslant i \leqslant n} a_i$. To complete the proof, it suffices to prove that $\alpha(\widehat{\mathcal{R}}_i) = a_i$ for each $1 \leqslant i \leqslant n$. To do this, we only need to verify that for every independent set $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{R}}$, $|\widehat{\mathcal{I}} \cap \widehat{\mathcal{R}}_i| \leqslant a_i$ for $1 \leqslant i \leqslant n$.

Let $\widehat{\mathcal{I}}$ be an independent set of $\widehat{\mathcal{R}}$. Then there exists a pair of cross-intersecting subfamilies $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{R}$ such that $\widehat{\mathcal{I}} \subseteq \mathcal{C} \times \mathcal{D}$. Since $|\mathcal{C}| + |\mathcal{D}| \leqslant 2n$, we may assume $|\mathcal{C}| = s \leqslant n$.

We first consider the simple case when $\mathcal{C}$ consists of consecutive elements of $\mathcal{R}$. Without loss of generality, assume $\mathcal{C} = \{R_n, R_{n+1}, \ldots, R_{n+s-1}\}$. For $1 \leqslant t \leqslant n$, set $\mathcal{C}_t = \{R_n, R_{n+1}, \ldots, R_{n+t-1}\}$. Then, $\mathcal{D} \subseteq \overline{N}(\mathcal{C}_s)$. For each $1 \leqslant t < n$ and $1 \leqslant i \leqslant n$, it is easy to verify that

$$\begin{aligned} \mathcal{C}_{t+1} \times \overline{N}(\mathcal{C}_{t+1}) &= [\mathcal{C}_t \times \overline{N}(\mathcal{C}_{t+1})] \cup [\{R_{n+t}\} \times \overline{N}(\mathcal{C}_{t+1})] \\ &= [\mathcal{C}_t \times \overline{N}(\mathcal{C}_t)] \cup [\{R_{n+t}\} \times \overline{N}(\mathcal{C}_{t+1})] - [\mathcal{C}_t \times \{R_t\}] \end{aligned}$$

and $(\{R_{n+t}\}, \overline{N}(\mathcal{C}_{t+1}))_i \geqslant (\mathcal{C}_t, \{R_t\})_i$, and consequently we have

$$(\mathcal{C}_{t+1}, \overline{N}(\mathcal{C}_{t+1}))_i = (\mathcal{C}_t, \overline{N}(\mathcal{C}_t))_i + (\{R_{n+t}\}, \overline{N}(\mathcal{C}_{t+1}))_i - (\mathcal{C}_t, \{R_t\})_i \geqslant (\mathcal{C}_t, \overline{N}(\mathcal{C}_t))_i.$$

Therefore, for $1 \leqslant i \leqslant n$,

$$(\mathcal{C}, \mathcal{D})_i \leqslant (\mathcal{C}_s, \overline{N}(\mathcal{C}_s))_i \leqslant (\mathcal{C}_{s+1}, \overline{N}(\mathcal{C}_{s+1}))_i \leqslant \cdots \leqslant (\mathcal{C}_n, \overline{N}(\mathcal{C}_n))_i = a_i$$

because $\mathcal{C}_n = \overline{N}(\mathcal{C}_n)$ is a maximum independent set of $\mathcal{R}$.

Now we consider the general case. Without loss of generality, assume $R_{2n} \in \mathcal{D}$. Then $\mathcal{C} \subseteq \overline{N}(\mathcal{D}) \subseteq \overline{N}(\{R_{2n}\}) = \{R_{n+1}, R_{n+2}, \ldots, R_{3n-1}\}$. Suppose $\mathcal{C} = \{R_{i_1}, R_{i_2}, \ldots, R_{i_s}\}$, where $n+1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant 3n-1$. Noting $p \geqslant 4$, if $R_j \in \overline{N}(\{R_{i_1}\}) \cap \overline{N}(\{R_{i_s}\})$, then it follows from definition that $|j - i_1| < n$ and $|j - i_s| < n$, that is, $i_s - n + 1 \leqslant j \leqslant i_1 + n - 1$. Therefore, $\overline{N}(\{R_{i_1}\}) \cap \overline{N}(\{R_{i_s}\}) = \{R_{i_s-n+1}, R_{i_s-n+2}, \ldots, R_{i_1+n-1}\} = \overline{N}(\mathcal{C})$. Set $\mathcal{C}' = \{R_{i_1}, R_{i_1+1}, \ldots, R_{i_s}\}$ and $\mathcal{D}' = \{R_{i_s-n+1}, R_{i_s-n+2}, \ldots, R_{i_1+n-1}\}$. Then, $\mathcal{C}' = \overline{N}(\mathcal{D}')$, and the above argument implies that the inequality $(\mathcal{C}', \mathcal{D}')_i \leqslant a_i$ holds for each $1 \leqslant i \leqslant n$. Note that $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{D} \subseteq \mathcal{D}'$. Hence, $(\mathcal{C}, \mathcal{D})_i \leqslant (\mathcal{C}', \mathcal{D}')_i \leqslant a_i$. □

Remark. In the above result, the condition that $p \geqslant 4$ is necessary. For example, assume $n = 6$ and $p = 3$, set $\mathcal{S} = \{R_1, R_2, R_3, R_4, R_5, R_6\}$, $\mathcal{C} = \{R_6, R_{14}\}$ and $\mathcal{D} = \{R_1, R_{11}\}$, it is easy to see that $\mathcal{S}$ is a maximum independent set of $\mathcal{R}$ and $\mathcal{C} \times \mathcal{D}$ is an independent set of $\widehat{\mathcal{R}}$, but $2 = (\mathcal{S}, \mathcal{S})_1 < (\mathcal{C}, \mathcal{D})_1 = 3 \leqslant \alpha(\widehat{\mathcal{R}}_1)$, and so $\sum_{i=1}^{6} \alpha(\widehat{\mathcal{R}}_i) > \sum_{i=1}^{6} (\mathcal{S}, \mathcal{S})_i = \alpha(\widehat{\mathcal{R}})$.

3 Proof of Theorem 3

In this section we complete the proof of Theorem 3.

Proof of Theorem 3. Take a maximum independent set $\mathcal{S}'$ of $\mathcal{L}_p$ and set $\widehat{\mathcal{I}}' = \mathcal{S}' \times \mathcal{S}'$. Then $\widehat{\mathcal{I}}'$ is an independent set of $\widehat{\mathcal{L}}_p$ with $|\widehat{\mathcal{I}}'| = p^{2n-2}$. Note that $\frac{\alpha(\mathcal{R})}{|\mathcal{R}|} = \frac{\alpha(\mathcal{L}_p)}{|\mathcal{L}_p|} = \frac{|\mathcal{S}'|}{|\mathcal{L}_p|} = \frac{1}{p}$. For each $\sigma \in \Gamma$, Lemma 5 implies $|\mathcal{S}' \cap \sigma(\mathcal{R})| = \alpha(\mathcal{R}) = n$, that is to say, $|\widehat{\mathcal{I}}' \cap \sigma(\widehat{\mathcal{R}})| = n^2$, and so the equalities $|\widehat{\mathcal{I}}' \cap \sigma(\widehat{\mathcal{R}})| = \alpha(\widehat{\mathcal{R}}) = \sum_{i=1}^{n} \alpha(\widehat{\mathcal{R}}_i)$ hold by Lemma 7. Then, it follows from Lemma 6 that $p^{2n-2} = |\widehat{\mathcal{I}}'| = \sum_{i=1}^{n} |\widehat{\mathcal{L}}_{p,i}| \frac{\alpha(\widehat{\mathcal{R}}_i)}{|\widehat{\mathcal{R}}_i|}$. Therefore, for every independent set $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{L}}_p$, we have $|\widehat{\mathcal{I}}| \leqslant \sum_{i=1}^{n} |\widehat{\mathcal{L}}_{p,i}| \frac{\alpha(\widehat{\mathcal{R}}_i)}{|\widehat{\mathcal{R}}_i|} = p^{2n-2}$. Furthermore, the equality holds if and only if $|\widehat{\mathcal{I}} \cap \sigma(\widehat{\mathcal{R}})| = \alpha(\widehat{\mathcal{R}})$ for all $\sigma \in \Gamma$. Then, for each $\sigma \in \Gamma$, by Lemma 7, $\widehat{\mathcal{I}} \cap \sigma(\widehat{\mathcal{R}}) = \mathcal{S}_\sigma \times \mathcal{S}_\sigma$ for some maximum independent set $\mathcal{S}_\sigma$ of $\sigma(\mathcal{R})$. Set $\mathcal{S} = \cup_{\sigma \in \Gamma} \mathcal{S}_\sigma$. Noting that the maximality of $\widehat{\mathcal{I}}$ implies that $\widehat{\mathcal{I}} = \mathcal{C} \times \mathcal{D}$ for a pair of cross-intersecting subfamilies $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{L}_p$. Then we have that $\mathcal{S}$ is an independent set and $\mathcal{S} \times \mathcal{S} \subseteq \widehat{\mathcal{I}}$. On the other hand, it is easy to see that $|\mathcal{S} \cap \sigma(\mathcal{R})| = \alpha(\mathcal{R})$ holds for all $\sigma \in \Gamma$, so Lemma 5 implies $\mathcal{S}$ is a maximum independent set of $\mathcal{L}_p$. Then we obtain $\widehat{\mathcal{I}} = \mathcal{S} \times \mathcal{S}$ since $|\widehat{\mathcal{I}}| = p^{2n-2} = |\mathcal{S} \times \mathcal{S}|$. This completes the proof of Theorem 3. □

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