Nonconvexity of the set of hypergraph degree sequences

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Abstract

It is well known that the set of possible degree sequences for a simple graph on n vertices is the intersection of a lattice and a convex polytope. We show that the set of possible degree sequences for a simple k-uniform hypergraph on n vertices is not the intersection of a lattice and a convex polytope for $k \geqslant 3$ and $n \geqslant k+13$. We also show an analogous nonconvexity result for the set of degree sequences of k-partite k-uniform hypergraphs and the generalized notion of λ-balanced k-uniform hypergraphs.

1 Introduction

The *degree sequence* of a graph G on vertices $v_1, v_2, \ldots, v_n$ is the sequence $d(G) = (d_1, d_2, \ldots, d_n)$, where d_i is the degree of the vertex v_i in G. The Erdős-Gallai Theorem [2] states that a sequence $(d_1, d_2, \ldots, d_n)$ is the degree sequence of a (simple) graph if and only if $\sum_i d_i$ is even and the d_i satisfy a certain set of inequalities. Koren [4] showed that these inequalities define a convex polytope $D_n(2)$, so that the sequences with even sum lying in this polytope are exactly the degree sequences of graphs on n vertices. (For more on this polytope, see [6].)

We consider the analogous question for (simple) k-uniform hypergraphs when $k > 2$. (We assume throughout that all graphs and hypergraphs are simple.) Klivans and Reiner [3] verified computationally that the set of degree sequences for k-uniform hypergraphs is the intersection of a lattice and a convex polytope for $k = 3$ and $n \leqslant 8$ and asked whether this holds in general. We will show in Section 2 that it does not hold for $k \geqslant 3$ and $n \geqslant k + 13$.

Similarly, we can associate to a bipartite graph a pair of degree sequences giving the degrees of the vertices in each part. The Gale-Ryser Theorem [5] gives necessary and sufficient conditions in the form of a system of linear inequalities for a pair of degree sequences to arise from a bipartite graph, so that the set of these pairs of degree sequences can again be described as the intersection of a lattice and a convex polytope. We will show in Section 3 that the analogous result does not hold for k-partite k-uniform hypergraphs if there exist three parts of sizes at least 5, 6, and 6, respectively. We also generalize the notion of k-partite k-uniform hypergraphs to that of λ-balanced k-uniform hypergraphs and prove a similar statement in this case.

2 Hypergraph degree sequences

A *(simple) k-uniform hypergraph* K on the set $[n] = \{1, 2, \ldots, n\}$ is a collection of distinct elements (called *hyperedges*) of $\binom{[n]}{k}$, the k-element subsets of $[n]$. The *degree sequence* of K is $d(K) = (d_1, d_2, \ldots, d_n)$, where d_i is the number of hyperedges in K containing i.

We consider degree sequences as points in $\mathbf{R}^n$. Let e_i be the ith standard basis vector, and for any $S = \{i_1, \ldots, i_k\} \subset [n]$, write $e_S = e_{i_1 i_2 \cdots i_k} = e_{i_1} + e_{i_2} + \cdots + e_{i_k}$. Each degree sequence $d(K)$ is the sum of some subset of the e_S's, so the convex hull of all such degree sequences is the zonotope

$$D = D_n(k) = \Big\{ \sum_{S \in \binom{[n]}{k}} c_S e_S \mid 0 \leqslant c_S \leqslant 1 \Big\}.$$

(For more on this polytope, see [1].) Moreover, if we let $L \subset \mathbf{Z}^n$ be the lattice generated by the e_S consisting of lattice points whose coordinates have sum divisible by k (as long as $n > k$), then each $d(K)$ lies in $D \cap L$. Our main result will be to show that $D \cap L$ contains a point that is not the degree sequence of a k-uniform hypergraph when $k \geqslant 3$ and $n \geqslant k + 13$.

As a remark, this is closely related to the weaker question of whether every point of L lying in the real cone generated by the e_S lies in the semigroup generated by the e_S. This is well known to be the case and is equivalent to normality of the monomial algebra generated by the $\mathbf{x}^S = x_{i_1} x_{i_2} \cdots x_{i_k}$. (See, for instance, [7].) It is also easy to derive the affirmative answer to this question for λ-balanced hypergraphs as defined in the next section. The essential difference with the present question is that here we are restricted to using each hyperedge at most once.

For a hypergraph K, we will define $D(K)$ to be the zonotope generated by the hyperedges in K, so

$$D(K) = \left\{ \sum_{S \in K} c_S e_S \mid 0 \leqslant c_S \leqslant 1 \right\}.$$

Lemma 2.1. *Let K be a k-uniform hypergraph on n vertices. For any $w \in (\mathbf{R}^*)^n$, let $F = F(w)$ be the face of $D(K)$ on which w is maximized. Then F is a translate of $D(K^0)$, where $K^0 = K^0(w) \subset K$ is the set of hyperedges $S \in K$ such that $w(e_S) = 0$. Moreover, $F \cap L$ contains a point that is not the degree sequence of a subhypergraph of K if and only if $D(K^0) \cap L$ contains a point that is not the degree sequence of a subhypergraph of K^0.*

Proof. To maximize $w\left(\sum c_S e_S\right) = \sum (c_S \cdot w(e_S))$ for $0 \leqslant c_S \leqslant 1$, we must take $c_S = 1$ when $w(e_S) > 0$ and $c_S = 0$ when $w(e_S) < 0$, while c_S can be arbitrary if $w(e_S) = 0$. Thus F is a translate of $D(K^0)$ by $\sum_{S \in K^+} e_S \in L$, where K^+ is the set of hyperedges S on which w is positive.

The same argument with the added condition that c_S equals either 0 or 1 for all S shows that the degree sequences in F are exactly the translations of degree sequences of subhypergraphs of K^0. □

Therefore, to find a point of $D \cap L$ that is not the degree sequence of a k-uniform hypergraph, it suffices, by Lemma 2.1, to take $K = \binom{[n]}{k}$ and to exhibit a weight vector w and a point of $D(K^0) \cap L$ that is not the degree sequence of a subhypergraph of K^0.

Proposition 2.2. *Let $k = 3$ and $n = 16$. If*

$$w = (8, 6, 6, 4, 1, 1, 0, 0, 0, 0, -2, -2, -3, -3, -5, -12),$$

then

$$p = (2, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 1)$$

lies in $D(K^0) \cap L$ but is not the degree sequence of a subhypergraph of K^0.

Proof. Since the sum of the entries of p is $21 = 3 \cdot 7$, p lies in L. Also,

$$p = \frac{1}{3}(e_{2,3,16} + e_{4,5,15} + e_{4,6,15} + e_{5,6,11} + e_{5,6,12} + e_{7,8,9} + e_{7,8,10} + e_{7,9,10} + e_{8,9,10}) \\ + \frac{2}{3}(e_{1,4,16} + e_{1,13,15} + e_{1,14,15} + e_{2,13,14} + e_{3,13,14} + e_{4,11,12}).$$

Since w vanishes on each e_S on the right side, it follows that $p \in D(K^0)$. However, p is not the degree sequence of a subhypergraph of K^0: since $w_7 = w_8 = w_9 = w_{10} = 0$ but otherwise $w_i \neq 0$ and $w_i \neq -w_j$ for distinct $i, j \notin \{7, 8, 9, 10\}$, we have $(e_7 + e_8 + e_9 + e_{10}) \cdot e_S$ is 0 or 3 for any $S \in K^0$ (where $\cdot$ indicates the usual inner product on $\mathbf{R}^n$). But $(e_7 + e_8 + e_9 + e_{10}) \cdot p = 4$, which is not divisible by 3, so p cannot be the sum of some e_S for $S \in K^0$. □

Using this, we can easily derive the following.

Theorem 2.3. *For $k \geqslant 3$ and $n \geqslant k + 13$, the set of degree sequences of k-uniform hypergraphs on n vertices is not the intersection of a lattice and a convex polytope.*

Proof. It suffices to show that there is a point in $D \cap L$ that is not a degree sequence (since D and L are the smallest convex polytope and lattice containing all degree sequences). Combining Lemma 2.1 and Proposition 2.2 gives the result for $k = 3$ and $n = 16$. Since $D_n(k)$ is the face of $D_{n+1}(k)$ with last coordinate 0, Lemma 2.1 also gives the result for $k = 3$ and $n \geqslant 16$.

Consider the map $f\colon (d_1, d_2, \ldots, d_n) \mapsto (d_1, d_2, \ldots, d_n, \frac{1}{k}(d_1 + \cdots + d_n))$. If d is a k-uniform hypergraph degree sequence on n vertices, then $f(d)$ is a $(k+1)$-uniform hypergraph degree sequence on $n+1$ vertices (simply add vertex $n+1$ to all hyperedges). Conversely, if the degree sequence of a $(k+1)$-uniform hypergraph satisfies $d_{n+1} = \frac{1}{k}(d_1 + \cdots + d_n)$, then every hyperedge must contain the vertex $n+1$, so it is of the form $f(d)$ for some k-uniform hypergraph degree sequence d. Since f is linear, it also sends $D_n(k)$ into $D_{n+1}(k+1)$, so any counterexample for (n, k) yields a counterexample for $(n+1, k+1)$. An easy induction completes the proof. $\square$

It is possible that with additional work or computation the constant 13 may be improved.

In the next section, we will prove an analogous result for k-partite k-uniform hypergraphs as well as the more general λ-balanced hypergraphs. (Our construction below can also be used to prove Theorem 2.3 but with a constant of 14 instead of 13.)

3 λ-balanced hypergraphs

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of k. We say a k-uniform hypergraph is λ-*balanced* if its vertex set can be partitioned into p sets $V_1, \ldots, V_p$ such that each hyperedge contains λ_i vertices from V_i. (We will also call a hyperedge λ-balanced if it satisfies this property.) A $(1, 1, \ldots, 1)$-balanced k-uniform hypergraph is called k-*partite*. Note that every k-uniform hypergraph is (k)-balanced.

Let $n_i = |V_i|$, and label the vertices in V_i by $v_1^i, v_2^i, \ldots, v_{n_i}^i$. We then associate to a λ-balanced hypergraph K a degree sequence

$$d = (d_1^1, d_2^1, d_3^1, \ldots; \quad d_1^2, d_2^2, \ldots; \quad \ldots; \quad d_1^p, d_2^p, \ldots),$$

where d_j^i gives the number of hyperedges in K containing vertex v_j^i. As before, this degree sequence is $\sum_{S \in K} e_S$, where e_S is the sum of the standard basis vectors in $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \cdots \times \mathbf{R}^{n_p}$ corresponding to vertices in the hyperedge S. When $n_i > \lambda_i$ for all i, the lattice L generated by all possible e_S consists of all sequences d for which there exists $q \in \mathbf{Z}$ such that $\sum_{j=1}^{n_i} d_j^i = \lambda_i q$ for all i. (In other words, the sum of the degrees of the vertices in V_i must be the same integer multiple of λ_i.)

As before, we let D be the zonotope generated by all e_S for λ-balanced hyperedges S and ask whether all points in $D \cap L$ are degree sequences for λ-balanced hypergraphs. We will again find that this is not the case for any λ when $k \geqslant 3$ and the n_i are sufficiently large. We first consider a special case.

Proposition 3.1. *Let $\lambda = (1, 1, 1)$ and $(n_1, n_2, n_3) = (5, 6, 6)$. Also let*

$$w = (-7, -7, -7, -7, -7; \quad 1, 1, 2, 2, 3, 3; \quad 6, 6, 5, 5, 4, 4),$$

and define K^0 as in Lemma 2.1. Then

$$p = (11, 9, 6, 3, 1; \quad 2, 4, 6, 8, 3, 7; \quad 2, 4, 6, 8, 3, 7)$$

lies in $D(K^0) \cap L$ but is not the degree sequence of a subhypergraph of K^0.

Proof. Define points

$$\begin{aligned} p^- &= (10,8,4,2,0;\quad 1,3,5,7,2,6;\quad 1,3,5,7,2,6),\\ p^+ &= (12,10,8,4,2;\quad 3,5,7,9,4,8;\quad 3,5,7,9,4,8), \end{aligned}$$

so $p = \frac{1}{2}(p^- + p^+)$. Note that the sum of the coordinates of the three parts of p^- are all 24, so $p^- \in L$. Likewise, p^+ and p also lie in L.

Let

$$A = (a_{rs}) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0\\ 2 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 4 & 0 & 0\\ 0 & 0 & 4 & 5 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 3 & 5 \end{pmatrix}.$$

Note that $K^0 = \{\{v_q^1, v_r^2, v_s^3\} \mid 1 \leqslant q \leqslant 5, 1 \leqslant r, s \leqslant 6, a_{rs} \neq 0\}$.

Then $p^- = \sum e_S$, where the sum ranges over all $S = \{v_q^1, v_r^2, v_s^3\}$ such that $q < a_{rs}$. Likewise $p^+ = \sum e_S$, where the sum now ranges over all $S = \{v_q^1, v_r^2, v_s^3\}$ with $q \leqslant a_{rs}$. Therefore p^-, p^+, and their midpoint p lie in $D(K^0)$.

We will now show that p is not the degree sequence of a hypergraph that uses only hyperedges in K^0. Suppose it were, so that we could write $p = \sum_{S \in K} e_S$ for some $K \subset K^0$. Let $B = (b_{rs})$ be the 6×6 matrix such that b_{rs} counts the number of q for which $\{v_q^1, v_r^2, v_s^3\} \in K$. Then the sequence of row and column sums of B must both be $(2,4,6,8,3,7)$. Since we also know that $0 \leqslant b_{rs} \leqslant 5$, this means that:

$$\begin{aligned} B_1 &= \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 & 2\\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1\\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0\\ 0 & 4 \end{pmatrix} \right\},\\ B_2 &= \begin{pmatrix} b_{33} & b_{34}\\ b_{43} & b_{44} \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 5\\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4\\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 3\\ 3 & 5 \end{pmatrix} \right\},\\ B_3 &= \begin{pmatrix} b_{55} & b_{56}\\ b_{65} & b_{66} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 & 3\\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2\\ 2 & 5 \end{pmatrix} \right\}. \end{aligned}$$

Moreover, for $1 \leqslant r, s \leqslant 6$, the pair $\{v_r^2, v_s^3\}$ can appear in at most $\min\{q, b_{rs}\}$ hyperedges with one of the vertices in $\{v_1^1, \ldots, v_q^1\}$. Therefore, if we let $\mu = (11,9,6,3,1)$, then $\mu_1 + \cdots + \mu_q \leqslant \sum_{r,s} \min\{q, b_{rs}\}$. In other words, if $\nu = (\nu_1, \ldots, \nu_5)$ is the partition such that ν_q counts the number of b_{rs} that are at least q, then $\mu_1 + \cdots + \mu_q \leqslant \nu_1 + \cdots + \nu_q$.

It is now straightforward to show that there are no possible choices of B_1, B_2, and B_3 satisfying these conditions: if $B_3 = \left(\begin{smallmatrix} 0 & 3\\ 3 & 4 \end{smallmatrix}\right)$, we cannot choose B_1 such that both $\mu_1 \leqslant \nu_1$ and $\mu_1 + \mu_2 \leqslant \nu_1 + \nu_2$. Similarly if $B_3 = \left(\begin{smallmatrix} 1 & 2\\ 2 & 5 \end{smallmatrix}\right)$, we cannot choose B_2 such that both $\mu_1 + \mu_2 + \mu_3 \leqslant \nu_1 + \nu_2 + \nu_3$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 \leqslant \nu_1 + \nu_2 + \nu_3 + \nu_4$. Thus $p \in D(K^0) \cap L$ is not the degree sequence of a hypergraph using only hyperedges in K^0. □

Combining Lemma 2.1 and Proposition 3.1 gives our desired result for 3-partite 3-uniform hypergraphs, and we can easily extend this result to k-partite k-uniform hypergraphs.

Theorem 3.2. *For $k \geqslant 3$, consider k-partite k-uniform hypergraphs with parts of sizes $n_1, n_2, \ldots, n_k$ for which $n_1 \geqslant 5$, $n_2 \geqslant 6$, $n_3 \geqslant 6$, and $n_i \geqslant 1$ otherwise. The corresponding set of degree sequences is not the intersection of a lattice and a convex polytope.*

Proof. As in Theorem 2.3, combining Lemma 2.1 and Proposition 3.1 gives the result for $k = 3$ and $(n_1, n_2, n_3) = (5, 6, 6)$. Also note that the polytopes and lattices for $k \geqslant 3$ with $(n_1, n_2, n_3, n_4, \ldots, n_k) = (5, 6, 6, 1, \ldots, 1)$ are all identical to the $k = 3$ case (by projecting away the last $k - 3$ coordinates) so this also proves those cases. Finally, increasing any n_i but restricting to the face of the zonotope where the new vertices have degree 0 again reduces to the same case by Lemma 2.1, completing the proof. □

Theorem 3.2 is also easy to extend to λ-balanced hypergraphs for all λ when $k \geqslant 3$. Consider a λ-balanced hypergraph on vertex sets $V_1, \ldots, V_p$ of sizes $n_1, n_2, \ldots, n_p$. We will say that $(n_1, \ldots, n_p)$ is a λ-*coarsening* of $(m_1, \ldots, m_k)$ if each V_i can be partitioned into λ_i sets such that the sizes of all the resulting sets are $m_1, \ldots, m_k$.

Theorem 3.3. *Consider λ-balanced hypergraphs with parts of sizes $n_1, \ldots, n_p$, where $(n_1, \ldots, n_p)$ is a λ-coarsening of $(m_1, m_2, \ldots, m_k)$ such that Theorem 3.2 holds for parts of sizes $m_1, \ldots, m_k$. (In particular, this will hold whenever the n_i are sufficiently large.) Then the corresponding set of degree sequences is not the intersection of a lattice and a convex polytope.*

Proof. Let the vertex sets $V_1, \ldots, V_p$ have corresponding coarsening $W_1, \ldots, W_k$. It suffices to exhibit a weight vector w such that the corresponding K^0 as in Lemma 2.1 is the complete k-partite k-uniform hypergraph on $W_1, \ldots, W_k$. Indeed, any hyperedge in K^0 will be λ-balanced by the definition of λ-coarsening, and the lattice generated by hyperedges in K^0 is a sublattice of the lattice generated by all λ-balanced hyperedges. Therefore any counterexample for K^0 will yield a counterexample for λ-balanced hypergraphs as in Lemma 2.1.

To exhibit such a weight vector, let N be an integer larger than any m_i. Then let the weight of vertices in W_1 be $-(1 + N + N^2 + \cdots + N^{k-2})$ and in W_i be N^{i-2} for $2 \leqslant i \leqslant k$. Then the only way to pick k vertices the sum of whose weights is 0 is to take one from each W_i. In other words, the only hyperedges in K^0 are those that have one vertex from each W_i, as desired. □

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