New approach to the k-independence number of a graph

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Abstract

Let $G = (V, E)$ be a graph and $k \geqslant 0$ an integer. A k-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by S is at most k. With $\alpha_k(G)$ we denote the maximum cardinality of a k-independent set of G. We prove that, for a graph G on n vertices and average degree d, $\alpha_k(G) \geqslant \frac{k+1}{\lceil d \rceil + k + 1} n$, improving the hitherto best general lower bound due to Caro and Tuza [Improved lower bounds on k-independence, J. Graph Theory 15 (1991), 99–107].

Keywords: k-independence; average degree

1 Introduction

Let $G = (V, E)$ be a graph on n vertices and $k \geqslant 0$ an integer. A *k-independent set* $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by S is at most k. With $\alpha_k(G)$ we denote the maximum cardinality of a k-independent set of G and it is called the *k-independence number* of G. In particular, $\alpha_0(G) = \alpha(G)$ is the usual independence number of G. The Caro-Wei bound $\alpha(G) \geqslant \sum_{v \in V} \frac{1}{\deg(v)+1}$ [11, 41] is an improvement of the well-known Turán bound for the independent number $\alpha(G) \geqslant \frac{n}{d(G)+1}$ [40], where $d(G)$ is the average degree of G. Various results concerning possible improvements and generalizations of the Caro-Wei bound are known (see [1, 2, 3, 6, 10, 22, 23, 25, 26, 33, 35, 37, 38]). A well known generalization to the k-independence number of r-uniform

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hypergraphs was obtained by Caro and Tuza [12] improving earlier results of Favaron [19] and was extended to non-uniform hypergraphs by Thiele [39]. See also the recent papers [15, 17] for updates. An extension of the notion of residue of a graph, notably developed by Fajtlowicz in [18] and Favaron et al. in [20], to the notion of k-residue has been developed by Jelen [29]. There has been also much interest in using the Caro-Tuza theorem to algorithmic aspects (see [24, 31, 36]). Yet all these lower bounds give asymptotically $\alpha_k(G) \geqslant \frac{k+2}{2(d+1)}n$ for k fixed and $d = d(G)$. It is easy to see that in general we cannot hope to get better than $\frac{k+1}{d+1}n$, as can be seen from the graph $G = mK_{d+1}$, consisting of m disjoint copies of the complete graph K_{d+1} and where $d \geqslant k$ and $n = m(d+1)$. So there is still an asymptotic multiplicative gap of a factor of $2\frac{k+1}{k+2}$. It is worth to mention that there is no known modification of the charming probabilistic proof of the lower bound of Caro-Wei theorem to the situation of k-independence that gives a better bound than the Caro-Tuza lower bound. Here, for the sake of being self-contained and to use the same notation, we restate and give the short proof of the Caro-Tuza theorem for graphs. Then we show how to improve this result using further ideas and, in particular, we close the multiplicative gap proving, as a corollary of our main result, that $\alpha_k(G) \geqslant \frac{k+1}{\lceil d(G) \rceil + k+1}n$. Doing so, we solve a "folklore" conjecture stated explicitly in [6].

All along this paper, we will use the following notation and definitions. Let G be a graph. By $V(G)$ we denote the set of vertices of G and $n(G) = |V(G)|$ is the order of G. $E(G)$ stands for the set of edges of G and $e(G)$ denotes its cardinality. For a vertex $v \in V(G)$, $\deg(v) = \deg_G(v)$ is the degree of v in G. By $\Delta(G)$ we denote the maximum degree of G and by $d(G)$ the average degree $\frac{1}{n(G)} \sum_{v \in V(G)} \deg(v)$. For a subset $S \subseteq V(G)$, we write $G[S]$ for the graph induced by S in G and $\deg_S(v)$ stands for the degree $\deg_{G[S]}(v)$ of v in $G[S]$. Lastly, for a vertex $v \in V(G)$, $G - v$ represents the graph G without vertex v and all the edges incident to v and, for an integer $m \geqslant 1$, mG is the graph consisting of m disjoint copies of G.

The paper is divided into five sections. After this introduction section, we deal in Section 2 with a first naive approach to obtain a lower bound on $\alpha_k(G)$ by deleting iteratively vertices of maximum degree until certain point where an old theorem of Lovász [32] is applied. In Section 3, we proceed the same way, taking however a better control on the number of vertices that are deleted and we prove that, for a graph G on n vertices and average degree d, $\alpha_k(G) \geqslant \frac{k+1}{\lceil d \rceil + k+1}n$, improving the hitherto best general lower bound due to Caro and Tuza. For this purpose, we define a parameter $f(k,d)$ which approaches from below the best possible ratio $\frac{\alpha(G)}{n(G)}$ for graphs G with $d(G) \leqslant d$, we calculate the exact value of $f(1,d)$ and prove some lower bounds on $f(k,d)$. In Section 4, we develop some upper bounds on $f(k,d)$. Finally, we present in Section 5 some open problems for further research.

2 The naive approach: first improvement

Let $f_k : [0,\infty) \to \mathbb{R}$ be the function defined by

$$f_k(x) = \begin{cases} 1 - \frac{x}{2(k+1)}, & \text{if } 0 \leqslant x \leqslant k+1 \\ \frac{k+2}{2(x+1)}, & \text{if } x \geqslant k+1. \end{cases}$$

Observe the following properties of $f_k(x)$:

(P1) $f_k(x)$ is a convex function and is strictly monotone decreasing on $[0,\infty)$.

(P2) $f_k(i) - f_k(i+1) \geqslant f_k(j) - f_k(j+1)$, for $j \geqslant i \geqslant 0$.

(P3) $i f_k(i-1) = (i+1) f_k(i)$, for $i \geqslant k+1$.

(P4) $f_k(0) = 1$ and $f_k(k+1) = \frac{1}{2}$.

Theorem 1 (Caro-Tuza for Graphs, [12]). *Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$. Then $\alpha_k(G) \geqslant \sum_{i=1}^n f_k(d_i)$.*

Proof. For a subset $X \subseteq V(G)$, define $s(X) = \sum_{x\in X} f_k(\deg_X(x))$. Among all subsets of $X \subseteq V(G)$ such that $s(X)$ is maximum, choose B such that B has the smallest cardinality. In particular, $|B| \geqslant s(B) \geqslant s(V(G)) = \sum_{x\in V(G)} f_k(\deg(x))$. We will show that B is a k-independent set of G. Suppose there is a vertex $y \in B$ such that $\deg_B(y) = d \geqslant k+1$. Let y be the vertex of maximum degree in $G[B]$. We will show that $s(B \setminus \{y\}) \geqslant s(B)$, a contradiction to the minimality of $|B|$. For $x \in B \setminus \{y\}$, let $z(x) = 1$ if xy is an edge in G and 0 otherwise. Then

$$\begin{aligned} s(B\setminus\{y\}) &= \sum_{x\in B\setminus\{y\}} f_k(\deg_{B\setminus\{y\}}(x)) = \sum_{x\in B\setminus\{y\}} f_k(\deg_B(x) - z(x)) \\ &= \left(\sum_{x\in B} f_k(\deg_B(x) - z(x))\right) - f_k(d) \\ &= s(B) - f_k(d) + \sum_{x\in B}(f_k(\deg_B(x) - z(x)) - f_k(\deg_B(x))) \\ &= s(B) - f_k(d) + \sum_{x\in B} z(x)\,(f_k(\deg_B(x)-1) - f_k(\deg_B(x))) \\ &= s(B) - f_k(d) + \sum_{x\in B\cap N(y)} (f_k(\deg_B(x)-1) - f_k(\deg(x)))\,. \end{aligned}$$

With (P2) we obtain that the last term is at least $s(B) - f_k(d) + d(f_k(d-1) - f_k(d)) = s(B) - (d+1)f_k(d) + d f_k(d-1)$ and, since $d f_k(d-1) = (d+1) f_k(d)$ by (P3), this is equal to $s(B)$. It follows that $s(B \setminus \{y\}) \geqslant s(B)$, which is a contradiction to the choice of B. Hence, B is a k-independent set and thus

$$\alpha_k(G) \geqslant |B| \geqslant s(B) \geqslant s(V) = \sum_{x \in V(G)} f_k(\deg(x)).$$

□

Note that, for $k = 0$, Theorem 1 yields the Caro-Wei bound. By convexity, the above bound yields also the following corollary.

Corollary 2. *For a graph G on n vertices, $\alpha_k(G) \geqslant f_k(d(G))n$.*

Note that, for $k = 0$, Corollary 2 yields the Turán bound $\alpha(G) \geqslant \frac{1}{d(G)+1}n$. Also, if $d(G) \geqslant k+1$, we obtain from this corollary the following one.

Corollary 3. *Let G be a graph on n vertices. If $d(G) \geqslant k+1$, then $\alpha_k(G) \geqslant \frac{k+2}{2(d(G)+1)}n$.*

For a graph G, we will denote with $\chi_k(G)$ the *k-chromatic number* of G, i.e. the minimum number t such that there is a partition $V(G) = V_1 \cup V_2 \cup \ldots V_t$ of the vertex set $V(G)$ such that $\Delta(G[V_i]) \leqslant k$ for all $1 \leqslant i \leqslant t$. The following theorem is due to Lovász.

Theorem 4 (Lovász [32], 1966)**.** *Let G be a graph with maximum degree Δ. If $k_1, k_2, \ldots, k_t \geqslant 0$ are integers such that $\Delta + 1 = \sum_{i=1}^{t}(k_i + 1)$, then there is a partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_t$ of the vertex set of G such that $\Delta(G[V_i]) \leqslant k_i$ for $1 \leqslant i \leqslant t$.*

Several proofs and generalizations of Lovász's theorem are known. We refer the reader to [8, 9, 13, 14, 34]. An algorithmic analysis of Lovász theorem with running time $O(n^3)$ is given in [24]. An immediate and well known corollary of Lovász's theorem is Corollary 5, which is useful in the study of defective colorings also known as improper colorings (see [4, 16, 21, 27]).

Corollary 5. *If G is a graph of maximum degree Δ, then $\chi_k(G) \leqslant \lceil \frac{\Delta+1}{k+1} \rceil$.*

Since $\alpha_k(G) \geqslant \frac{n}{\chi_k(G)}$, the following bound proved in 1986 by Hopkins and Staton follows trivially from the above corollary.

Theorem 6 (Hopkins, Staton [28] 1986)**.** *Let G be a graph of order n and maximum degree Δ. Then*

$$\alpha_k(G) \geqslant \frac{n}{\lceil \frac{\Delta+1}{k+1} \rceil}.$$

The following theorem is a direct consequence of Theorem 6 which generalizes and improves several results concerning relations between $\alpha_p(G)$ and $\alpha_q(G)$ (see e.g. [5]).

Theorem 7. *Let G be a graph and $q \geqslant p \geqslant 0$ two integers. Then $\alpha_q(G) \leqslant \left\lceil \frac{q+1}{p+1} \right\rceil \alpha_p(G)$.*

Proof. Let S be a maximum q-independent set of G. Then $\Delta(G[S]) \leqslant q$ and, by Theorem 6,

$$\alpha_p(G) \geqslant \alpha_p(G[S]) \geqslant \frac{|S|}{\left\lceil \frac{\Delta(G[S])+1}{p+1} \right\rceil} \geqslant \frac{\alpha_q(G)}{\left\lceil \frac{q+1}{p+1} \right\rceil},$$

which implies the statement. □

Completing $\Delta + 1$ to the next multiple of $k + 1$, the following observation is straightforward from Theorem 6.

Observation 8. *Let G be a graph of order n and maximum degree Δ and let r be an integer such that $0 \leqslant r \leqslant k$ and $\Delta + 1 + r \equiv 0 \pmod{k+1}$. Then*

$$\alpha_k(G) \geqslant \frac{k+1}{\Delta + r + 1} n.$$

Proof. As clearly $\lceil \frac{\Delta+1}{k+1} \rceil = \frac{\Delta+r+1}{k+1}$, Theorem 6 implies then $\alpha_k(G) \geqslant \frac{k+1}{\Delta+r+1} n$. □

When the graph is d-regular, we can set $\Delta = d = d(G)$ in Observation 8 and we obtain the following one.

Observation 9. *Let G be a d-regular graph on n vertices and let r be an integer such that $0 \leqslant r \leqslant k$ and $d + 1 + r \equiv 0 \pmod{k+1}$. Then $\alpha_k(G) \geqslant \frac{k+1}{d+r+1} n$.*

So this observation shows that indeed, for d-regular graphs, we can close the multiplicative gap of $2\frac{k+1}{k+2}$ using Lovász's theorem. This serves as an inspiration to trying to close the multiplicative gap in general.

Note that, in practice, the Hopkins-Staton bound can be poor if the maximum degree is far from the average degree. So, our first naive strategy will be to delete a vertex with large degree and, if possible, use induction on the number of vertices. Otherwise, if $\Delta(G)$ is near to the average degree $d(G)$, we will apply Theorem 6. This is precisely what is done in the next result.

Theorem 10. *Let G be a graph on n vertices. Then $\alpha_k(G) > \frac{k+1}{d(G)+2k+2} n$.*

Proof. We will proceed by induction on n. If $n = 1$, the statement is trivial. If $n = 2$, G is either K_2 or $\overline{K_2}$. If $G = K_2$, then $d(G) = 1$ and $\frac{k+1}{d(G)+2k+2} n = \frac{2(k+1)}{3+2k} < 1 \leqslant \alpha_k(G)$ for any $k \geqslant 0$. If $G = \overline{K_2}$, then $d(G) = 0$ and thus $\frac{k+1}{d(G)+2k+2} n = 1 < 2 = \alpha_k(G)$ for all $k \geqslant 0$. Suppose now that $n \geqslant 3$ and that the statement holds for $n - 1$. Let G be a graph on n vertices and $v \in V(G)$ a vertex of maximum degree Δ. Define $G^* = G - v$. Since any k-independent set of G^* is also a k-independent set of G, $\alpha_k(G) \geqslant \alpha_k(G^*)$. We distinguish two cases.

Case 1. Suppose that $\Delta \leqslant \lceil d(G) \rceil + k$. Then, by Observation 8, we have

$$\alpha_k(G) \geqslant \frac{k+1}{\Delta + k + 1} n \geqslant \frac{k+1}{\lceil d(G) \rceil + 2k + 1} n > \frac{k+1}{d(G) + 2k + 2} n$$

and we are done.
Case 2. Suppose that $\Delta \geqslant \lceil d(G) \rceil + k + 1$*.* By induction and with $\Delta \geqslant \lceil d(G) \rceil + k + 1 \geqslant d(G) + k + 1$, we obtain

$$\begin{aligned}
\alpha_k(G) \;\geqslant\; \alpha_k(G^*) \;&>\; \frac{(k+1)(n-1)}{d(G^*) + 2k + 2} = \frac{(k+1)(n-1)}{\frac{2e(G^*)}{n-1} + 2k + 2} \\
&= \frac{(k+1)(n-1)}{\frac{2e(G) - 2\Delta}{n-1} + 2k + 2} = \frac{(k+1)(n-1)}{\frac{d(G)n - 2\Delta}{n-1} + 2k + 2} \\
&\geqslant \frac{(k+1)(n-1)}{\frac{d(G)n - 2(d(G)+k+1)}{n-1} + 2k + 2} = \frac{(k+1)n}{(d(G) + 2k + 2)\frac{(n-2)n}{(n-1)^2}} \\
&> \frac{k+1}{d(G) + 2k + 2} n
\end{aligned}$$

and the statement follows. □

Note that the bound in the previous theorem is better than the Caro-Tuza bound for $k = 1$ and $d \geqslant 8$ and for $k \geqslant 2$ and $d \geqslant 2k + 5$. Note also that Theorem 10 already closes the multiplicative factor of $2\frac{k+1}{k+2}$ for fixed k as $d(G)$ grows. However, to obtain an even better lower bound, we need to get more control on the number of vertices of large degrees that are deleted and to apply Observation 8 in its full accuracy. This will be done in the next section.

We close this section with the following algorithm for obtaining a k-independent set of cardinality at least $\frac{k+1}{d(G)+2k+2} n$ for any graph G on n vertices that yields us the proof of Theorem 10.

Algorithm 1

INPUT: a graph G on n vertices and m edges.

(1) Compute $\Delta(G)$ and $d(G)$. GO TO (2).

(2) If $\Delta(G) \leqslant \lceil d(G) \rceil + k$, perform a Lovász partition into k-independent sets, choose the largest class S and END. Otherwise choose a vertex v of maximum degree $\Delta(G)$, set $G := G - v$ and GO TO (1).

OUTPUT: S

The algorithm terminates as, at some step, $\Delta(G) \leqslant \lceil d(G) \rceil + k$ must hold (the latest when G is the empty graph). As already mentioned, the Lovász partition requires a running time of $O(n^3)$, while each other step takes at most $O(n)$ time and the number of iteration steps before performing Lovász partition is at most n. Hence, the algorithm runs in at most $O(n^3)$ time.

3 Deletions, partitions and a better lower bound on $\alpha_k(G)$ - second improvement

Definition 11. Let $d, k \geqslant 0$ be two integers. We define

$$f(k,d) = \inf\left\{\frac{\alpha_k(G)}{n(G)} : G \text{ is a graph with } d(G) \leqslant d\right\}.$$

Observation 12. *Let $d, k \geqslant 0$ be two integers. For every graph G on n vertices and average degree $d(G) \leqslant d$, $\alpha_k(G) \geqslant f(k,d)n$.*

The next theorem shows that $f(k,d)$ is convex as a function of d.

Theorem 13. *Let $d, k, t \geqslant 0$ be integers and $t \leqslant d$. Then $2f(k,d) \leqslant f(k,d-t)+f(k,d+t)$.*

Proof. We will show that for any two graphs G_1 and G_2 such that $d(G_1) \leqslant d-t$ and $d(G_2) \leqslant d+t$, there is a graph G with $d(G) \leqslant d$ such that $2\frac{\alpha_k(G)}{n(G)} \leqslant \frac{\alpha_k(G_1)}{n(G_1)} + \frac{\alpha_k(G_2)}{n(G_2)}$. Let G_1 and G_2 be such graphs and let $n(G_i) = n_i$ and $V(G_i) = V_i$, $i = 1, 2$. Define the graph $G = n_2G_1 \cup n_1G_2$. Then

$$\begin{aligned} 2n_1n_2d(G) = n(G)d(G) &= n_2\sum_{x\in V_1}\deg_{G_1}(x) + n_1\sum_{y\in V_2}\deg_{G_2}(y) \\ &= n_2n_1d(H_1) + n_1n_2d(G_2) \\ &\leqslant n_2n_1(d-t) + n_1n_2(d+t) = 2n_1n_2d, \end{aligned}$$

implying that $d(G) \leqslant d$ and thus $f(k,d) \leqslant \frac{\alpha_k(G)}{n(G)}$. Moreover,

$$2f(k,d) \leqslant 2\frac{\alpha_k(G)}{n(G)} = \frac{n_2\alpha_k(G_1) + n_1\alpha_k(G_2)}{n_1n_2} = \frac{\alpha_k(G_1)}{n_1} + \frac{\alpha_k(G_2)}{n_1}.$$

As G_1 and G_2 were arbitrarily chosen, it follows that $2f(k,d) \leqslant f(k,d-t)+f(k,d+t)$. □

Before coming to the main theorems of this section, we need the following lemmas.

Lemma 14. *Let $d, t \geqslant 0$ be two integers and let G be a graph on n vertices with average degree $d(G) \leqslant d$. Then G has a subgraph H such that either $\Delta(H) \leqslant d+t-1$ and $n(H) \geqslant n - \lfloor\frac{n}{d+2t+1}\rfloor$ or $d(H) \leqslant d-1$ and $n(H) = n - \lceil\frac{n}{d+2t+1}\rceil$.*

Proof. For an $r \geqslant 0$, let $\{v_1, v_2, \ldots, v_r\}$ be a set of vertices of maximum cardinality such that $\deg_{G_{i+1}}(v_i) \geqslant d+t$, where $G_{i+1} = G_i - v_i$ and $G_0 = G$. Suppose first that $r \leqslant \lfloor\frac{n}{d+2t+1}\rfloor$ and let $H = G_{r+1}$. Then H has at least $n - r \geqslant n - \lfloor\frac{n}{d+2t+1}\rfloor$ vertices and $\Delta(H) \leqslant d+t-1$. Now suppose that $r \geqslant \lceil\frac{n}{d+2t+1}\rceil$. Let now $H = G_{q+1}$, where $q = \lceil\frac{n}{d+2t+1}\rceil$. Then $n(H) = n - \lceil\frac{n}{d+2t+1}\rceil$. Further,

$$d(H) = \frac{2e(H)}{n(H)} \leqslant \frac{2(e(G)-(d+t)q)}{n-q} \leqslant \frac{dn - 2(d+t)q}{n-q} = \frac{d(n - \frac{2(d+t)}{d}q)}{n-q}.$$

Since, for any real numbers $a \geqslant 0$ and $b \geqslant 1$, the function $\frac{a-bx}{a-x}$ is monotonically decreasing in $[0, \infty)$, setting $a = n$ and $b = \frac{2(d+t)}{d}$, we obtain with $q = \lceil \frac{n}{d+2t+1} \rceil \geqslant \frac{n}{d+2t+1}$

$$\begin{aligned} d(H) &\leqslant \frac{d(n - \frac{2(d+t)}{d} q)}{n - q} \leqslant \frac{d(n - \frac{2(d+t)}{d} \frac{n}{d+2t+1})}{n - \frac{n}{d+2t+1}} \\ &= \frac{d(d+2t+1) - 2(d+t)}{d+2t} = \frac{d(d+2t) - (d+2t)}{d+2t} = d - 1. \end{aligned}$$

Hence, we have shown that G has a subgraph H with either $d(H) \leqslant d - 1$ and $n(H) = n - \lceil \frac{n}{d+2t+1} \rceil$ or $\Delta(H) \leqslant d + t - 1$ and $n(H) = n - \lfloor \frac{n}{d+2t+1} \rfloor$. □

The following corollary is straightforward from this lemma.

Corollary 15. *Let $d, t \geqslant 0$ be two integers. Let G be a graph on n vertices with average degree $d(G) \leqslant d$ and such that $d + 2t + 1$ divides n. Then G has a subgraph H on $n(H) \geqslant \frac{d+2t}{d+2t+1} n$ vertices such that either $d(H) \leqslant d - 1$ or $\Delta(H) \leqslant d + t - 1$.*

Lemma 16. *Let G be a graph on n vertices with average degree $d(G) \leqslant d$ and such that $d + 2s + 1$ does not divide n. Then there is a graph H such that $d + 2s + 1$ divides $m = n(H)$, $d(H) = d(G) \leqslant d$ and $\frac{\alpha_k(H)}{m} = \frac{\alpha_k(G)}{n}$.*

Proof. Let $H = (d + 2s + 1)G$. Then $m = n(H) = (d + 2s + 1)n$ is multiple of $d + 2s + 1$, $d(H) = d(G)$ and $\frac{\alpha_k(H)}{m} = \frac{(d+2s+1)\alpha_k(G)}{(d+2s+1)n} = \frac{\alpha_k(G)}{n}$. □

Let n be an even integer. We denote by J_n the graph consisting of a complete graph on n vertices minus a 1-factor. We are now ready to present the exact value of $f(1, d)$ and the consequences of this result.

Theorem 17. *Let $d \geqslant 0$ be an integer. Then the following statements hold.*

(1) $f(1, d) = \begin{cases} \frac{2}{d+2}, & \text{if } d \equiv 0 \pmod 2 \\ \frac{2(d+2)}{(d+1)(d+3)}, & \text{if } d \equiv 1 \pmod 2. \end{cases}$

(2) The equality $f(1, d) = \frac{\alpha_1(G)}{n(G)}$ is attained by the graph $G = J_{d+2}$, when d is even, and by $G = (d + 3)J_{d+1} \cup (d + 1)J_{d+3}$, when d is odd.

(3) $f(1, d) \geqslant \frac{2}{d+2}$.

(4) For every graph G on n vertices, $\alpha_1(G) \geqslant \frac{2n}{\lceil d(G) \rceil + 2}$.

Proof. (1) We will prove by induction on d that

$$f(1, d) \geqslant \begin{cases} \frac{2}{d+2}, & \text{if } d \equiv 0 \ (\bmod\ 2) \\ \frac{2(d+2)}{(d+1)(d+3)}, & \text{if } d \equiv 1 \ (\bmod\ 2). \end{cases}$$

For $d = 0$, clearly $f(1, 0) = 1 = \frac{2}{0+2}$, as the only possible graph G with $d(G) \leqslant 0$ is the empty graph. For $d = 1$, let G be a graph with $d(G) \leqslant 1$. Setting $s = 1$, we can

suppose by Lemma 16 that 4 divides $n(G) = n$. Hence, Corollary 15 implies that there is a subgraph H of G on $n(H) \geqslant \frac{3}{4}n$ vertices with $d(H) \leqslant 0$ or $\Delta(H) \leqslant 1$. In both cases we have clearly $\alpha_1(G) \geqslant \alpha_1(H) = n(H) \geqslant \frac{3}{4}n$ and hence $f(1,1) = \inf\{\frac{\alpha_1(G)}{n(G)} : G$ graph with $d(G) \leqslant 1\} \geqslant \frac{3}{4} = \frac{2(1+2)}{(1+1)(1+3)}$.
Assume we have proved the statement for $f(1, d-1)$. Now we will prove it for $f(1,d)$, where $d > 1$. Let G be a graph on n vertices such that $d(G) \leqslant d$. We distinguish two cases.
Case 1. Suppose that $d \equiv 0 \pmod 2$. Setting $s = 0$, we can suppose by Lemma 16 that $d+1$ divides n. By Corollary 15, there is a subgraph H of G on at least $\frac{d}{d+1}n$ vertices with either $d(H) \leqslant d-1$ or $\Delta(H) \leqslant d-1$. Hence, in both cases $d(H) \leqslant d-1$ and thus, by induction, we have

$$\alpha_1(G) \geqslant \alpha_1(H) \geqslant f(1, d-1)n(H) \geqslant \frac{2(d+1)d}{d(d+2)(d+1)}n = \frac{2}{d+2}n.$$

Hence, $f(1,d) = \inf\{\frac{\alpha_1(G)}{n(G)} : G$ graph with $d(G) \leqslant d\} \geqslant \frac{d}{d+2}$ and we are done.
Case 2. Suppose that $d \equiv 1 \pmod 2$. Set $s = 1$. By Lemma 16, we can suppose that $d+3$ divides n. By Corollary 15, there is a subgraph H of G on at least $\frac{d+2}{d+3}n$ vertices with either $d(H) \leqslant d-1$ or $\Delta(H) \leqslant d$. If $d(H) \leqslant d-1$, we can apply the induction hypothesis on H and we obtain

$$\alpha_1(G) \geqslant \alpha_1(H) \geqslant f(1, d-1)n(H) \geqslant \frac{2(d+2)}{(d+1)(d+3)}n$$

and we are done. Suppose finally that $\Delta(H) \leqslant d$. Then, by Theorem 6 and as d is odd, we have

$$\alpha_1(G) \geqslant \alpha_1(H) \geqslant \frac{n(H)}{\left\lceil \frac{\Delta(H)+1}{2} \right\rceil} \geqslant \frac{\frac{d+2}{d+3}n}{\lceil \frac{d+1}{2} \rceil} = \frac{2(d+2)}{(d+1)(d+3)}n.$$

Thus, in both cases, $f(1,d) = \inf\{\frac{\alpha_1(G)}{n(G)} : G$ graph with $d(G) \leqslant d\} \geqslant \frac{2(d+2)}{(d+1)(d+3)}$ Hence, by induction, the statement holds.
Let d be even. Clearly $\alpha_1(J_{d+2}) = 2$ and hence, $f(1,d) \leqslant \frac{\alpha_1(J_{d+2})}{d+2} = \frac{2}{d+2}$. For d odd, the graph $G = (d+3)J_{d+1} \cup (d+1)J_{d+3}$ has $\alpha_1(G) = (d+3)2 + (d+1)2 = 4(d+2)$. Hence $f(1,d) \leqslant \frac{\alpha_1(G)}{n(G)} = \frac{4(d+2)}{2(d+1)(d+3)}$. Together with the inequalities proven above, it follows

$$f(1,d) = \begin{cases} \frac{2}{d+2}, & \text{if } d \equiv 0 \ (\bmod 2) \\ \frac{2(d+2)}{(d+1)(d+3)}, & \text{if } d \equiv 1 \ (\bmod 2). \end{cases}$$

(2) This follows from the discussion in (1).
(3) It is easily seen that $\frac{2(d+2)}{(d+1)(d+3)} \geqslant \frac{2}{d+2}$. Hence we have always $f(1,d) \geqslant \frac{2}{d+2}$.
(4) From item (3), we obtain $\alpha_1(G) \geqslant f(1, \lceil d(G) \rceil)n \geqslant \frac{2}{\lceil d(G) \rceil + 2}$ n. □

We can now state and prove our main result generalizing the proof of Theorem 17 to arbitrary k and d.

Theorem 18. *Let $d, k \geqslant 0$ be two integers. Then the following statements hold.*

(1) $f(k,d) \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \geqslant \frac{k+1}{d+k+1}$, *where t is such that $d \equiv k+1-t \pmod{k+1}$ and $1 \leqslant t \leqslant k+1$.*

(2) For $k \geqslant d$, $f(k,d) \geqslant \frac{2k+2-d}{2k+2}$. For $k \geqslant d = 1$, the bound is realized by the graph $K_{1,k+1} \cup kK_1$ and thus $f(k,1) = \frac{2k+1}{2k+2}$.

(3) For any graph G on n vertices, $\alpha_k(G) \geqslant \frac{k+1}{\lceil d(G) \rceil + k+1} n$.

Proof. (1) We will proceed to prove the inequality $f(k,d) \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}$ by induction on d. If $d = 0$, then $d \equiv (k+1) - (k+1)$ and clearly $f(k,0) = 1 = \frac{(k+1)(0+2(k+1))}{(0+k+(k+1)+1)(0+(k+1))}$, as the only possible graph G with $d(G) \leqslant 0$ is the empty graph.

Assume $f(k,d-1) \geqslant \frac{(k+1)(d-1+2t')}{(d+k+t')(d-1+t')}$, where $d-1 \equiv k+1-t' \pmod{k+1}$, $1 \leqslant t' \leqslant k+1$, and $d \geqslant 1$. We will prove the statement for d. Herefor, we distinguish two cases.
Case 1. Suppose that $d \equiv 0 \pmod{k+1}$. Then $t = k+1$. Let G be a graph on n vertices such that $d(G) \leqslant d$. By Lemma 16, setting there $s = 0$, we can suppose that $d+1$ divides n. Then from Corollary 15 it follows that there is a subgraph H of G on at least $\frac{d}{d+1}n$ vertices such that $d(H) \leqslant d-1$ or $\Delta(H) \leqslant d-1$. In both cases we have $d(H) \leqslant d-1$. Then, as $d - 1 \equiv (k+1) - 1 \pmod{k+1}$, we obtain by induction

$$\begin{aligned} \alpha_k(G) &\geqslant \alpha_k(H) \geqslant \frac{(k+1)(d+1)}{(d+k+1)d} n(H) \geqslant \frac{k+1}{d+k+1} n \\ &= \frac{(k+1)(d+2t)}{(d+2t)(d+k+1)} n = \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} n. \end{aligned}$$

Thus, $f(k,d) = \inf\{\frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leqslant d\} \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}$ and we are done.
Case 2. Suppose that $d \equiv k+1-t \pmod{k+1}$ *for some t with* $1 \leqslant t \leqslant k$. Using Lemma 16 with $s = t$, we can suppose that $d+2t+1$ divides n. By Corollary 15, there is a subgraph H of G on $n(H) \geqslant \frac{d+2t}{d+2t+1}n$ vertices with either $d(H) \leqslant d-1$ or $\Delta(H) \leqslant d+t-1$. If $\Delta(H) \leqslant d+t-1$, then Theorem 6 yields

$$\begin{aligned} \alpha_k(G) \geqslant \alpha_k(H) \geqslant \frac{n(H)}{\left\lceil \frac{\Delta(H)+1}{k+1} \right\rceil} \geqslant \frac{\frac{d+2t}{d+2t+1}n}{\left\lceil \frac{d+t}{k+1} \right\rceil} &= \frac{(k+1)(d+2t)}{(d+2t+1)(d+t)} n \\ &\geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} n. \end{aligned}$$

Hence, $f(k,d) = \inf\{\frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leqslant d\} \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}$ and we are done. Suppose now that $d(H) \leqslant d-1$. Since $d - 1 \equiv (k+1) - (t+1)$, we obtain by induction

$$\begin{aligned} \alpha_k(G) \geqslant \alpha_k(H) &\geqslant \frac{(k+1)((d-1)+2(t+1))}{((d-1)+k+(t+1)+1)((d-1)+(t+1))} n(H) \\ &\geqslant \frac{(k+1)(d+2t+1)}{(d+k+t+1)(d+t)} \cdot \frac{d+2t}{d+2t+1} n \\ &= \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} n. \end{aligned}$$

Thus, again, $f(k,d) \geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+k+1)}$ and Case 2 is done.
Hence, by induction, the statement holds. Finally, the inequality $\frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \geqslant \frac{k+1}{d+k+1}$ follows easily.
(2) Let $k \geqslant d$ and let t be such that $d \equiv k+1-t \pmod{k+1}$ and $1 \leqslant t \leqslant k+1$. Then $d = k+1-t$. Hence, with (1),

$$\begin{aligned} f(d,k) &\geqslant \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \\ &= \frac{(k+1)(d+2(k+1-d))}{(d+k+(k+1-d)+1)(d+(k+1-d))} = \frac{2k+2-d}{2k+2}. \end{aligned}$$

Let $G = K_{1,k+1} \cup kK_1$. Then $\alpha_k(G) = 2k+1$, $n(G) = 2k+2$ and $d(G) = \frac{2k+2}{2k+2} = 1$. Hence, $\frac{2k+1}{2k+2} \leqslant f(k,1) \leqslant \frac{\alpha_k(G)}{n(G)} = \frac{2k+1}{2k+2}$, obtaining thus equality.
(3) If G is a graph on n vertices, then, using (1), we obtain

$$\frac{\alpha_k(G)}{n} \geqslant f(k, \lceil d(G) \rceil) \geqslant \frac{k+1}{\lceil d(G) \rceil + k + 1}. \qquad \square$$

The proofs of Lemma 14 and Theorem 18 yield us an algorithm for finding, for any graph G on n vertices, a k-independent set of cardinality at least $\frac{k+1}{\lceil d(G) \rceil + k+1} n$. It works the following way. It computes $d = d(G)$ and $\Delta(G)$ and finds the integer t such that $0 \leqslant t \leqslant k$ and $d \equiv k+1-t \pmod{k+1}$ (note that the case $t=0$ corresponds here to the case $t = k+1$ of Theorem 18). Then it checks if the graph satisfies the condition $\Delta(G) \leqslant d+t-1$. If so, then it performs a Lovász partition into k-independent sets, selects the largest set from it and gives this as output. If not, then a vertex of maximum degree is deleted and the condition on the maximum degree is checked again on the remaining graph. This deletion step is repeated up to $\lceil \frac{n}{d+2t+1} \rceil$ times, as, by Lemma 14, if the maximum degree is still larger than $d+t-1$, then we are left with a graph with smaller average degree, with which the algorithm starts over again, doing here the inductive step of Theorem 18.

Algorithm 2

INPUT: a graph G on n vertices and m edges.

(1) Compute $\Delta(G)$ and $d(G)$. Set $d = \lceil d(G) \rceil$ and determine t such that $0 \leqslant t \leqslant k$ and $d \equiv k+1-t \pmod{k+1}$. Set $r := 0$ and GO TO (2).

(2) If $\Delta(G) \leqslant d+t-1$, perform a Lovász partition into k-independent sets, choose the largest class S and END. Otherwise GO TO (3).

(3) Set $r := r+1$. If $r > \lceil \frac{n}{d+2t+1} \rceil$, set $n := n - \lceil \frac{n}{d+2t+1} \rceil$ and GO TO (1). Otherwise choose a vertex v of maximum degree $\Delta(G)$, set $G := G - v$, compute $\Delta(G)$ and GO TO (2).

OUTPUT: S

The algorithm terminates as, at some step, $\Delta(G) \leqslant \lceil d(G) \rceil + t - 1$ must hold (the latest when G is the empty graph). Again, the algorithm has a running time of at most $O(n^3)$.

4 Upper bounds on $f(k,d)$ and determination of $f(k,d)$ for further small values

Observe that after Theorems 17(1) and 18(2), we know the exact value of $f(k,d)$ in case $\min\{d,k\} \leqslant 1$. The first pair (k,d) for which an exact value of $f(k,d)$ is not known yet is $(2,2)$. In this section, we develop several upper bounds on $f(k,d)$ as a starting point to future research to obtain further exact values of $f(k,d)$. We will use the following theorem.

Theorem 19 (see [7], p.108)**.** *Let $r, g \geqslant 3$ be two integers. If m is an integer with $m \geqslant \frac{(r-1)^{(g-1)}-1}{r-2}$, then there exists an r-regular graph of girth at least g and order $2m$.*

Define the function $h(r) = \frac{(r-1)^{r+3}-1}{r-2}$. We will use the particular form of this theorem with $m \geqslant h(r)$, implying that there is an r-regular graph of girth at least $r+4$ and order $2m$.

In the proof of the following theorem, we use the following notation. $\overline{G}$ denotes the complementary graph of G. If $F \subseteq E(G)$, then $G - F$ represents the graph G without the edges contained in F. For a graph H on at most n vertices, $K_n - E(H)$ stands for the complete graph K_n without the edges of a subgraph H. Further, given two graphs G and H, $G \cup H$ is the graph consisting of one copy of H and one copy of G. Finally, the girth of a graph G is denoted by $g(G)$.

Theorem 20. *Let $d, k \geqslant 0$ be two integers. Then the following statements hold.*

(1) For $d \geqslant k$, $\frac{k+1}{d+k+1} \leqslant f(k,d) \leqslant \frac{k+1}{d+1}$.

(2) For $d > k$, $d \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$, $f(k,d) \leqslant \frac{k+1}{d+2}$.

(3) For $d > k$, $f(k,d) \leqslant \frac{k+2}{d+3}$.

(4) For $k \geqslant 3$, $d \geqslant 2h(k) - k - 1$ and $d + k + 1 \equiv 0 \pmod 2$, $f(k,d) \leqslant \frac{k+2}{d+k+1}$.

(5) For $2 \leqslant d \leqslant 4 + 6q$, where $q \geqslant 0$ is an integer, $\frac{3}{d+3} \leqslant f(2,d) \leqslant \frac{3}{d+1+\frac{1}{q+1}}$.

(6) For $k \geqslant 2$, $\frac{k}{k+1} \leqslant f(k,2) \leqslant \frac{k+1}{k+2+\frac{1}{k+1}}$.

(7) For $k \geqslant 3$, there is a constant $c > 0$ auch that $f(k,d) < \frac{k+2}{d+c(\frac{d}{2})^{\frac{1}{k+2}}+1}$.

Proof. (1) The lower bound follows from Theorem 18. The upper bound follows from $f(k,d) \leqslant \frac{\alpha_k(K_{d+1})}{d+1} = \frac{k+1}{d+1}$.
(2) Let G be the graph K_{d+2} minus a 1-factor (this is possible, as d is assumed even). Then $d(G) = d$. Let $T \subseteq V(G)$ be any subset of $k+2$ vertices. As $k+1 \equiv 0 \pmod 2$, not every vertex of T is covered by the edges of the 1-factor in $\overline{G}[T]$. Hence, at least one vertex from T is adjacent in G to all other vertices from T. Hence, no subset of $k+2$

vertices can be a k-independent set and thus $\alpha_k(G) \leqslant k+1$. This implies $f(k,d) \leqslant \frac{k+1}{d+2}$.
(3) Let $d > k$. Consider the graph $G = K_{d+3} - E(C_{d+3})$, where C_{d+3} is a cycle of length $d+3$ in K_{d+3}. Then $d(G) = d$. Let $T \subseteq V(G)$ a subset of $k+3$ vertices. Since $k+3 < d+3 = n(G)$, the graph $\overline{G}[T]$ contains no cycles. Hence there is at least one vertex in $v \in V(T)$ which is adjacent in $G[T]$ to all but at most one vertex and hence $\deg_{G[T]}(v) \geqslant k+1$. This implies that $\alpha_k(G) \leqslant k+2$ and thus $f(k,d) \leqslant \frac{k+2}{d+3}$.
(4) Let $k \geqslant 3$, $d \geqslant 2h(k) - k - 1$ and $d + k + 1 \equiv 0 \pmod 2$. By Theorem 19, there is a k-regular graph H with $g(H) \geqslant k+4$ and $n(H) = d+k+1 = n$. Consider now the graph $G = K_n - E(H)$. Then $d(G) = n-1-k = d$. Let $T \subseteq V(G)$ be a subset of $k+3$ vertices. Since $g(H) \geqslant k+4$, $\overline{G}[T]$ is a forest. Hence there is at least one vertex in $v \in V(T)$ which is adjacent in $G[T]$ to all but at most one vertex and hence $\deg_{G[T]}(v) \geqslant k+1$. Thus, $\alpha_k(G) \leqslant k+2$ and we obtain $f(k,d) \leqslant \frac{k+2}{d+k+1}$.
(5) Consider the graph $G = (K_{d+2} - E(K_3)) \cup q(K_{d+1} - E(K_3))$. Then $n(G) = (q+1)d + q + 2$ and $d(G)n(G) = (d-1)(d+1) + 3(d-1) + q((d-2)d + 3(d-2)) = (q+1)d^2 + (q+3)d - (4+6q)$. Since $d \leqslant 4 + 6q$, it follows that

$$d(G) = \frac{(q+1)d^2 + (q+3)d - (4+6q)}{(q+1)d + q + 2} \leqslant \frac{(q+1)d^2 + (q+3)d - d}{(q+1)d + q + 2} = d.$$

As clearly $\alpha_2(G) = 3(q+1)$, we obtain therefore, together with Theorem 18 (1),

$$\frac{3}{d+3} \leqslant f(2,d) \leqslant \frac{3(q+1)}{(q+1)d + q + 2} = \frac{3}{d + \frac{q+2}{q+1}} = \frac{3}{d + 1 + \frac{1}{q+1}}.$$

(6) Let $k \geqslant 2$ and consider the graph $G = (K_{k+3} - E(K_{k+1})) \cup kK_{1,k+1}$. Then $n(G) = k+3+k(k+2) = k^2+3k+3$ and $d(G)n(G) = 2(k+2)+(k+1)2+2k(k+1) = 2(k^2+3k+3) = 2n(G)$. Hence, $d(G) = 2$. Moreover, it is easy to see that $\alpha_k(G) = (k+1)^2$. Thus this implies that

$$f(k,2) \leqslant \frac{\alpha_k(G)}{n(G)} = \frac{(k+1)^2}{k^2+3k+3} = \frac{k+1}{k+2+\frac{1}{k+1}}.$$

Together with the bound from item(2) of Theorem 18, we obtain

$$\frac{k}{k+1} \leqslant f(k,2) \leqslant \frac{k+1}{k+2+\frac{1}{k+1}}.$$

(7) By Theorem 19 there is an r-regular graph H with $g(H) \geqslant k+4$ and $n = n(H) \geqslant \frac{2((r-1)^{k+3}-1)}{r-2}$. Take n even and let $G = K_n - E(H)$. Then $d = d(G) = n - 1 - r$. Let $T \subseteq V(G)$ be a subset of $k+3$ vertices. As $g(H) \geqslant k+4$, $\overline{G}[T]$ is a forest and thus there is a vertex in T which is adjacent in G to all other vertices from T with the exception of at most one. Hence, T cannot be a k-independent set and thus $\alpha_k(G) \leqslant k+2$. This implies that $f(k,d) \leqslant \frac{\alpha_k(G)}{n} \leqslant \frac{k+2}{d+r+1}$. As $d \sim 2r^{k+2}$, we have $r \sim (\frac{d}{2})^{\frac{1}{k+2}}$, and thus there is a constant $c > 0$ such that $r = c(\frac{d}{2})^{\frac{1}{k+2}}$, implying that $f(k,d) \leqslant \frac{k+2}{d+c(\frac{d}{2})^{\frac{1}{k+2}}+1}$. □

5 Open problems

We close this paper with the following open problems.

Problem 21. Is $f(k,d)$ in fact a minimum for every k and d? Namely, does

$$\inf\left\{\frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leqslant d\right\} = \min\left\{\frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leqslant d\right\}$$

hold?

In case the answer to this problem is positive, this may have several consequences in computing $f(k,d)$.

Problem 22. Is the bound $f(k,d) \geqslant \frac{2k+2-d}{2k+2}$ of Theorem 18 (2) sharp for $k \geqslant d \geqslant 2$?

Below are the best possible bounds on $f(2,d)$ we have for $d = 0, 1, \ldots, 10$.

d	lower bound*	upper bound	graph for upper bound	theorem used for upper bound
0	1	1	K_1	-
1	5/6	5/6	$K_{1,k+1} \cup kK_1$	18(2)
2	2/3	9/13	$(K_5 - E(K_3)) \cup 2K_{1,3}$	20 (6)
3	1/2	3/5	$K_5 - E(K_3)$	20 (5), $q = 0$
4	4/9	1/2	$K_6 - E(K_3)$	20 (5), $q = 0$
5	7/18	6/13	$(K_7 - E(K_3)) \cup (K_6 - E(K_3))$	20 (5), $q = 1$
6	1/3	2/5	$(K_8 - E(K_3)) \cup (K_7 - E(K_3))$	20 (5), $q = 1$
7	11/36	6/17	$(K_9 - E(K_3)) \cup (K_8 - E(K_3))$	20 (5), $q = 1$
8	5/18	6/19	$(K_{10} - E(K_3)) \cup (K_9 - E(K_3))$	20 (5), $q = 1$
9	1/4	2/7	$(K_{11} - E(K_3)) \cup (K_{10} - E(K_3))$	20 (5), $q = 1$
10	7/30	6/23	$(K_{12} - E(K_3)) \cup (K_{11} - E(K_3))$	20 (5), $q = 1$

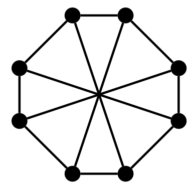
*Lower bounds are from Thm. 18(2) in case $d = 1$ and Thm. 18(1) else.

Problem 23. Improve upon the values given in the table.

In order to better understand $f(k,d)$, we can define

$$f(k,d,\Delta) = \inf\left\{\frac{\alpha_k(G)}{n(G)} : G \text{ is a graph with } d(G) \leqslant d \text{ and } \Delta(G) \leqslant \Delta\right\},$$

where $\Delta \geqslant d$, and k are all nonnegative integers. Observe that $f(k,d) = \inf\{f(k,d,\Delta) : \Delta \geqslant d\}$ and hence a knowledge on $f(k,d,\Delta)$ may help in obtaining better bounds on $f(k,d)$. For instance, let us take $f(2,2,3)$. Observe that, from Theorem 18 (2), $f(2,2,3) \geqslant f(2,2) \geqslant \frac{2}{3}$. Further, consider the graph $G = R_8 \cup 4K_{1,3}$ on 24 vertices, where R_8 is the graph depicted below (note that R_8 is the extremal graph for Reed's upper bound of $\frac{3}{8}n$ on the domination



The graph R_8.

number for graphs on n vertices with minimum degree at least 3), and observe that $\alpha_2(G) = 17$, $n(G) = 24$ and $\Delta(G) = 3$. Then, it follows that $\frac{2}{3} \leqslant f(2,2) \leqslant f(2,2,3) \leqslant \frac{17}{24}$.

But if we consider for instance the graph $H = (K_5 - E(K_3)) \cup 2K_{1,3}$, then we have there $\alpha_2(H) = 9$, $n(H) = 13$ and $\Delta(H) = 4$ and thus $\frac{2}{3} \leqslant f(2,2) \leqslant f(2,2,4) \leqslant \frac{9}{13}$, which is better than the bound $\frac{17}{24}$ obtained with the graph G. Thus, we would like to state the following question.

Problem 24. Obtain lower and upper bounds on $f(k,d,\Delta)$.

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