Arc-transitive pentavalent graphs of order $4pq$*

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Abstract

This paper determines all arc-transitive pentavalent graphs of order $4pq$, where $q > p \geqslant 5$ are primes. The cases $p = 1, 2, 3$ and $p = q$ is a prime have been treated previously by Hua et al. [Pentavalent symmetric graphs of order $2pq$, *Discrete Math.* **311** (2011), 2259-2267], Hua and Feng [Pentavalent symmetric graphs of order $8p$, *J. Beijing Jiaotong University* **35** (2011), 132-135], Guo et al. [Pentavalent symmetric graphs of order $12p$, *Electronic J. Combin.* **18** (2011), #P233] and Huang et al. [Pentavalent symmetric graphs of order four time a prime power, submitted for publication], respectively.

Keywords: arc-transitive graph; normal quotient; automorphism group.

1 Introduction

For a simple, connected and undirected graph Γ, denoted by $V\Gamma$ and $A\Gamma$ the vertex set and arc set of Γ, respectively. Let G be a subgroup of the full automorphism group $\mathsf{Aut}\Gamma$ of Γ. Then Γ is called *G-vertex-transitive* and *G-arc-transitive* if G is transitive on $V\Gamma$ and $A\Gamma$, respectively. An arc-transitive graph is also called *symmetric*. It is well known that Γ is G-arc-transitive if and only if G is transitive on $V\Gamma$ and the stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ for some $\alpha \in V\Gamma$ is transitive on the neighbor set $\Gamma(\alpha)$ of α in Γ.

The cubic and tetravalent graphs have been studied extensively in the literature. It would be a natural next step toward a characterization of pentavalent graphs. In recent years, a series of results regarding this topic have been obtained. For example, a classification of arc-transitive pentavalent abelian Cayley graphs is given in [1], a classification of

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1-regular pentavalent graph (that is, the full automorphism group acts regularly on its arc set) of square-free order is presented in [12], and all the possibilities of vertex stabilizers of pentavalent arc-transitive graphs are determined in [7, 18]. Also, for distinct primes p and q, classifications of arc-transitive pentavalent graphs of order $8p, 12p, 2pq, 2p^2$ and $4p^n$ are presented in [9, 8, 10, 14, 11], respectively. In the present paper, we shall classify arc-transitive pentavalent graphs of order $4pq$ with $q > p \geqslant 5$ primes. By using the *Fitting subgroup* (that is, the largest nilpotent normal subgroup) and the *soluble radical* (that is, the largest soluble normal subgroup), the method used in this paper is more simple than some relative papers.

We now give some necessary preliminary results. The first one is a property of the Fitting subgroup, see [17, P. 30, Corollary].

Lemma 1.1. *Let F be the Fitting subgroup of a group G. If G is soluble, then $F \neq 1$ and the centralizer $\mathbf{C}_G(F) \leqslant F$.*

The maximal subgroups of $\mathsf{PSL}(2, q)$ are known, see [4, Section 239].

Lemma 1.2. *Let $T = \mathsf{PSL}(2, q)$, where $q = p^n \geqslant 5$ with p a prime. Then a maximal subgroup of T is isomorphic to one of the following groups, where $d = (2, q - 1)$.*

(1) $\mathsf{D}_{2(q-1)/d}$, *where* $q \neq 5, 7, 9, 11$;

(2) $\mathsf{D}_{2(q+1)/d}$, *where* $q \neq 7, 9$;

(3) $\mathbb{Z}_q{:}\mathbb{Z}_{(q-1)/d}$;

(4) A_4, *where* $q = p = 5$ *or* $q = p \equiv 3, 13, 27, 37 \pmod{40}$;

(5) S_4, *where* $q = p \equiv \pm 1 \pmod 8$

(6) A_5, *where* $q = p \equiv \pm 1 \pmod 5$, *or* $q = p^2 \equiv -1 \pmod 5$ *with p an odd prime;*

(7) $\mathsf{PSL}(2, p^m)$ *with n/m an odd integer;*

(8) $\mathsf{PGL}(2, p^{n/2})$ *with n an even integer.*

For a graph Γ and a positive integer s, an *s-arc* of Γ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of vertices such that α_{i-1}, α_i are adjacent for $1 \leqslant i \leqslant s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant s - 1$. In particular, a 1-arc is just an arc. Then Γ is called (G, s)*-arc-transitive* with $G \leqslant \mathsf{Aut}\Gamma$ if G is transitive on the set of s-arcs of Γ. A (G, s)-arc-transitive graph is called (G, s)-*transitive* if it is not $(G, s + 1)$-arc-transitive. In particular, a graph Γ is simply called *s-transitive* if it is $(\mathsf{Aut}\Gamma, s)$-transitive.

The following lemma determines the stabilizers of arc-transitive pentavalent graphs, refer to [7, 18].

Lemma 1.3. *Let Γ be a pentavalent (G, s)-transitive graph, where $G \leqslant \mathsf{Aut}\Gamma$ and $s \geqslant 1$. Let $\alpha \in V\Gamma$. Then one of the following holds, where D_{10}, D_{20} and F_{20} denote the dihedral groups of order 10 and 20, and the Frobenius group of order 20, respectively.*

(a) *If G_α is soluble, then $s \leqslant 3$ and $|G_\alpha| \mid 80$. Further, the couple (s, G_α) lies in the following table.*

s	1	2	3
G_α	$\mathbb{Z}_5,\ \mathsf{D}_{10},\ \mathsf{D}_{20}$	$\mathsf{F}_{20},\ \mathsf{F}_{20}\times\mathbb{Z}_2$	$\mathsf{F}_{20}\times\mathbb{Z}_4$

(b) *If G_α is insoluble, then $2 \leqslant s \leqslant 5$, and $|G_\alpha| \mid 2^9 \cdot 3^2 \cdot 5$. Further, the couple (s, G_α) lies in the following table.*

s	2	3	4	5
G_α	$\mathsf{A}_5, \mathsf{S}_5$	$\mathsf{A}_4\times\mathsf{A}_5, (\mathsf{A}_4\times\mathsf{A}_5){:}\mathbb{Z}_2,$ $\mathsf{S}_4\times\mathsf{S}_5$	$\mathsf{ASL}(2,4), \mathsf{AGL}(2,4),$ $\mathsf{A\Sigma L}(2,4), \mathsf{A\Gamma L}(2,4)$	$\mathbb{Z}_2^6{:}\mathsf{\Gamma L}(2,4)$

The next result may easily follow from [10, Proposition 2.3] and its proof.

Lemma 1.4. *Let $q > p \geqslant 5$ be primes, and let T be a nonabelian simple group of order $2^i \cdot 3^j \cdot 5 \cdot p \cdot q$, where $1 \leqslant i \leqslant 11$ and $0 \leqslant j \leqslant 2$. Then T lies in the following Table* 1.

4−PD	Order	5−PD	Order
$\mathsf{PSL}(2,5^2)$	$2^3\cdot 3\cdot 5^2\cdot 13$	M_{22}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$
$\mathsf{PSU}(3,4)$	$2^6\cdot 3\cdot 5^2\cdot 13$	$\mathsf{PSL}(5,2)$	$2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 31$
$\mathsf{PSp}(4,4)$	$2^8\cdot 3^2\cdot 5^2\cdot 17$	$\mathsf{PSL}(2,2^6)$	$2^6\cdot 3^2\cdot 5\cdot 7\cdot 13$
		$\mathsf{PSL}(2,2^8)$	$2^8\cdot 3\cdot 5\cdot 17\cdot 257$
		$\mathsf{PSL}(2,q)$	*q an odd prime*

Table 1.

A typical method for studying vertex-transitive graphs is taking normal quotients. Let Γ be a G-vertex-transitive graph, where $G \leqslant \mathsf{Aut}\Gamma$. Suppose that G has a normal subgroup N which is intransitive on $V\Gamma$. Let $V\Gamma_N$ be the set of N-orbits on $V\Gamma$. The *normal quotient graph* Γ_N of Γ induced by N is defined as the graph with vertex set $V\Gamma_N$, and B is adjacent to C in Γ_N if and only if there exist vertices $\beta \in B$ and $\gamma \in C$ such that β is adjacent to γ in Γ. In particular, if $\mathsf{val}(\Gamma) = \mathsf{val}(\Gamma_N)$, then Γ is called a *normal cover* of Γ_N.

A graph Γ is called *G-locally primitive* if, for each $\alpha \in V\Gamma$, the stabilizer G_α acts primitively on $\Gamma(\alpha)$. Obviously, an arc-transitive pentavalent graph is locally primitive. The following theorem gives a basic method for studying vertex-transitive locally primitive graphs, see [15, Theorem 4.1] and [13, Lemma 2.5].

Theorem 1.5. *Let Γ be a G-vertex-transitive locally primitive graph, where $G \leqslant \mathsf{Aut}\Gamma$, and let $N \lhd G$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

(i) N is semi-regular on $V\Gamma$, $G/N \leqslant \mathsf{Aut}\Gamma_N$, and Γ is a normal cover of Γ_N;

(ii) $G_\alpha \cong (G/N)_\gamma$, where $\alpha \in V\Gamma$ and $\gamma \in V\Gamma_N$;

(iii) Γ is (G,s)-transitive if and only if Γ_N is $(G/N,s)$-transitive, where $1 \leqslant s \leqslant 5$ or $s = 7$.

For reduction, we need some information of arc-transitive pentavalent graphs of order $2pq$, stated in the following proposition, see [10, Theorem 4.2], where $\mathcal{C}_n$, following the notation in [10], denotes the corresponding graph of order n. Noting that a G-arc-transitive graph is bipartite if and only if G has a normal subgroup with index 2 which has exactly two orbits on the vertex set.

Proposition 1.6. *Let Γ be an arc-transitive pentavalent graph of order $2pq$, where $q > p \geqslant 5$ are primes. Then either $\mathsf{Aut}\Gamma$ is soluble, or the couple $(\mathsf{Aut}\Gamma, (\mathsf{Aut}\Gamma)_\alpha)$ lies in the following Table 2, where $\alpha \in V\Gamma$.*

Row	Γ	(p,q)	$\mathsf{Aut}\Gamma$	$(\mathsf{Aut}\Gamma)_\alpha$	Transitivity	Remark
1	$\mathcal{C}_{574}$	$(7,41)$	$\mathsf{PSL}(2,41)$	A_5	$2-$ transitive	not bipartite
2	$\mathcal{C}_{406}$	$(7,29)$	$\mathsf{PGL}(2,29)$	A_5	$2-$ transitive	bipartite
3	$\mathcal{C}_{3422}$	$(29,59)$	$\mathsf{PGL}(2,59)$	A_5	$2-$ transitive	bipartite
4	$\mathcal{C}_{3782}$	$(31,61)$	$\mathsf{PGL}(2,61)$	A_5	$2-$ transitive	bipartite
5	$\mathcal{C}_{170}$	$(5,17)$	$\mathsf{PSp}(4,4).\mathbb{Z}_4$	$\mathbb{Z}_2^6{:}\Gamma\mathsf{L}(2,4)$	$5-$ transitive	bipartite

Table 2.

2 Examples

In this section, we give two examples of arc-transitive pentavalent graphs of order $4pq$ with $p, q \geqslant 5$ distinct primes.

The standard double cover is a method to construct arc-transitive graphs from small arc-transitive graphs. Let Γ be a graph. Its *standard double cover*, denoted by $\Gamma^{(2)}$, is defined as a graph with vertex set $V\Gamma \times \{1,2\}$ (Cartesian product) such that vertices (α, i) and (β, j) are adjacent if and only if $i \neq j$ and α is adjacent to β in Γ. The following facts are well known: $\mathsf{val}(\Gamma) = \mathsf{val}(\Gamma^{(2)})$, $\Gamma^{(2)}$ is connected s-transitive if and only if Γ is connected s-transitive and is not a bipartite graph.

Thus, by Proposition 1.6, the standard double cover $\mathcal{C}_{574}^{(2)}$ is a connected 2-transitive pentavalent graph of order $1148 = 4 \cdot 7 \cdot 41$.

For the proof of Theorem 3.1 in Section 3, we need a necessary and sufficient condition for a graph to be the standard double cover of its normal quotient graph, which can be easily derived from [8, Proposition 2.6].

Lemma 2.1. *Let Γ be a G-arc-transitive graph with $G \leqslant \mathsf{Aut}\Gamma$. Suppose that $N \lhd G$ acts semi-regularly on $V\Gamma$. Then Γ is the standard double cover of the normal quotient graph*

Γ_N if and only if $N \cong \mathbb{Z}_2$, and there is $H \lhd X$ such that $G = N \times H$ and H has exactly two orbits on $V\Gamma$.

Another useful tool for constructing and studying arc-transitive graphs is the coset graph. For a group G, a *core-free subgroup* H of G (that is, H contains no nontrivial normal subgroup of G), and an element $g \in G \setminus H$, the *coset graph* $\mathsf{Cos}(G, H, HgH)$ is defined as the graph with vertex set $[G{:}H]{:} = \{Hx \mid x \in G\}$ and Hx is adjacent to Hy if and only if $yx^{-1} \in HgH$. The following lemma is well known, see [16].

Lemma 2.2. *Using notation as above. Then the coset graph $\Gamma := \mathsf{Cos}(G, H, HgH)$ is G-arc-transitive and $\mathsf{val}(\Gamma) = |H{:}H \cap H^g|$. Moreover, Γ is undirected if and only if $g^2 \in H$, and Γ is connected if and only if $\langle H, g\rangle = G$.*

Conversely, each G-arc-transitive graph Σ with $G \leqslant \mathsf{Aut}\Sigma$ is isomorphic to the coset graph $\mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$, where $\alpha \in V\Sigma$, and $g \in \mathbf{N}_G(G_{\alpha\beta})$ with $\beta \in \Gamma(\alpha)$ is a 2-element.

Example 2.3. *Let $T = \mathsf{PSL}(2, 79)$. Then T has two maximal subgroups $H \cong \mathsf{A}_5$ and $K \cong \mathsf{S}_4$ such that $H \cap K \cong \mathsf{A}_4$. Take an involution $g \in K \setminus H$ and define the coset graph $\mathcal{C}_{4108} = \mathsf{Cos}(T, H, HgH)$. Then $\mathcal{C}_{4108}$ is a connected arc-transitive pentavalent graph of order 4108 and $\mathsf{Aut}(\mathcal{C}_{4108}) = T$. Further, any connected arc-transitive pentavalent graph of order 4108 admitting T as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{4108}$.*

Proof. By Lemma 1.2, T has a maximal subgroup $H \cong \mathsf{A}_5$. Let $L \cong \mathsf{A}_4$ be a subgroup of H. Then $K{:} = \mathbf{N}_T(L) \cong \mathsf{S}_4$ is a maximal subgroup of T and $H \cap K = L$. Let $g \in K \setminus H$ be an involution and define the coset graph $\mathcal{C}_{4108} = \mathsf{Cos}(T, H, HgH)$. Since $\langle H, g\rangle = T$ and $|H{:}H \cap H^g| = 5$, $\mathcal{C}_{4108}$ is a connected arc-transitive pentavalent graph of order 4108.

Now, let Γ be a connected arc-transitive pentavalent graph of order 4108 admitting T as an arc-transitive automorphism group. Then, for $\alpha \in V\Gamma$, $|T_\alpha| = |T|/4108 = 60$ and so $T_\alpha \cong \mathsf{A}_5$ by Lemma 1.2. Noting that T has two conjugate classes of subgroups isomorphic to A_5, and let $H_1 = H$ and H_2 be representatives of the two classes. Then up to isomorphism of the graphs, we may assume that $T_\alpha = H_1$ or H_2.

Suppose $T_\alpha = H_1$. By Lemma 2.2, $\Gamma \cong \mathsf{Cos}(T, H, HfH)$ for some $f \in T \setminus H$ such that $H \cap H^f \cong \mathsf{A}_4$. Since $H \cong \mathsf{A}_5$ has unique conjugate class of subgroups isomorphic to A_4, $L = (H \cap H^f)^h$ for some $h \in H$. Then, as $H \cap H^{fh} = (H \cap H^f)^h = L$, $HfH = HfhH$ and $\mathsf{Cos}(T, H, HfH) = \mathsf{Cos}(T, H, HfhH)$, without lose of generality, we may assume that $H \cap H^f = L$ and $f \in \mathbf{N}_T(L) \setminus L$. Now, since $\mathbf{N}_T(L)/L \cong \mathsf{S}_4/\mathsf{A}_4 \cong \mathbb{Z}_2$, we have $\mathbf{N}_T(L) = L \cup Lg$, so $f \in Lg$. It follows that $HfH = HgH$, and hence $\Gamma \cong \mathsf{Cos}(T, H, HfH) = \mathsf{Cos}(T, H, HgH) = \mathcal{C}_{4108}$. Moreover, by [2], $|\mathsf{Aut}\Gamma| = 246480 = |T|$, we have $\mathsf{Aut}\Gamma = \mathsf{PSL}(2, 79)$. Since $(\mathsf{Aut}\Gamma)_\alpha \cong \mathsf{A}_5$, $\Gamma \cong \mathcal{C}_{4108}$ is 2-transitive.

Suppose next $T_\alpha = H_2$. Arguing similarly as above, there also exists unique T-arc-transitive pentavalent graph. Further, by [2], this graph and $\mathcal{C}_{4108}$ are isomorphic, thus completes the proof. ◻

3 Classification

For a given group G, the *socle* of G, denoted by $\mathsf{soc}(G)$, is the product of all minimal normal subgroups of G. Obviously, $\mathsf{soc}(G)$ is a characteristic subgroup of G. Now, we prove the main result of this paper.

Theorem 3.1. *Let Γ be an arc-transitive pentavalent graph of order $4pq$, where $q > p \geqslant 5$ are primes. Then Γ lies in the following Table 3, where $\alpha \in V\Gamma$.*

Γ	(p,q)	$\mathsf{Aut}\Gamma$	$(\mathsf{Aut}\Gamma)_\alpha$	Transitivity
$\mathcal{C}_{574}^{(2)}$	$(7,41)$	$\mathsf{PSL}(2,41)\times\mathbb{Z}_2$	A_5	$2-$transitive
$\mathcal{C}_{4108}$	$(13,79)$	$\mathsf{PSL}(2,79)$	A_5	$2-$transitive

Table 3.

Proof. Set $\mathsf{A} = \mathsf{Aut}\Gamma$. By Lemma 1.3, $|\mathsf{A}_\alpha| \mid 2^9\cdot 3^2\cdot 5$, and hence $|\mathsf{A}| \mid 2^{11}\cdot 3^2\cdot 5\cdot p\cdot q$. We divide our discussion into the following two cases.

Case 1. Assume A has a soluble normal subgroup.

Let R be the soluble radical of A and let F be the Fitting subgroup of A. Then $R \neq 1$ and F is also the Fitting subgroup of R. By Lemma 1.1, $F \neq 1$ and $\mathbf{C}_R(F) \leqslant F$. As $|V\Gamma| = 4pq$, A has no nontrivial normal Sylow s-subgroup where $s \neq 2, p$ or q. So $F = \mathbf{O}_2(\mathsf{A}) \times \mathbf{O}_p(\mathsf{A}) \times \mathbf{O}_q(\mathsf{A})$, where $\mathbf{O}_2(\mathsf{A}), \mathbf{O}_p(\mathsf{A})$ and $\mathbf{O}_q(\mathsf{A})$ denote the largest normal Sylow 2-, p- and q-subgroups of A, respectively.

For each $r \in \{2, p, q\}$, since $q > p \geqslant 5$, $\mathbf{O}_r(\mathsf{A})$ has at least 4 orbits on $V\Gamma$, by Proposition 1.5, $\mathbf{O}_r(\mathsf{A})$ is semi-regular on $V\Gamma$. Therefore, $|\mathbf{O}_2(\mathsf{A})| \leqslant 4$, $|\mathbf{O}_p(\mathsf{A})| \leqslant p$ and $|\mathbf{O}_q(\mathsf{A})| \leqslant q$.

If $|\mathbf{O}_2(\mathsf{A})| = 4$, by Proposition 1.5, the normal quotient graph $\Gamma_{\mathbf{O}_2(\mathsf{A})}$ is an $\mathsf{A}/\mathbf{O}_2(\mathsf{A})$-arc-transitive pentavalent graph of odd order pq, not possible.

If $|\mathbf{O}_p(\mathsf{A})| = p$, then $\Gamma_{\mathbf{O}_p(\mathsf{A})}$ is an arc-transitive pentavalent graph of order $4q$, by [9, Theorem 4.1], we have $q = 3$, which is not the case. Similarly, we may exclude the case where $|\mathbf{O}_q(\mathsf{A})| = q$.

Thus, $F \cong \mathbb{Z}_2$. Then $\mathbf{C}_R(F) = F$ and $R/F = R/\mathbf{C}_R(F) \leqslant \mathsf{Aut}(F) = 1$, it follows that $R = F \cong \mathbb{Z}_2$. In particular, $\mathsf{A} \neq R$, that is, A is insoluble.

Now, Γ_R is an A/R-arc-transitive pentavalent graph of order $2pq$. Since $\mathsf{Aut}(\Gamma_R) \geqslant \mathsf{A}/R$ is insoluble, by Proposition 1.6, Γ_R and $\mathsf{Aut}(\Gamma_R)$ lie in Table 2, and as A/R is transitive on $A\Gamma_R$, we have $10pq \mid |\mathsf{A}/R|$, then checking the subgroups of $\mathsf{Aut}(\Gamma_R)$ in the Atlas [3], we easily conclude that $\mathsf{soc}(\mathsf{A}/R) = \mathsf{soc}(\mathsf{Aut}\Gamma_R)$. Set $T = \mathsf{soc}(\mathsf{A}/R)$. We consider all the possibilities of T lying in Table 2 one by one.

For row 1, $(p,q) = (7,41)$ and $\mathsf{A} = R.\mathsf{PSL}(2,41) \cong \mathbb{Z}_2.\mathsf{PSL}(2,41)$. It follows that either $\mathsf{A} = \mathbb{Z}_2 \times H\colon = \mathbb{Z}_2 \times \mathsf{PSL}(2,41)$ or $\mathsf{A} = \mathsf{SL}(2,41)$. For the former, by Theorem 1.5, H has at most two orbits on $V\Gamma$. Further, if H is transitive on $V\Gamma$, then $|H_\alpha| = \frac{|H|}{4\cdot 7\cdot 41} = 30$, which is not possible as $H = \mathsf{PSL}(2,41)$ has no subgroup of order 30. Thus, H has exactly two orbits on $V\Gamma$, it then follows from Lemma 2.1 that $\Gamma = \mathcal{C}_{574}^{(2)}$, as in row 1 of Table 3. For the latter, by Theorem 1.5(ii), $\mathsf{A} = \mathsf{SL}(2,41) \geqslant \mathsf{A}_\alpha \cong \mathsf{A}_5$. However, by [5, Lemma

2.7], the group $\mathsf{GL}(2,a)$ for each prime $a \geqslant 5$ contains no nonabelian simple subgroup, which is a contradiction.

For row 2, $(p,q)=(7,29)$, and $T=\mathsf{PSL}(2,29)$ has exactly two orbits on $V\Gamma_R$. Since $\mathsf{Out}(\mathsf{PSL}(2,29))=\mathbb{Z}_2$, we have $\mathsf{A}/R=\mathsf{Aut}(\Gamma_R)=\mathsf{PGL}(2,29)$, and by Theorem 1.5 (ii), $\mathsf{A}_\alpha \cong \mathsf{A}_5$. Since $\mathsf{A} \cong \mathbb{Z}_2.\mathsf{PSL}(2,29).\mathbb{Z}_2$, we have that either $\mathsf{A}=(\mathbb{Z}_2\times\mathsf{PSL}(2,29)).\mathbb{Z}_2$ or $\mathsf{SL}(2,29).\mathbb{Z}_2$. For the former case, A has a normal subgroup $M_1 \cong \mathsf{PSL}(2,29)$. By Theorem1.5 (i), M_1 has at most two orbits on $V\Gamma$. If M_1 is transitive on $V\Gamma$, then $T\cong (R\times M_1)/R$ is transitive on $V\Gamma$, a contradiction. Therefore, M_1 has exactly two orbits on $V\Gamma$, and hence $|(M_1)_\alpha| = \frac{|M_1|}{2\cdot 7\cdot 29} = 30$. By [3], $(M_1)_\alpha \cong \mathsf{D}_{30}$, which is not possible as $(M_1)_\alpha \leqslant \mathsf{A}_\alpha \cong \mathsf{A}_5$. For the latter case, A has a normal subgroup $M_2 \cong \mathsf{SL}(2,29)$ which has exactly two orbits on $V\Gamma$. It follows that $|(M_2)_\alpha| = \frac{|M_2|}{2\cdot 7\cdot 29} = 60$. Then as $(M_2)_\alpha \leqslant \mathsf{A}_\alpha \cong \mathsf{A}_5$, we obtain that $(M_2)_\alpha \cong \mathsf{A}_5$. which is not possible as $\mathsf{SL}(2,29)$ has no nonabelian simple subgroup by [5, Lemma 2.7].

Similarly, we may exclude the cases where $T=\mathsf{PSL}(2,59)$ and $\mathsf{PSL}(2,61)$, lying in rows 3 and 4 of Table 2.

Finally, we treat the case $T=\mathsf{PSp}(4,4)$, as in row 5 of Table 2. Then $(p,q)=(5,17)$ and $\mathsf{PSp}(4,4).\mathbb{Z}_2 \leqslant \mathsf{A}/R \leqslant \mathsf{PSp}(4,4).\mathbb{Z}_4$ as $T=\mathsf{PSp}(4,4)$ is not transitive on $V\Gamma_R$. Since the Schur Multiplier of $\mathsf{PSp}(4,4)$ is trivial, we conclude that $\mathsf{A}=(\mathbb{Z}_2\times\mathsf{PSp}(4,4)).\mathbb{Z}_2$ or $(\mathbb{Z}_2\times\mathsf{PSp}(4,4)).\mathbb{Z}_4$. Thus A always has a normal subgroup M such that $M\cong\mathsf{PSp}(4,4)$. By Theorem 1.5, M has at most two orbits on $V\Gamma$. However, by [3], $\mathsf{PSp}(4,4)$ has no subgroup with index 170 or 340, which is a contradiction as $|V\Gamma|=340$.

Case 2. Assume A has no soluble normal subgroup.

Let N be a minimal normal subgroup of A. Then $N=S^d$, where S is a nonabelian simple group and $d\geqslant 1$.

If N is semi-regular on $V\Gamma$, then $|N|$ divides $4pq$, we conclude that $|S|=4pq$ because S is insoluble. Noting that $q>p\geqslant 5$, by [6, P. 12-14], no such simple group exists, a contradiction.

Hence, N is not semi-regular on $V\Gamma$. Then by Theorem 1.5, N has at most two orbits on $V\Gamma$, so $2pq$ divides $|\alpha^N|$. Moreover, since Γ is connected and $1\neq N \lhd \mathsf{A}$, we have $1\neq N_\alpha^{\Gamma(\alpha)} \lhd \mathsf{A}_\alpha^{\Gamma(\alpha)}$, it follows that $5 \mid |N_\alpha|$, we thus have $10pq \mid |N|$. Since $q>5$, $q \mid |N|$ and q^2 does not divide $|N|$ as $|\mathsf{A}| \mid 2^{11}\cdot 3^2\cdot 5\cdot p\cdot q$, we conclude that $d=1$ and $N=S$ is a nonabelian simple group. Let $C=\mathbf{C}_{\mathsf{A}}(S)$. Then $C\lhd\mathsf{A}$, $C\cap S=1$ and $\langle C,S\rangle = C\times S$. Because $|C\times S|$ divides $2^{11}\cdot 3^2\cdot 5\cdot p\cdot q$ and $10pq \mid |S|$, C is a $\{2,3\}$-group, and hence soluble. So $C=1$ as $R=1$. This implies $\mathsf{A}=\mathsf{A}/C\leqslant\mathsf{Aut}(S)$, that is, A is almost simple with socle S.

Thus, $\mathsf{soc}(\mathsf{A})=S$ is a nonabelian simple group and satisfies the following condition.

Condition $(*)$**:** $|S|$ lies in Table 1 such that $10pq \mid |S|$, and $|S{:}S_\alpha| = 2pq$ or $4pq$.

Suppose first that S has exactly four prime factors. Then $S=\mathsf{PSL}(2,5^2), \mathsf{PSU}(3,4)$ or $\mathsf{PSp}(4,4)$. By Condition $(*)$ and [3], the only possibility is $(S,S_\alpha)=(\mathsf{PSL}(2,25),\mathsf{A}_5)$. Now, $(p,q)=(5,13)$, $|S{:}S_\alpha|=130$ and S has two orbits on $V\Gamma$. Since $\mathsf{Out}(\mathsf{PSL}(2,25))=\mathbb{Z}_2^2$, $\mathsf{A}\leqslant\mathsf{PSL}(2,5^2).\mathbb{Z}_2^2$, hence either $\mathsf{A}=\mathsf{P\Gamma L}(2,25)$ and $\mathsf{A}_\alpha=\mathsf{S}_5$, or $\mathsf{A}=\mathsf{PSL}(2,25){:}\mathbb{Z}_2\leqslant$

$\mathsf{P\Gamma L}(2,25)$ and $\mathsf{A}_\alpha \cong \mathsf{A}_5$. For the former, A has three conjugate classes of subgroups isomorphic to S_5. By [2], for each case, A has no suborbit of length 5, that is, there is no pentavalent graph admitting A as an arc-transitive automorphism group, no example appears. For the latter, $\mathsf{P\Gamma L}(2,25)$ has three subgroups which are semi-products $\mathsf{PSL}(2,25){:}\mathbb{Z}_2$, and A is isomorphic to one of the three. Then by using [2], a direct computation shows that A also has no suborbit of length 5 for each of the three cases, and thus can not give rise example.

Suppose now that S has five prime divisors, as in column 3 of Table 1. Assume $S \neq \mathsf{PSL}(2,q)$. If $S = \mathsf{PSL}(5,2)$, then by [3], we have $S_\alpha = \mathbb{Z}_2^4{:}\mathsf{S}_6$. As $\mathsf{Out}(\mathsf{PSL}(5,2)) = \mathbb{Z}_2$, $\mathsf{A}_\alpha = \mathbb{Z}_2^4{:}\mathsf{S}_6$ or $\mathbb{Z}_2^4{:}(\mathsf{S}_6 \times \mathbb{Z}_2)$, both have no permutation representation of degree 5, not possible. If $S = \mathsf{M}_{22}$, then $(p,q) = (7,11)$. By [3], M_{22} has no subgroup with index 154 or 308, that is, S does not satisfy the Condition $(*)$, not the case. If $S = \mathsf{PSL}(2,2^6)$, then $(p,q) = (7,13)$, and either $|S_\alpha| = |S|/2pq = 1440$ or $|S_\alpha| = |S|/4pq = 720$. However, by Lemma 1.2, it is easy to verify that $\mathsf{PSL}(2,2^6)$ has no subgroup with order 720 or 1440, a contradiction. For $S = \mathsf{PSL}(2,2^8)$, then $(p,q) = (17,257)$, and either $|S_\alpha| = |S|/2pq = 1920$ or $|S_\alpha| = |S|/4pq = 960$. By Lemma 1.2, the only possibility is that $S_\alpha \leqslant \mathbb{Z}_{2^8}{:}\mathbb{Z}_{2^8-1}$ is soluble, then as $\mathsf{Out}(S) = \mathbb{Z}_8$, $\mathsf{A}_\alpha \leqslant S_\alpha.\mathbb{Z}_8$ is also soluble. However, as $|\mathsf{A}_\alpha| \geqslant 960$, Lemma 1.3 implies that A_α is insoluble, which is a contradiction.

Now, assume that $S = \mathsf{PSL}(2,q)$ has five prime divisors. Then $q > p > 5$ and S is a $\{2,3,5,p,q\}$-group. Since $|S{:}S_\alpha| = 2pq$ or $4pq$, we obtain $3 \mid |S_\alpha|$, so $|\mathsf{A}_\alpha| \nmid 80$. hence A_α is insoluble by Lemma 1.3. Since $\mathsf{Out}(\mathsf{PSL}(2,q)) = \mathbb{Z}_2$, $\mathsf{A}_\alpha \leqslant S_\alpha.\mathbb{Z}_2$, we have S_α is insoluble, that is, S_α is an insoluble subgroup of $\mathsf{PSL}(2,q)$. It then follows from Lemma 1.2 that $S_\alpha = \mathsf{A}_\alpha = \mathsf{A}_5$ as $\mathsf{Aut}(\mathsf{PSL}(2,q)) = \mathsf{PGL}(2,q)$ has no subgroup isomorphic to $\mathsf{A}_5.\mathbb{Z}_2$. Since S has at most two orbits on $V\Gamma$, $|S| = 60 \cdot 2pq$ or $60 \cdot 4pq$. So $|S|$ has exactly one 3-divisor, one 5-divisor, and three or four 2-divisors. Moreover, as 8 divides $|S| = |\mathsf{PSL}(2,q)| = \frac{q(q-1)(q+1)}{2}$, $16 \mid (q^2-1)$, it implies $q \equiv \pm 1 \ (mod\ 8)$. Now, since $(\frac{q-1}{2}, \frac{q+1}{2}) = 1$, if $p \mid (q-1)$, then $q+1 = 2^i 3^j 5^k$, where $1 \leqslant i \leqslant 4, 0 \leqslant j,k \leqslant 1$, we easily conclude that $q \in \{7,23,47,79,239\}$ as $q \equiv \pm 1 \ (mod\ 8)$. Similarly, if $p \mid (q+1)$, then $q \in \{7,17,31,41,241\}$. Further, as $|S| = |\mathsf{PSL}(2,q)|$ has exactly five prime divisors, a simple computation shows that $q \neq 7,17,23,31,47,239$ or 241, we finally conclude $q = 41$ or 79.

Now, by [3], the only possibilities are as in the following table.

S	$\mathsf{PSL}(2,41)$	$\mathsf{PSL}(2,79)$
S_α	A_5	A_5
(p,q)	$(13,41)$	$(13,79)$

If $S = \mathsf{PSL}(2,41)$, then S has two orbits on $V\Gamma$, so $\mathsf{A} = \mathsf{PGL}(2,41)$ and $\mathsf{A}_\alpha = S_\alpha = \mathsf{A}_5$. Let $\beta \in \Gamma(\alpha)$. By Lemma 2.2, we may suppose $\Gamma = \mathsf{Cos}(\mathsf{A}, \mathsf{A}_\alpha, \mathsf{A}_\alpha g \mathsf{A}_\alpha)$ for some $g \in \mathbf{N}_{\mathsf{A}}(\mathsf{A}_{\alpha\beta})$. Since $\mathsf{val}(\Gamma) = 5$, $\mathsf{A}_{\alpha\beta} \cong \mathsf{A}_4$, it follows that $\mathbf{N}_{\mathsf{A}}(\mathsf{A}_{\alpha\beta}) = \mathbf{N}_S(\mathsf{A}_{\alpha\beta}) \cong \mathsf{S}_4$. Hence $\langle \mathsf{A}_\alpha, g \rangle \subseteq S \subset \mathsf{A}$, which contradicts the connectivity of Γ.

Finally, for $S = \mathsf{PSL}(2,79)$, S is transitive on $V\Gamma$, and so $\mathsf{A} = \mathsf{PSL}(2,79)$ and $\mathsf{A}_\alpha = \mathsf{A}_5$. By Example 2.3, $\Gamma \cong \mathcal{C}_{4108}$ is a 2-transitive graph. This completes the proof. $\square$

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