

On the binomial edge ideal of a pair of graphs

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Abstract

We characterize all pairs of graphs (G_1, G_2) , for which the binomial edge ideal J_{G_1, G_2} has linear relations. We show that J_{G_1, G_2} has a linear resolution if and only if G_1 and G_2 are complete and one of them is just an edge. We also compute some of the graded Betti numbers of the binomial edge ideal of a pair of graphs with respect to some graphical terms. In particular, we show that for every pair of graphs (G_1, G_2) with girth (i.e. the length of a shortest cycle in the graph) greater than 3, $\beta_{i, i+2}(J_{G_1, G_2}) = 0$, for all i . Moreover, we give a lower bound for the Castelnuovo-Mumford regularity of any binomial edge ideal J_{G_1, G_2} and hence the ideal of adjacent 2-minors of a generic matrix. We also obtain an upper bound for the regularity of J_{G_1, G_2} , if G_1 is complete and G_2 is a closed graph.

Keywords: Binomial edge ideal of a pair of graphs, Linear resolutions, Linear relations, Castelnuovo-Mumford regularity

1 Introduction

The binomial edge ideal of a graph was introduced in [7], and at about the same time in [13]. Let G be a finite simple graph with vertex set $[n]$ and edge set $E(G)$. Also, let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring over a field K . Then the **binomial edge ideal** of G in S , denoted by J_G , is generated by binomials of the form $f_{ij} = x_i y_j - x_j y_i$, where $i < j$ and $\{i, j\} \in E(G)$. This ideal also could be seen as the ideal generated by a collection of 2-minors of a $(2 \times n)$ -matrix whose entries are all indeterminates. In [7], the authors characterized those graphs, which, for certain labeling of their edges, have a

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quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. These graphs are called **closed** graphs. Many of the other algebraic properties of such ideals were studied in [1], [2], [7], [17] and [19]. In [3], the authors introduced the binomial edge ideal of a pair of graphs, as a generalization of the binomial edge ideal of a graph. Let G_1 be a graph on the vertex set $[m]$ and G_2 a graph on the vertex set $[n]$, and let $X = (x_{ij})$ be an $(m \times n)$ -matrix of indeterminates. Let $K[X]$ be the polynomial ring in the variables x_{ij} , where $i = 1, \dots, m$ and $j = 1, \dots, n$. Let $e = \{i, j\}$ for some $1 \leq i < j \leq m$ and $f = \{t, l\}$ for some $1 \leq t < l \leq n$. To the pair (e, f) , the following 2-minor of X is assigned:

$$p_{e,f} = [i, j | t, l] = x_{it}x_{jl} - x_{il}x_{jt}.$$

Then, the ideal

$$J_{G_1, G_2} = (p_{e,f} : e \in E(G_1), f \in E(G_2))$$

is called the **binomial edge ideal of the pair** (G_1, G_2) . Throughout the paper, by the binomial generators of J_{G_1, G_2} , we mean elements of the form $p_{e,f}$, as above, in J_{G_1, G_2} . If G_1 is a complete graph, then J_{G_1, G_2} is the generalized binomial edge ideal attached to G_2 , which studied in [18]. If G_1 and G_2 are two paths, then J_{G_1, G_2} is the ideal of adjacent 2-minors of X , which studied for example in [6], [8] and [15]. In [3], those pairs of graphs (G_1, G_2) were characterized, for which for a certain labeling of their edges, J_{G_1, G_2} has a quadratic Gröbner basis with respect to the lexicographic order, induced by $x_{11} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{m1} > \cdots > x_{mn}$, were characterized. The only pairs with this property, are the pairs (G_1, G_2) in which G_1 is complete and G_2 is closed, or vice versa. In [3], it was shown that J_{G_1, G_2} is a radical ideal if and only if either G_1 or G_2 is complete. Also, it was proved that J_{G_1, G_2} is a prime ideal if and only if G_1 and G_1 are complete. Moreover, the authors determined all minimal prime ideals of J_{G_1, G_2} , and hence characterized all unmixed binomial edge ideal of pairs of graphs.

In this paper, we study some other algebraic properties and invariants of J_{G_1, G_2} . In particular, when G_1 is just an edge, we can recover the results of [19] on binomial edge ideals.

Associated to the graph G is also a quadratic squarefree monomial ideal $I(G) = (x_i x_j : \{i, j\} \in E(G))$, in the polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , called the **edge ideal** of G . In [4], Fröberg characterized all graphs whose edge ideals have a linear resolution. He showed that $I(G)$ has a linear resolution if and only if the complementary graph $\overline{G}$ is chordal. In [19], the authors determined all graphs whose binomial edge ideals have a linear resolution. They showed that J_G has a linear resolution if and only if J_G has linear relations, if and only if G is a complete graph. The question arises whether there is a graphical characterization for binomial edge ideals of pairs of graphs to have a linear resolution. In this paper, we give a positive answer to this question. In Section 2, we show that J_{G_1, G_2} has linear relations only if G_1 and G_2 are both complete graphs. Then, we deduce that J_{G_1, G_2} has a linear resolution, if and only if G_1 and G_2 are complete graphs and one of them is just an edge. Also, in this section, we determine some of the Betti numbers of the binomial edge ideal of a pair of graphs. Actually, we show that $\beta_{1,3}(J_{G_1, G_2}) =$

$2e(G_1)k_3(G_2) + 2e(G_2)k_3(G_1) + k_3(G_1)(p_3(G_2) - k_3(G_2)) + k_3(G_2)(p_3(G_1) - k_3(G_1))$, where $e(G)$, $p_3(G)$ and $k_3(G)$ are the number of edges, 3-paths and 3-cycles of a graph G , respectively. Then, we deduce that for all $i > 0$, $\beta_{i,i+2}(J_{G_1,G_2}) = 0$, for every pairs of graphs with girth greater than 3. In particular, we deduce that $\beta_{i,i+2}(I) = 0$, for all $i > 0$, whenever I is the ideal generated by adjacent 2-minors of the matrix X . In addition, we show that if one of G_1 or G_2 is a non-complete connected graph, then $\beta_{1,4}(J_{G_1,G_2}) \neq 0$.

In Section 3, we give some bounds for the Castelnuovo-Mumford regularity of the binomial edge ideal of a pair of graphs. We give a lower bound for the Castelnuovo-Mumford regularity of J_{G_1,G_2} , for every pair (G_1, G_2) of graphs. Consequently, we give a lower bound for the Castelnuovo-Mumford regularity of the ideal of adjacent 2-minors of an $(m \times n)$ generic matrix. Also, by using an important result of Kalai and Meshulam on the regularity of monomial ideals, we gain an upper bound for the Castelnuovo-Mumford regularity of the binomial edge ideal of a pair of graphs, in which one of the graphs is complete and the other one is closed. Precisely, we show that the regularity of the binomial edge ideal of a pair of graphs (K_m, G) , where K_m is the complete graph on $[m]$ and G is a closed graph, is less than or equal to $\min\left\{\binom{m}{2}c(G), e(G)\right\} + 1$, where $c(G)$ is the number of maximal cliques of G .

Throughout the paper, we mean by a graph G , a simple graph over n vertices, with no isolated vertices. Whenever we say that G is a graph on $[n]$, we mean that the set of vertices of G is $\{v_1, \dots, v_n\}$. Also, by $<$, we mean the lexicographic order induced by $x_{11} > \dots > x_{1n} > x_{21} > \dots > x_{2n} > \dots > x_{m1} > \dots > x_{mn}$. Moreover, we consider S to be standard graded, unless we mention something else. None of the results of this paper depends on the characteristic of the field K .

2 The binomial edge ideal of a pair of graphs with linear resolution

In this section, we study the graded Betti numbers $\beta_{1,3}(J_{G_1,G_2})$ and $\beta_{1,4}(J_{G_1,G_2})$, and we characterize all pairs of graphs (G_1, G_2) , in which J_{G_1,G_2} has linear relations. Then, we classify all pairs of graphs (G_1, G_2) , in which J_{G_1,G_2} has a linear resolution. The following theorem is one of two main theorems of this section:

Theorem 1. *Let G_1 and G_2 be two graphs on $[m]$ and $[n]$, respectively. Then the following conditions are equivalent:*

- (a) J_{G_1,G_2} has linear relations.
- (b) J_{G_1,G_2} is a toric ideal, and G_1 and G_2 are connected.
- (c) G_1 and G_2 are complete graphs.

In order to prove Theorem 1, we need some facts that we will mention in the following. We denote the number of edges, 3-paths and 3-cycles of a graph G , by $e(G)$, $p_3(G)$ and $k_3(G)$, respectively. In the next result, we determine the first initial graded Betti number of the binomial edge ideal of a graph:

Theorem 2. Let G_1 and G_2 be two graphs on $[m]$ and $[n]$, respectively. Then we have

$$(a) \quad \beta_{1,3}(J_{G_1,G_2}) = 2e(G_1)k_3(G_2) + 2e(G_2)k_3(G_1) + k_3(G_1)(p_3(G_2) - k_3(G_2)) + k_3(G_2)(p_3(G_1) - k_3(G_1)).$$

$$(b) \quad \beta_{1,4}(J_{G_1,G_2}) \neq 0, \text{ if either } G_1 \text{ or } G_2 \text{ is non-complete and connected.}$$

$$(c) \quad \beta_{i-1,j}(J_{G_1,G_2}) = 0, \text{ for } j > 2i, \text{ if } G_1 \text{ is closed and } G_2 \text{ is complete, or vice versa. In particular, } \beta_{1,j}(J_{G_1,G_2}) = 0, \text{ for } j \neq 3, 4, \text{ if } G_1 \text{ is closed and } G_2 \text{ is complete, or vice versa.}$$

$$(d) \quad \beta_{i,j}(J_{G_1,G_2}) = 0, \text{ for } j > mn, \text{ if either } G_1 \text{ or } G_2 \text{ is a complete graph.}$$

Proof. (a) Note that we can consider two different $\mathbb{Z}$ -gradings for S . One is the standard grading and the other is grading by the weight $\mathbf{w} = (2, \dots, 2) \in \mathbb{N}^{mn}$. Thus, for every p, q , $\beta_{p,q}(J_{G_1,G_2})$ in the standard grading coincides with $\beta_{p,2q}(J_{G_1,G_2})$ in the weighted one. So, here, instead of computing $\beta_{1,3}(J_{G_1,G_2})$ in the standard grading, we will compute $\beta_{1,6}(J_{G_1,G_2})$ in the weighted grading. For every $e = \{i, j\} \in E(G_1)$ and $f = \{k, l\} \in E(G_2)$, we set $p_{ij,kl} := p_{e,f}$. Suppose that

$$\dots \longrightarrow S^{e(G_1)e(G_2)}(-4) \xrightarrow{\psi} S \longrightarrow S/J_{G_1,G_2} \longrightarrow 0$$

is the minimal graded free resolution of $S/J_{G_1,G_2}$, in which $\psi(\epsilon_{ij,kl}) = p_{ij,kl}$ such that $\epsilon_{ij,kl}$ is an element of the standard basis of the free S -module $S^{e(G_1)e(G_2)}(-4)$. Moreover, S is also $\mathbb{Z}^{m+n}$ -multigraded, with $\text{mdeg}(x_{ij}) = \varepsilon_{i,j+m}$, where $\varepsilon_{i,j+m}$ is the sum of the i -th and the $(j+m)$ -th canonical basis vectors of $\mathbb{Z}^{m+n}$. So, $\text{mdeg}(\epsilon_{ij,kl}) = \text{mdeg}(p_{ij,kl}) = \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. Let Z_1 be the relation module of $S/J_{G_1,G_2}$, and consider a relation $r = \sum g_{ij,kl}\epsilon_{ij,kl}$ of degree 6 (in the weighted grading), that is, an element in $(Z_1)_6$. Since $S/J_{G_1,G_2}$ is $\mathbb{Z}^{m+n}$ -graded, it follows that $(Z_1)_6$ is also $\mathbb{Z}^{m+n}$ -graded, and hence is generated by multihomogeneous elements. Thus we may assume that r is multihomogeneous, say of multidegree $a \in \mathbb{Z}^{m+n}$. Then all nonzero summands $g_{ij,kl}\epsilon_{ij,kl}$ are of multidegree a , with $|a| = 6$ (here $|a|$ is the sum of the components of a). Let $g_{ij,kl}\epsilon_{ij,kl} \neq 0$. Then $a = \text{mdeg}(g_{ij,kl}) + \text{mdeg}(\epsilon_{ij,kl}) = \text{mdeg}(g_{ij,kl}) + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. Therefore, $\text{mdeg}(g_{ij,kl}) = \varepsilon_{s,t+m}$ for some s, t . If $s = i$ and $t = k$ or l , then $a = 2\varepsilon_{i,k+m} + \varepsilon_{j,l+m}$ or $\varepsilon_{i,l+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$, and hence there is only one summand in r with this multidegree and $r \notin Z_1$, a contradiction. Similarly, if $s = j$ and $t = k$ or l , then $r \notin Z_1$, a contradiction. So, it remains to consider the following cases:

Case (1). Suppose that $s = i$ and $t \neq k, l$. Let $t < k < l$. Then, $a = \varepsilon_{i,t+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. So, r has exactly three summands and hence $r = g_{ij,tk}\epsilon_{ij,tk} + g_{ij,tl}\epsilon_{ij,tl} + g_{ij,kl}\epsilon_{ij,kl}$. But, it happens if and only if G_2 contains a 3-cycle over the vertices t, k and l . Thus r is a relation of the ideal $(p_{ij,tk}, p_{ij,tl}, p_{ij,kl})$, which is the ideal of 2-minors of the matrix $\begin{bmatrix} x_{it} & x_{ik} & x_{il} \\ x_{jt} & x_{jk} & x_{jl} \end{bmatrix}$. So, the generating relations

are $x_{il}\epsilon_{ij,tk} - x_{ik}\epsilon_{ij,tl} + x_{it}\epsilon_{ij,kl}$ and $x_{jl}\epsilon_{ij,tk} - x_{jk}\epsilon_{ij,tl} + x_{jt}\epsilon_{ij,kl}$, by Hilbert-Burch theorem. But, multidegree of the latter is not equal to a . Hence, in this multidegree, we just consider $x_{il}\epsilon_{ij,tk} - x_{ik}\epsilon_{ij,tl} + x_{it}\epsilon_{ij,kl}$. Therefore, in this case, we obtain $e(G_1)k_3(G_2)$ elements in $(Z_1)_6$.

Case (2). Suppose that $s = j$ and $t \neq k, l$. Let $t < k < l$. Then, $a = \varepsilon_{j,t+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. So, r has exactly three summands and hence $r = g_{ij,tk}\epsilon_{ij,tk} + g_{ij,tl}\epsilon_{ij,tl} + g_{ij,kl}\epsilon_{ij,kl}$. But, it happens if and only if G_2 contains a 3-cycle over the vertices t, k and l . By repeating the discussion in Case (1), we obtain that $x_{jl}\epsilon_{ij,tk} - x_{jk}\epsilon_{ij,tl} + x_{jt}\epsilon_{ij,kl}$ is the only possible generating relation in this case. Thus, in this case, we get $e(G_1)k_3(G_2)$ elements in $(Z_1)_6$.

Case (3). Suppose that $s \neq i, j$ and $t = k$. Let $s < i < j$. Then, $a = \varepsilon_{s,k+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. So, r has exactly three summands and hence $r = g_{si,kl}\epsilon_{si,kl} + g_{sj,kl}\epsilon_{sj,kl} + g_{ij,kl}\epsilon_{ij,kl}$. But, it happens if and only if G_1 contains a 3-cycle over the vertices s, i and j . Thus r is a relation of the ideal $(p_{si,kl}, p_{sj,kl}, p_{ij,kl})$, which is the ideal of 2-minors of the matrix $\begin{bmatrix} x_{sk} & x_{ik} & x_{jk} \\ x_{sl} & x_{il} & x_{jl} \end{bmatrix}$. So, the generating relations are $x_{jk}\epsilon_{si,kl} - x_{ik}\epsilon_{sj,kl} + x_{sk}\epsilon_{ij,kl}$ and $x_{jl}\epsilon_{ij,tk} - x_{il}\epsilon_{ij,tl} + x_{sl}\epsilon_{ij,kl}$, by Hilbert-Burch theorem. But, multidegree of the latter is not equal to a . Hence, in this multidegree, we just consider $x_{jk}\epsilon_{si,kl} - x_{ik}\epsilon_{sj,kl} + x_{sk}\epsilon_{ij,kl}$. Therefore, in this case, we obtain $e(G_2)k_3(G_1)$ elements in $(Z_1)_6$.

Case (4). Suppose that $s \neq i, j$ and $t = l$. Let $s < i < j$. Then, $a = \varepsilon_{s,l+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. So, r has exactly three summands and hence $r = g_{si,kl}\epsilon_{si,kl} + g_{sj,kl}\epsilon_{sj,kl} + g_{ij,kl}\epsilon_{ij,kl}$. But, it happens if and only if G_1 contains a 3-cycle over the vertices s, i and j . By repeating the discussion in Case (3), we obtain that $x_{jl}\epsilon_{ij,tk} - x_{il}\epsilon_{ij,tl} + x_{sl}\epsilon_{ij,kl}$ is the only possible generating relation in this case. Thus, in this case, we get $e(G_2)k_3(G_1)$ elements in $(Z_1)_6$.

Case (5). Suppose that $s \neq i, j$ and $t \neq k, l$. Let $s < i < j$ and $t < k < l$. Then, $a = \varepsilon_{s,t+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$. Thus, we have $r = g_{si,tk}\epsilon_{si,tk} + g_{si,tl}\epsilon_{si,tl} + g_{si,kl}\epsilon_{si,kl} + g_{sj,tk}\epsilon_{sj,tk} + g_{sj,tl}\epsilon_{sj,tl} + g_{sj,kl}\epsilon_{sj,kl} + g_{ij,tk}\epsilon_{ij,tk} + g_{ij,tl}\epsilon_{ij,tl} + g_{ij,kl}\epsilon_{ij,kl}$. Since $\text{mdeg}(r) = a = \varepsilon_{s,t+m} + \varepsilon_{i,k+m} + \varepsilon_{j,l+m}$, we have $r = c_1x_{jl}\epsilon_{si,tk} + c_2x_{jk}\epsilon_{si,tl} + c_3x_{jt}\epsilon_{si,kl} + c_4x_{il}\epsilon_{sj,tk} + c_5x_{ik}\epsilon_{sj,tl} + c_6x_{it}\epsilon_{sj,kl} + c_7x_{sl}\epsilon_{ij,tk} + c_8x_{sk}\epsilon_{ij,tl} + c_9x_{st}\epsilon_{ij,kl}$, where $c_1, \dots, c_9 \in K$. By easy computations, we have that the generating relations in this case, are exactly correspond to the solution space of the homogeneous system of equations $c_1 + c_5 + c_9 = 0$, $-c_1 + c_6 + c_8 = 0$, $c_2 + c_4 - c_9 = 0$, $c_2 + c_6 - c_7 = 0$, $c_3 - c_4 - c_8 = 0$ and $c_3 + c_5 + c_7 = 0$, whose dimension is 4. Thus, the generating relations are as follows: $r_1 = x_{jl}\epsilon_{si,tk} - x_{jk}\epsilon_{si,tl} + x_{jt}\epsilon_{si,kl} + x_{il}\epsilon_{sj,tk} - x_{ik}\epsilon_{sj,tl} + x_{it}\epsilon_{sj,kl}$, where the vertices i, s, j make a 3-path with edges $\{s, i\}$ and $\{s, j\}$ in G_1 , and the vertices t, k, l induce a 3-cycle in G_2 ; $r_2 = x_{sl}\epsilon_{ij,tk} - x_{sk}\epsilon_{ij,tl} + x_{st}\epsilon_{ij,kl} + x_{il}\epsilon_{sj,tk} - x_{ik}\epsilon_{sj,tl} + x_{it}\epsilon_{sj,kl}$, where the vertices i, s, j make a 3-path with edges $\{s, j\}$ and $\{i, j\}$ in G_1 , and the vertices t, k, l induce a

3-cycle in G_2 ; $r_3 = x_{jl}\epsilon_{si,tk} - x_{il}\epsilon_{sj,tk} + x_{sl}\epsilon_{ij,tk} + x_{jk}\epsilon_{si,tl} - x_{ik}\epsilon_{sj,tl} + x_{sk}\epsilon_{ij,tl}$, where the vertices i, s, j induce a 3-cycle in G_1 , and the vertices t, k, l make a 3-path with edges $\{t, k\}$ and $\{t, l\}$ in G_2 ; $r_4 = x_{jk}\epsilon_{si,tl} - x_{ik}\epsilon_{sj,tl} + x_{sk}\epsilon_{ij,tl} + x_{jt}\epsilon_{si,kl} - x_{it}\epsilon_{sj,kl} + x_{st}\epsilon_{ij,kl}$, where the vertices i, s, j induce a 3-cycle in G_1 , and the vertices t, k, l make a 3-path with edges $\{t, l\}$ and $\{k, l\}$ in G_2 . Note that if the vertices i, s, j do not induce any cycles in G_1 , then just one of the elements r_1 and r_2 could appear. Similarly, if the vertices t, k, l do not induce any cycles in G_2 , then just one of the elements r_3 and r_4 could appear. Therefore, in this case, we have $k_3(G_2)(p_3(G_1) - k_3(G_1))$ elements in $(Z_1)_6$, regarding r_1 and r_2 , and also $k_3(G_1)(p_3(G_2) - k_3(G_2))$ elements in $(Z_1)_6$, regarding r_3 and r_4 .

Comparing the multidegrees in these 5 cases, we obtain that the minimal generating relations of degree 6 are of the above forms, and hence the result follows.

(b) Without loss of generality, we may assume that G_2 is connected and non-complete. Thus, there exist 3 vertices k, t, l in $[n]$ with $k < t < l$ which induce a 3-path in G_2 with edges $\{k, t\}$ and $\{t, l\}$. Suppose that $\{i, j\}$ is an edge in G_1 . Let $\gamma := p_{ij,kt}\epsilon_{ij,tl} - p_{ij,tl}\epsilon_{ij,kt}$. Clearly, $\gamma \in Z_1$ and $\deg(\gamma) = 4$ in the standard grading. We show that γ is a minimal relation. Then we have $\beta_{1,4}(J_{G_1,G_2}) > 0$. Note that $\deg_{\mathbf{w}}(\gamma) = 8$ (with the weighted grading mentioned in part (a)), and using $\mathbb{Z}^{m+n}$ -grading introduced in part (a), we have $\text{mdeg}(\gamma) = \varepsilon_{i,k+m} + \varepsilon_{j,t+m} + \varepsilon_{i,t+m} + \varepsilon_{j,l+m}$. If γ is not a minimal relation, then it must be reduced by elements of $(Z_1)_6$. By comparing the multidegrees of γ and the generating relations in $(Z_1)_6$, one obtains that non of the generating relations of the form of Case (3), Case (4) and Case (5) could occur in the expression of γ in terms of the elements of $(Z_1)_6$. If the generating relations of the form of Case (1) and Case (2) occur in that expression of γ , then the vertices k, t, l induce a 3-cycle in G_2 , which is a contradiction, since $\{k, l\}$ is not an edge of G_2 .

(c) Suppose that G_1 is complete and G_2 is closed. Then we have $\text{in}_{<}(J_{G_1,G_2}) = (x_{ik}x_{jl} : i < j, k < l, \{i, j\} \in E(G_1), \{k, l\} \in E(G_2))$, by [3, Theorem 1.3]. Thus, it can be seen as the edge ideal of an n -partite graph over the vertex set $V = \bigcup_{p=1}^n V_p$, where $V_p = \{x_{1p}, \dots, x_{mp}\}$, for all $p = 1, \dots, n$. We denote this n -partite graph by $\text{in}_{<}(G_1, G_2)$. So, we have $\text{in}_{<}(J_{G_1,G_2}) = I(\text{in}_{<}(G_1, G_2))$. But, $\beta_{i-1,j}(I(\text{in}_{<}(G_1, G_2))) = 0$, for all $j > 2i$, by [10, Lemma 2.2]. On the other hand, we have $\beta_{i-1,j}(J_{G_1,G_2}) \leq \beta_{i-1,j}(\text{in}_{<}(J_{G_1,G_2}))$, for all i, j , by [5, Corollary 3.3.3]. So, if G_1 is complete and G_2 is closed, then $\beta_{i-1,j}(J_{G_1,G_2}) = 0$, for all $j > 2i$.

(d) By [3, Theorem 1.2], $\text{in}_{<}(J_{G_1,G_2})$ is a squarefree monomial ideal in S . Thus, the result follows by Hochster's formula, since $\beta_{i,j}(J_{G_1,G_2}) \leq \beta_{i,j}(\text{in}_{<}(J_{G_1,G_2}))$, for all i, j . $\square$

Notice that, by setting $G_1 = K_2$, Theorem 2 yields all parts of [19, Theorem 2.2].

The length of any shortest cycle (if any) in a graph G is called the **girth** of G . The girth of acyclic graphs, i.e. graphs with no cycle, is considered as infinity.

Corollary 3. *If G_1 and G_2 are graphs with girth greater than 3, then $\beta_{i,i+2}(J_{G_1,G_2}) = 0$, for all $i > 0$. In particular, if G_1 and G_2 are bipartite graphs, one has $\beta_{i,i+2}(J_{G_1,G_2}) = 0$, for all $i > 0$.*

A (2×2) adjacent minor of X is the determinant of a submatrix with row indices $i, i + 1$ and column indices $j, j + 1$. We call the ideal generated by all of the (2×2) adjacent minors of X , the *ideal of adjacent 2-minors* of X .

Corollary 4. *Let I be the ideal of adjacent 2-minors of an $(m \times n)$ generic matrix. Then $\beta_{1,4}(I) \neq 0$, and $\beta_{i,i+2}(I) = 0$, for all $i > 0$.*

Proof. It is enough to note that $I = J_{P_m, P_n}$. □

Applying Theorem 2, part (a), we gain the following:

Corollary 5. *Let $m, n \geq 3$ and $t \geq 4$. Then*

$$(a) \beta_{1,3}(J_{K_m, K_n}) = 2 \left(\binom{m}{3} \binom{n+1}{3} + \binom{n}{3} \binom{m+1}{3} \right).$$

$$(b) \beta_{1,3}(J_{K_m, C_t}) = 3t \binom{m}{3}, \text{ where } C_t \text{ is a cycle over } t \text{ vertices.}$$

$$(c) \beta_{1,3}(J_{K_m, T}) = (2n + p_3(T) - 2) \binom{m}{3}, \text{ where } T \text{ is a tree over } n \text{ vertices. In particular, } \beta_{1,3}(J_{K_m, P_n}) = (3n - 4) \binom{m}{3}.$$

Remark 6. If G_1 is a closed graph and G_2 is complete, or vice versa, we apply **consecutive cancellations** to show that $\beta_{1,3}(J_{G_1, G_2}) = \beta_{1,3}(\text{in}_{<}(J_{G_1, G_2}))$. Actually, we have $\beta_{0,3}(J_{G_1, G_2}) = \beta_{0,3}(\text{in}_{<}(J_{G_1, G_2})) = 0$ and $\beta_{2,3}(J_{G_1, G_2}) = \beta_{2,3}(\text{in}_{<}(J_{G_1, G_2})) = 0$, by minimality of the free resolutions. On the other hand, by [16, Theorem 22.12], the sequence of graded Betti numbers of J_{G_1, G_2} is obtained from the sequence of graded Betti numbers of $\text{in}_{<}(J_{G_1, G_2})$ by **consecutive cancellations**. So, we have $\beta_{1,3}(J_{G_1, G_2}) = \beta_{1,3}(\text{in}_{<}(J_{G_1, G_2}))$. A sequence $q_{i,j}$ of numbers is said to be obtained from a sequence $p_{i,j}$ by a consecutive cancellation if there exist indices s and r such that $q_{s,r} = p_{s,r} - 1$, $q_{s+1,r} = p_{s+1,r} - 1$ and $q_{i,j} = p_{i,j}$ for all other values of i, j .

Recall that a homogeneous ideal I whose generators all have degree d is said to have a **d -linear resolution** (or simply linear resolution) if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$. Also, if $\beta_{1,j}(I) = 0$ for all $j \neq d + 1$, then we say that I has **linear relations**.

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. (a) $\Rightarrow$ (c): Suppose that J_{G_1, G_2} has linear relations. Thus, $\beta_{1,j}(J_{G_1, G_2}) = 0$, for all $j > 3$. In particular, $\beta_{1,4}(J_{G_1, G_2}) = 0$. So that G_1 and G_2 are both connected, since if one of them, say G_1 , has connected components $H_1, \dots, H_c$, then the minimal graded free resolution of $S/J_{G_1, G_2}$ is the tensor product of those of $S/J_{G_1, H_1}, \dots, S/J_{G_1, H_c}$, and hence $\beta_{1,4}(J_{G_1, G_2}) \geq \sum_{1 \leq i < j \leq c} \beta_{0,2}(J_{G_1, H_i}) \beta_{0,2}(J_{G_1, H_j}) > 0$. Therefore, G_1 and G_2 are both complete graphs, by Theorem 2, part (b).

(c) $\Rightarrow$ (a): Suppose that G_1 and G_2 are complete graphs. Then $J_{G_1, G_2} = I_2(X)$ is the ideal of 2-minors of X , the $(m \times n)$ -matrix of indeterminates. Thus, by Kurano's theorem, J_{G_1, G_2} has linear relations (see [11]).

(b) $\Rightarrow$ (c): Suppose that J_{G_1, G_2} is a toric ideal. Then, it is a prime ideal. So, G_1 and G_2 are both complete graphs, by [3, Corollary 2.2].

(c) $\Rightarrow$ (b): Let G_1 and G_2 be complete graphs. Then, by [21, Proposition 9.1.2], $J_{G_1, G_2} = I_2(X)$ is the toric ideal of $K[K_{m, n}]$, where $K[K_{m, n}] = K[\{s_i t_j : 1 \leq i \leq m, 1 \leq j \leq n\}]$ and $K_{m, n}$ is the complete bipartite graph over the set of vertices $\{s_1, \dots, s_m, t_1, \dots, t_n\}$. More precisely, J_{G_1, G_2} is the kernel of the graded homomorphism of K -algebras

$$\varphi : K[\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}] \longrightarrow K[K_{m, n}],$$

in which $\varphi(x_{ij}) = s_i t_j$, for all i, j . $\square$

Remark 7. By [19, Theorem 2.1], when $G_1 = K_2$, we have that J_{G_1, G_2} has linear relations if and only if $\text{in}_<(J_{G_1, G_2})$ does. But this is not true, in general, that is when G_1 is not an edge. For example, computations by *CoCoA* show that $\beta_{1,4}(\text{in}_<(J_{K_3, K_3})) = 1$, and hence $\text{in}_<(J_{K_3, K_3})$ does not have linear relations.

Let B be a K -algebra and A a K -subalgebra of B . Recall that A is called an **algebra retract** of B , if there exists a surjective K -algebra homomorphism $\pi : B \rightarrow A$ whose composition with the inclusion map $A \rightarrow B$ is the identity on A .

Let G_1 and G_2 be graphs on $[m]$ and $[n]$, and let H_1 and H_2 be subgraphs of G_1 and G_2 over $m_1 \leq m$ and $n_1 \leq n$ vertices, respectively. So, one could consider an $m_1 \times n_1$ submatrix Y of X , correspond to the vertices of H_1 and H_2 . Here, we also use Y to denote the set of variables appeared in the matrix Y . So that the binomial generators of J_{H_1, H_2} are contained in the ring $K[Y]$. With these notations, we have the following proposition:

Proposition 8. *Let G_1 and G_2 be two graphs over $[m]$ and $[n]$, respectively. If H_1 and H_2 are induced subgraphs of G_1 and G_2 , respectively, then we have*

- (a) $\beta_{i,j}^{K[Y]}(J_{H_1, H_2}) \leq \beta_{i,j}^{K[X]}(J_{G_1, G_2})$, for all i, j .
- (b) $\text{reg}_{K[Y]}(J_{H_1, H_2}) \leq \text{reg}_{K[X]}(J_{G_1, G_2})$.
- (c) $\text{pd}_{K[Y]}(J_{H_1, H_2}) \leq \text{pd}_{K[X]}(J_{G_1, G_2})$.

Proof. (a) Let H_1 and H_2 be induced subgraphs of G_1 and G_2 over $m_1 \leq m$ and $n_1 \leq n$ vertices, respectively. So, one could consider an $m_1 \times n_1$ submatrix Y of X , correspond to the vertices of H_1 and H_2 . Note that J_{H_1, H_2} is an ideal of $K[X]$ whose binomial generators are in $K[Y]$. By $J_{H_1, H_2}K[Y]$, we mean an ideal of $K[Y]$, whose generators are the same as J_{H_1, H_2} (as an ideal of $K[X]$). We have $J_{G_1, G_2} \cap K[Y] = J_{H_1, H_2}K[Y]$. Because, obviously, $J_{H_1, H_2}K[Y] \subseteq J_{G_1, G_2} \cap K[Y]$. For the other containment, let $f \in J_{G_1, G_2} \cap K[Y]$. So, $f = \sum_{s=1}^t r_s p_s$, for some binomial generators p_s of J_{G_1, G_2} and $r_s \in K[X]$, where $s = 1, \dots, t$. Now, it is enough to set all variables x_{ij} in X , which do not belong to Y , equal to zero in $f = \sum_{s=1}^t r_s p_s$. The left hand side of this equality does not change, since $f \in K[Y]$. But, in the right hand side, if $p_i = x_{jk}x_{ql} - x_{jl}x_{qk} \notin K[Y]$, for some i , then one

of the variables appeared in p_i does not belong to Y , say $x_{jk} \notin Y$. So, $j \notin [m_1]$ or $k \notin [n_1]$. Hence, $x_{jl} \notin Y$ or $x_{qk} \notin Y$. So, after substituting desired variables by zero, p_i will be omitted in the expression of f . Thus, we get $f = \sum_{i=1}^{t'} r'_{s_i} p_{s_i}$, where r'_{s_i} 's are obtained by putting zero instead of variables of $X \setminus Y$ in r_s 's, and p_{s_i} 's belong to $K[Y]$. Since H_1 and H_2 are induced subgraphs of G_1 and G_2 , p_{s_i} 's belong to $J_{H_1, H_2} K[Y]$, and hence $f \in J_{H_1, H_2} K[Y]$. Now, set $A = K[Y]/J_{H_1, H_2} K[Y]$ and $B = K[X]/J_{G_1, G_2}$. Thus, A is a K -subalgebra of B . Let $\pi : B \rightarrow A$ be the epimorphism induced by setting all variables x_{ij} in X , which do not belong to Y , equal to zero. So, we have the maps $A \hookrightarrow B \xrightarrow{\pi} A$ whose composition is the identity on A . Hence, A is an algebra retract of B . Now, applying [14, Corollary 2.8], the result follows. Parts (b) and (c) follow immediately from (a). $\square$

Corollary 9. *Let G_1 and G_2 be two graphs over $[m]$ and $[n]$, respectively, and let $S_1 = [x_i, y_i : 1 \leq i \leq m]$ and $S_2 = [x_i, y_i : 1 \leq i \leq n]$. Then we have*

- (a) $\beta_{ij}^{K[X]}(J_{G_1, G_2}) \geq \max\{\beta_{ij}^{S_1}(J_{G_1}), \beta_{ij}^{S_2}(J_{G_2})\}$, for all i, j .
- (b) $\text{reg}_{K[X]}(J_{G_1, G_2}) \geq \max\{\text{reg}_{S_1}(J_{G_1}), \text{reg}_{S_2}(J_{G_2})\}$.
- (c) $\text{pd}_{K[X]}(J_{G_1, G_2}) \geq \max\{\text{pd}_{S_1}(J_{G_1}), \text{pd}_{S_2}(J_{G_2})\}$.

Proof. since G_1 and G_2 contain at least an edge, by Proposition 8, we have that $\beta_{ij}^{K[Y_1]}(J_{G_1, K_2}) \leq \beta_{ij}^{K[X]}(J_{G_1, G_2})$ and $\beta_{ij}^{K[Y_2]}(J_{K_2, G_2}) \leq \beta_{ij}^{K[X]}(J_{G_1, G_2})$, for all i, j , where Y_1 and Y_2 are appropriate subsets of X . But, obviously, $\beta_{ij}^{K[Y_1]}(J_{G_1, K_2}) = \beta_{ij}^{S_1}(J_{G_1})$ and $\beta_{ij}^{K[Y_2]}(J_{K_2, G_2}) = \beta_{ij}^{S_2}(J_{G_2})$. So, we get part (a). Parts (b) and (c) follow immediately from (a). $\square$

Now, we go to the second main result of this section:

Theorem 10. *Let G_1 and G_2 be two graphs over $[m]$ and $[n]$, respectively. Then J_{G_1, G_2} has a linear resolution if and only if G_1 and G_2 are complete graphs, and $m = 2$ or $n = 2$.*

Proof. If G_1 and G_2 are complete graphs such that one of them is an edge, then J_{G_1, G_2} has a linear resolution, by [19, Theorem 2.1]. Conversely, suppose that J_{G_1, G_2} has a linear resolution. So, it has linear relations, and hence G_1 and G_2 are both complete graphs, on $[m]$ and $[n]$, respectively, by Theorem 1. Suppose on the contrary that $m, n \geq 3$. Then, both of G_1 and G_2 have an induced 3-cycle. Thus, the graded Betti numbers of J_{G_1, G_2} is greater than or equal to the graded Betti numbers of J_{K_3, K_3} , by Proposition 8. On the other hand, by [20, Theorem 5.4.6], $S/J_{K_3, K_3}$ is Gorenstein, and hence its minimal graded free resolution is symmetric. So that J_{K_3, K_3} has no linear resolution, since it is generated by quadratic forms. Hence, J_{G_1, G_2} has no linear resolution as well, a contradiction. So, we have $m = 2$ or $n = 2$. $\square$

3 The Castelnuovo-Mumford regularity of the binomial edge ideal of a pair of graphs

In this section, we study the Castelnuovo-Mumford regularity (or regularity, for short,) of the binomial edge ideal of a pair of graphs. Indeed, we give a lower bound for the

regularity of the binomial edge ideal of an arbitrary pair of graphs. Consequently, we obtain a lower bound for the ideals of adjacent 2-minors. Also, we obtain an upper bound for the regularity of the binomial edge ideal of a pair of graphs (K_m, G) , in which G is a closed graph. In order to prove the main theorem of this section, we need some facts which we will mention in the sequel.

Notice that if G is a closed graph, then we have $\text{in}_<(J_G) = (x_i y_j : i < j, \{v_i, v_j\} \in E(G))$. Thus, it can be seen as the edge ideal of a bipartite graph over the vertex set $V = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. We denote this bipartite graph by $\text{in}_<(G)$. So, we have $\text{in}_<(J_G) = I(\text{in}_<(G))$ (see also [19, Theorem 2.2, part (c)]). Moreover, as we mentioned in the proof of Theorem 2, if G_1 is complete and G_2 is closed, then we have $\text{in}_<(J_{G_1, G_2}) = (x_{ik} x_{jl} : i < j, k < l, \{i, j\} \in E(G_1), \{k, l\} \in E(G_2))$, by [3, Theorem 1.3]. So that it can be seen as the edge ideal of an n -partite graph over the vertex set $V = \bigcup_{p=1}^n V_p$, where $V_p = \{x_{1p}, \dots, x_{mp}\}$, for all $p = 1, \dots, n$. We denote this n -partite graph by $\text{in}_<(G_1, G_2)$. Thus, we have $\text{in}_<(J_{G_1, G_2}) = I(\text{in}_<(G_1, G_2))$.

A graph G is called **chordal** if each induced cycle in G has length 3, and G is called **co-chordal** if the complementary graph $\overline{G}$ is chordal. The **co-chordal cover number** of a graph G , which is denoted by $\text{cochord}(G)$, is the minimum number of subgraphs $H_1, \dots, H_s$ of G such that every H_i is cochordal and $\bigcup_{i=1}^s E(H_i) = E(G)$.

In [22], Woodroffe posed an upper bound for the regularity of the edge ideal of a graph:

Theorem 11. [22, Theorem 11] *For any graph G , we have $\text{reg}(I(G)) \leq \text{cochord}(G) + 1$.*

The following theorem is a special case of the result proved by Kalai and Meshulam. Their result is on simplicial complexes, in general.

Theorem 12. [9, Theorem 1.2] *If $G_1, \dots, G_s$ are graphs on the same vertex set, then $\text{reg}(S/I(\bigcup_{i=1}^s G_i)) \leq \sum_{i=1}^s \text{reg}(S/I(G_i))$.*

We denote by $c(G)$, the number of maximal cliques of the graph G . Here, we mean by a maximal clique of a graph G , an induced subgraph of G which is a complete graph and is also maximal with this property. Now, we are ready to prove the main theorem of this section:

Theorem 13. *Let G be a closed graph on $[n]$ and $m, n \geq 2$. Then we have*

$$\text{reg}(J_{K_m, G}) \leq \min \left\{ \binom{m}{2} c(G), e(G) \right\} + 1.$$

Proof. Since G is closed, we have $\text{in}_<(J_G) = I(\text{in}_<(G))$ and $\text{in}_<(J_{K_m, G}) = I(\text{in}_<(K_m, G))$, as we mentioned above. Note that $\text{in}_<(K_m, G)$ could be seen as a multipartite graph in the following two ways:

(1) Consider $\text{in}_<(K_m, G)$ as an n -partite graph over the vertex set $V = \bigcup_{p=1}^n V_p$, where $V_p = \{x_{1p}, \dots, x_{mp}\}$, for all $p = 1, \dots, n$. It can be easily checked that for every i, j with $1 \leq i < j \leq n$, the induced subgraph of $\text{in}_<(K_m, G)$ on $V_{i,j} := V_i \cup V_j$ is isomorphic to

$\text{in}_<(K_m)$. On the other hand, we have that $\text{in}_<(K_m, G) = \bigcup_{\{i,j\} \in E(G)} \left(\text{in}_<(K_m, G) \right)_{v_{ij}}$. Thus, by Theorem 12, we obtain

$$\text{reg}(S/I(\text{in}_<(K_m, G))) \leq e(G) \text{reg}(S/I(\text{in}_<(K_m))) = e(G),$$

where the last equality holds, since $\text{reg}(S/I(\text{in}_<(K_m))) = \text{reg}(S/\text{in}_<(J_{K_m})) = \text{reg}(S/J_{K_m}) = 1$, by [19, Theorem 2.1].

(2) Consider $\text{in}_<(K_m, G)$ as an m -partite graph over the vertex set $W = \bigcup_{p=1}^m W_p$, where $W_p = \{x_{p1}, \dots, x_{pn}\}$, for all $p = 1, \dots, m$. It can be easily checked that for every i, j with $1 \leq i < j \leq m$, the induced subgraph of $\text{in}_<(K_m, G)$ on $W_{i,j} := W_i \cup W_j$ is isomorphic to $\text{in}_<(G)$. On the other hand, we have that $\text{in}_<(K_m, G) = \bigcup_{1 \leq i < j \leq m} \left(\text{in}_<(K_m, G) \right)_{W_{i,j}}$. Thus, by Theorem 12 and Theorem 11, we obtain

$$\text{reg}(S/I(\text{in}_<(K_m, G))) \leq \binom{m}{2} \text{reg}(S/I(\text{in}_<(G))) \leq \binom{m}{2} \text{cochord}(\text{in}_<(G)).$$

Now, similar to the proof of [19, Theorem 3.2], we show that $\text{cochord}(\text{in}_<(G)) \leq c(G)$. Let H be a maximal clique of G . Then $\text{in}_<(H)$ is an induced subgraph of $\text{in}_<(G)$. By [19, Theorem 2.1], $I(\text{in}_<(H))$ has a linear resolution. Hence, by Fröberg's theorem, [4, Theorem 1], the complementary graph of $\text{in}_<(H)$ is chordal. On the other hand, all maximal cliques of G , say $H_1, \dots, H_{c(G)}$, cover all edges of G . So, clearly, $\text{in}_<(H_1), \dots, \text{in}_<(H_{c(G)})$ cover all edges of $\text{in}_<(G)$. Thus, by definition, we have $\text{cochord}(\text{in}_<(G)) \leq c(G)$. Hence, $\text{reg}(S/I(\text{in}_<(K_m, G))) \leq \binom{m}{2} c(G)$.

Therefore, by the above two cases, we have

$$\text{reg}(I(\text{in}_<(K_m, G))) = \text{reg}(S/I(\text{in}_<(K_m, G))) + 1 \leq \min \left\{ \binom{m}{2} c(G), e(G) \right\} + 1,$$

and hence the desired result follows, since $\text{reg}(J_{K_m, G}) \leq \text{reg}(\text{in}_<(J_{K_m, G}))$, by [5, Corollary 3.3.4]. $\square$

Note that, by setting $m = 2$, one could see that Theorem 13 yields the result of [19] on the regularity of the binomial edge ideal of a graph (see [19, Theorem 3.2]).

Corollary 14. *Let G be a closed graph on $[n]$ and $m, n \geq 2$. Then $\beta_{i,2i}(J_{K_m, G}) = 0$, for all $i > \min \left\{ \binom{m}{2} c(G), e(G) \right\} + 1$. In particular, we have $\beta_{i,2i}(J_G) = 0$, for all $i > c(G) + 1$.*

Proof. Note that $\text{reg}(J_{K_m, G}) = \max \{j - i : \beta_{i,j}(J_{K_m, G}) \neq 0\}$. So, if there exists some $i > \min \left\{ \binom{m}{2} c(G), e(G) \right\} + 1$ with $\beta_{i,2i}(J_{K_m, G}) \neq 0$, then we have $\text{reg}(J_{K_m, G}) \geq i$. Hence, $\text{reg}(J_{K_m, G}) > \min \left\{ \binom{m}{2} c(G), e(G) \right\} + 1$, which is a contradiction, by Theorem 13. For the second part, it is enough to set $m = 2$. $\square$

The following corollary shows that the upper bound posed in Theorem 13 is sharp.

Corollary 15. *Let P_n be the path of length $n - 1$ and $m \geq 2$ be an integer. Then we have $\text{reg}(J_{K_m, P_n}) = n$. In particular, the regularity of J_{K_m, P_n} does not depend on m .*

Proof. Since P_n is a closed graph, we can apply Theorem 13. So that we have $\text{reg}(J_{K_m, P_n}) \leq \min\left\{\binom{m}{2}c(P_n), e(P_n)\right\} + 1 = \min\left\{\binom{m}{2}(n - 1), (n - 1)\right\} + 1 = n$. On the other hand, by Corollary 9, part (b), we have $\text{reg}(J_{P_n}) \leq \text{reg}(J_{K_m, P_n})$. But, one has $\text{reg}(J_{P_n}) = n$ (see [19, Remark 3.3]). Therefore, $\text{reg}(J_{K_m, P_n}) = n$, as desired. $\square$

Remark 16. The bound for the regularity in Theorem 13 might be strict. For instance, by using *CoCoA*, one can see that $\text{reg}(J_{K_3, K_3}) = 3$, but Theorem 13 gives 4 as an upper bound.

The following corollary gives a lower bound for the regularity of the binomial edge ideal of a pair of graphs.

Corollary 17. *Let G_1 and G_2 be two graphs on $[m]$ and $[n]$, respectively. If $p_1 - 1$ and $p_2 - 1$ are the lengths of the longest induced paths in G_1 and G_2 , respectively, then we have $\text{reg}(J_{G_1, G_2}) \geq \max\{p_1, p_2\}$.*

Proof. It is enough to apply Proposition 8 and Corollary 15. $\square$

Corollary 18. *Let I be the ideal of adjacent 2-minors of an $m \times n$ generic matrix with $m \leq n$. Then we have $\text{reg}(I) \geq n$.*

We end this section by the following question about an upper bound for the regularity in a more general case, without the assumption of closedness:

Question. Let G be a graph. Is it true that

$$\text{reg}(J_{K_m, G}) \leq \min\left\{\binom{m}{2}c(G), e(G)\right\} + 1?$$

In particular, by setting $m = 2$, is it true that $\text{reg}(J_G) \leq c(G) + 1$?

Note that the latter bound is true if G is a tree, as it was shown in [12] that for any graph G on n vertices, one has $\text{reg}(J_G) \leq n$.

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