Some remarks on the joint distribution of descents and inverse descents

Mirkó Visontai

Department of Mathematics
Kungl. Tekniska Högskolan
S-100 44 Stockholm, Sweden

visontai@kth.se

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In memoriam Herb Wilf

Abstract

We study the joint distribution of descents and inverse descents over the set of permutations of n letters. Gessel conjectured that the two-variable generating function of this distribution can be expanded in a given basis with nonnegative integer coefficients. We investigate the action of the Eulerian operators that give the recurrence for these generating functions. As a result we devise a recurrence for the coefficients in question but are unable to settle the conjecture.

We examine generalizations of the conjecture and obtain a type B analog of the recurrence satisfied by the two-variable generating function. We also exhibit some connections to cyclic descents and cyclic inverse descents. Finally, we propose a combinatorial model for the joint distribution of descents and inverse descents in terms of statistics on inversion sequences.

Keywords: Permutations, descents, inverse descents, Eulerian numbers

1 Introduction

Let $\mathfrak{S}_n$ denote the set of permutations of $\{1, \ldots, n\}$. The number of *descents* in a permutation $\pi = \pi_1 \cdots \pi_n$ is defined as $\operatorname{des}(\pi) = |\{i : \pi_i > \pi_{i+1}\}|$. Our object of study is the two-variable generating function of descents and *inverse descents*:

$$A_n(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi^{-1})+1} t^{\operatorname{des}(\pi)+1} .$$

The specialization of this polynomial to a single variable reduces to the classical *Eulerian polynomial*:

$$A_n(t) = A_n(1,t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle t^k \,.$$

Eulerian polynomials and their coefficients play an important role (not only) in enumerative combinatorics. The univariate polynomials, $A_n(t)$, are quite well-studied—see, for example, [3, 6] and references therein. This cannot be said for the bivariate generating function for the pair of statistics $(\mathrm{des}, \mathrm{ides})$. Here and throughout this note we will use the shorthand $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$.

Our main motivation to study these bivariate polynomials is the following conjecture of Gessel which appeared in a recent article by Brändén [2]; see also a nice exposition by Petersen [15].

Conjecture 1 (Gessel)**.** For all $n \geqslant 1$,

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^j (1+st)^{n+1-j-2i} \,,$$

where $\gamma_{n,i,j}$ are nonnegative integers for all $i, j \in \mathbb{N}$.

If true, the above decomposition would refine the following classical result, the γ-*nonnegativity* for the Eulerian polynomials $A_n(t)$. (For background on γ-*nonnegativity* we refer the reader to the works of Brändén [1] and Gal [9].)

Theorem 2 (Théorème 5.6 of [6])**.**

$$A_n(t) = \sum_{i=1}^{\lceil n/2 \rceil} \gamma_{n,i} t^i (1+t)^{n+1-2i} \,,$$

where $\gamma_{n,i}$ are nonnegative integers for all $i \in \mathbb{N}$.

Before giving their proof, let us recall the recurrence satisfied by the Eulerian polynomials:

$$A_n(t) = ntA_{n-1}(t) + t(1-t)\frac{\partial}{\partial t} A_{n-1}(t) \,, \qquad \text{for } n \geqslant 2, \tag{1}$$

with initial value $A_1(t) = t$.

Foata and Schützenberger [6, Chapitre V] give a purely algebraic proof of Theorem 2 by considering the *homogenized* Eulerian polynomial, of degree $n+1$,

$$\begin{aligned} A_n(t;y) &= y^{n+1} A_n(t/y) \\ &= \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} y^{\mathrm{asc}(\pi)+1} \,, \end{aligned} \tag{2}$$

where $\mathrm{asc}(\pi)$ denotes the number of ascents ($\pi_i < \pi_{i+1}$) in the permutation $\pi = \pi_1 \cdots \pi_n$. Note that this polynomial is different from and therefore should not be confused with $A_n(s,t)$. To avoid confusion we use a semicolon and different variables. We include their proof next, as we will be applying the same idea to the joint generating polynomial of descents and inverse descents in Section 3.

Proof of Theorem 2. The homogenized Eulerian polynomials defined in (2) satisfy the recurrence

$$A_n(t;y) = ty\left(\frac{\partial}{\partial t}A_{n-1}(t;y) + \frac{\partial}{\partial y}A_{n-1}(t;y)\right), \quad \text{for } n \geqslant 2\,, \tag{3}$$

which follows from observing the effect on the number of descents and ascents of inserting the letter n into a permutation of $\{1,\ldots,n-1\}$. Compare this with recurrence (1).

It is clear from symmetry observations and homogeneity that $A_n(t;y)$ can be written (uniquely) in the basis

$$\left\{(ty)^i(t+y)^{n+1-2i} : i = 1,\ldots,\lceil n/2\rceil\right\}$$

with some coefficients $\gamma_{n,i}$. To show that $\gamma_{n,i}$ are in fact nonnegative integers consider the action of the recurrence operator $T = ty\,(\partial/\partial t + \partial/\partial y)$ on a basis element. Applying T on the ith basis element we get that

$$T\left[(ty)^i(t+y)^{n+1-2i}\right] = i(ty)^i(t+y)^{n+2-2i} + 2(n+1-2i)(ty)^{i+1}(t+y)^{n-2i},$$

which in turn implies the following recurrence on the coefficients:

$$\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1}. \tag{4}$$

The statement of Theorem 2 now follows, since the initial values are nonnegative integers, in particular, $\gamma_{1,1} = 1$ and $\gamma_{1,i} = 0$ for $i \neq 1$. Furthermore, the constraint $1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil$ assures that both positivity and integrality are preserved by recurrence (4). □

Remark 3. The study of these so-called Eulerian operators goes back to Carlitz as it was pointed out to the author by I. Gessel. See [4] for a slightly different variant of T. Also, the operator $t\,(n + (1-t)(\partial/\partial t))$ is closely related to a special case of a generalized derivative operator already studied by Laguerre, called *émanant* or polar derivative; see, for example, Section 6 in [13].

Finally, we must also mention the "valley-hopping" proof of Theorem 2 by Shapiro, Woan, Getu [17, Proposition 4] which is a beautiful construction that proves that the coefficients $\gamma_{n,i}$ are not only nonnegative integers but that they are, in fact, cardinalities of certain equivalence classes of permutations. Their proof is part of a more general phenomenon, an action of transformation groups on the symmetric group $\mathfrak{S}_n$ studied by Foata and Strehl [7].

2 A homogeneous recurrence

The polynomials $A_n(s,t)$ were first studied by Carlitz, Roselle, and Scoville [5]. They proved a recurrence for the coefficients of $A_n(s,t)$ (see equation (7.8) in their article—note there is an obvious typo in the last row of the equation, cf. equation (7.7) in the same article). The recurrence they provide for the coefficients is equivalent to the following one for the generating functions.

Theorem 4 (Equation (9) of [15])**.** *For $n \geqslant 2$,*

$$\begin{aligned} nA_n(s,t) = &\left(n^2st + (n-1)(1-s)(1-t)\right) A_{n-1}(s,t) \\ &+ nst(1-s)\frac{\partial}{\partial s}A_{n-1}(s,t) + nst(1-t)\frac{\partial}{\partial t}A_{n-1}(s,t) \\ &+ st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}A_{n-1}(s,t)\,, \end{aligned}$$

with initial value $A_1(s,t) = st$.

At first glance, this recurrence might not seem very useful at all. However, if we introduce additional variables—to count ascents (asc) and inverse ascents (iasc)—we obtain a more transparent recurrence. So, let us first define

$$A_n(s,t;x,y) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{ides}(\pi)+1} t^{\mathrm{des}(\pi)+1} x^{\mathrm{iasc}(\pi)+1} y^{\mathrm{asc}(\pi)+1}\,.$$

Proposition 5. *The polynomial $A_n(s,t;x,y)$ is homogeneous of degree $2n+2$ and is invariant under the action of the Klein 4-group $V \cong \langle \mathrm{id}, (12)(34), (13)(24), (14)(23) \rangle$, where the action of $\sigma \in V$ on $A_n(s,t;x,y)$ is permutation of the variables accordingly (for example, $\sigma = (13)(24)$ swaps x with s and y with t, simultaneously).*

Proof. The homogeneity is immediate from

$$\begin{aligned} A_n(s,t;x,y) &= \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{ides}(\pi)+1} t^{\mathrm{des}(\pi)+1} x^{n-\mathrm{ides}(\pi)} y^{n-\mathrm{des}(\pi)} \\ &= (xy)^{n+1} A_n(s/x, t/y)\,. \end{aligned}$$

The invariance is a consequence of the symmetry properties of $A_n(s,t)$, such as $A_n(s,t) = A_n(t,s)$; see, for example, equations (12–14) in [15]. Note that, due to the introduction of the new variables, for $n \geqslant 4$, the polynomial $A_n(s,t;x,y)$ is *not* symmetric. ☐

Now we are in position to give our homogeneous recurrence.

Theorem 6. *For $n \geqslant 2$,*

$$\begin{aligned} nA_n(s,t;x,y) = &(n-1)(s-x)(t-y)A_{n-1}(s,t;x,y) \\ &+ stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) A_{n-1}(s,t;x,y) \end{aligned} \tag{5}$$

with initial value $A_1(s,t;x,y) = stxy$.

Proof. Consider the bivariate recurrence given in Theorem 4 and observe that it can be rewritten as

$$nA_n(s,t) = \left((n-1)(1-s)(1-t) + st\left(n + (1-s)\frac{\partial}{\partial s}\right)\left(n + (1-t)\frac{\partial}{\partial t}\right)\right) A_{n-1}(s,t).$$

Let τ_{n-1} denote the operator on the right-hand side, that is, $nA_n(s,t) = \tau_{n-1}[A_{n-1}(s,t)]$. Similarly, let

$$T_n = n(s-x)(t-y) + stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right). \tag{6}$$

Finally, let h_n denote the homogenization operator which maps the monomial $s^a t^b$ to the monomial $(s/x)^a (t/y)^b (xy)^n$. In order to prove the theorem, it suffices to show that the action of the operators τ_{n-1} and T_{n-1} agrees on the corresponding monomials, that is,

$$T_{n-1}\left(h_n\left[s^a t^b\right]\right) = h_{n+1}\left[\tau_{n-1}(s^a t^b)\right].$$

For the multiplicative part we have that

$$\begin{aligned}(n-1)(x-s)(y-t)h_n\left[s^a t^b\right] &= (n-1)(x-s)(y-t)s^a t^b x^{n-a} y^{n-b}\\ &= (n-1)\left(1-\frac{s}{x}\right)\left(1-\frac{t}{y}\right)\left(\frac{s}{x}\right)^a\left(\frac{t}{y}\right)^b (xy)^{n+1}\\ &= h_{n+1}\left[(n-1)(1-s)(1-t)s^a t^b\right].\end{aligned}$$

And for the differential part we have

$$stxy\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)s^a t^b x^{n-a} y^{n-b} = sx\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)s^a x^{n-a}\, ty\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)t^b y^{n-b}$$

and as it was already observed in [6]:

$$\begin{aligned}sx\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)s^a x^{n-a} &= as^a x^{n+1-a} + (n-a)s^{a+1}x^{n-a}\\ &= h_{n+1}\left[(a + s(n-a))\, s^a\right]\\ &= h_{n+1}\left[s\left(n + (1-s)\partial/\partial s\right) s^a\right].\end{aligned}$$

□

Remark 7. The invariance of $A_n(s,t;x,y)$ under the Klein-group action also follows easily from recurrence (5) directly. Clearly, $A_1(s,t;x,y) = stxy$ is invariant under the action of the group (in fact, it is symmetric) and also T_n—the operator acting on $A_n(s,t;x,y)$—is invariant under the action of the Klein-group.

Finally, Theorem 6 allows for a (homogenized) restatement of Gessel's conjecture:

Conjecture 8.

$$A_n(s,t;x,y) = \sum_{i,j} \gamma_{n,i,j}(stxy)^i (st+xy)^j (tx+sy)^{n+1-2i-j},$$

where $\gamma_{n,i,j} \in \mathbb{N}$ for all $i,j \in \mathbb{N}$.

For example, we have (cf. page 18 of [15]):

$$\begin{aligned}
A_1(s,t;x,y) &= stxy \\
A_2(s,t;x,y) &= stxy(st+xy) \\
A_3(s,t;x,y) &= stxy(st+xy)^2 + 2(stxy)^2 \\
A_4(s,t;x,y) &= stxy(st+xy)^3 + 7(stxy)^2(st+xy) + (stxy)^2(tx+sy) \\
A_5(s,t;x,y) &= stxy(st+xy)^4 + 16(stxy)^2(st+xy)^2 + 6(stxy)^2(st+xy)(tx+sy) \\
&\quad + 16(stxy)^3
\end{aligned}$$

Remark 9. It is not too hard to see that Theorem 6 is, in fact, equivalent to Theorem 4. At the same time, the symmetric nature of the homogeneous operator is more suggestive to combinatorial interpretation. It would be nice to find such an interpretation (perhaps in terms of non-attacking rook placements on a rectangular board).

3 A recurrence for the coefficients $\gamma_{n,i,j}$

Following the ideas in [6, Chapitre V] that were used to devise a recurrence for $\gamma_{n,i}$, we apply the operator T_n to the basis elements to obtain a recurrence for the coefficients $\gamma_{n,i,j}$. As a result, we obtain the following recurrence.

Theorem 10. *Let $n \geqslant 1$. For all $i \geqslant 1$ and $j \geqslant 0$, we have*

$$\begin{aligned}
(n+1)\gamma_{n+1,i,j} &= (n+i(n+2-i-j))\gamma_{n,i,j-1} + (i(i+j)-n)\gamma_{n,i,j} \\
&\quad + (n+4-2i-j)(n+3-2i-j)\gamma_{n,i-1,j-1} \\
&\quad + (n+2i+j)(n+3-2i-j)\gamma_{n,i-1,j} \\
&\quad + (j+1)(2n+2-j)\gamma_{n,i-1,j+1} + (j+1)(j+2)\gamma_{n,i-1,j+2},
\end{aligned} \tag{7}$$

with $\gamma_{1,1,0} = 1$, $\gamma_{1,i,j} = 0$ (unless $i = 1$ and $j = 0$) and $\gamma_{n,i,j} = 0$ if $i < 1$ or $j < 0$.

Proof. Denote the basis elements by $B_{i,j}^{(n)} = (stxy)^i(st+xy)^j(tx+sy)^{n+1-2i-j}$ for convenience, and recall the definition of T_n given in (6).

A quick calculation shows that

$$n(s-x)(t-y)B_{i,j}^{(n)} = n\left(B_{i,j+1}^{(n+1)} - B_{i,j}^{(n+1)}\right). \tag{8}$$

To calculate the action of the differential operators on the basis elements, we use the product rule. After some calculations, this gives the following:

$$\begin{aligned}
stxy\left(\frac{\partial^2}{\partial s\partial t} + \frac{\partial^2}{\partial x\partial y}\right)B_{i,j}^{(n)} &= i(n+1-i-j)B_{i,j+1}^{(n+1)} + j(2n+3-j)B_{i+1,j-1}^{(n+1)} \\
&\quad + (n+1-2i-j)(n-2i-j)B_{i+1,j+1}^{(n+1)}.
\end{aligned} \tag{9}$$

$$
\begin{aligned}
stxy\left(\frac{\partial^2}{\partial s\partial y}+\frac{\partial^2}{\partial t\partial x}\right)B_{i,j}^{(n)} &= i(i+j)B_{i,j}^{(n+1)}+j(j-1)B_{i+1,j-2}^{(n+1)} \\
&\quad + (n+1-2i-j)(n+2+2i+j)B_{i+1,j}^{(n+1)}\,.
\end{aligned}
\tag{10}
$$

Summing (8), (9) and (10) we arrive at the following expression.

$$
\begin{aligned}
T_n[B_{i,j}^{(n)}] &= (n+i(n+1-i-j))B_{i,j+1}^{(n+1)}+(i(i+j)-n)B_{i,j}^{(n+1)} \\
&\quad + (n+1-2i-j)(n-2i-j)B_{i+1,j+1}^{(n+1)} \\
&\quad + (n+2+2i+j)(n+1-2i-j)B_{i+1,j}^{(n+1)} \\
&\quad + j(2n+3-j)B_{i+1,j-1}^{(n+1)}+j(j-1)B_{i+1,j-2}^{(n+1)}\,.
\end{aligned}
$$

Finally, collecting together all terms $T_n[B_{k,\ell}^{(n)}]$ which contribute to $B_{i,j}^{(n+1)}$ we obtain (7). □

Remark 11. If we sum up both sides of (7) for all possible j then we get (4) back.

One could study the generating function

$$
G(u,v,w)=\sum_{i,j}\gamma_{n,i,j}u^n v^i w^j
$$

with coefficients satisfying the above recurrence. Gessel's conjecture is equivalent to saying that its coefficients are nonnegative integers. Unfortunately, these properties are not immediate from the recurrence (7): note that the left-hand side has a multiplicative factor of $(n+1)$ and the coefficient $(i(i+j)-n)$ may assume negative values.

4 Generalizations of the conjecture

Gessel [10] noted that the following equality of Carlitz, Roselle, and Scovelle [5]

$$
\sum_{i,j=0}^{\infty}\binom{ij+n-1}{n}s^i t^j=\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}
$$

can be generalized as follows.

Let $\tau\in\mathfrak{S}_n$ with $\operatorname{des}(\tau)=k-1$. Define $A_n^{(k)}(s,t)$ by

$$
\sum_{i,j=0}^{\infty}\binom{ij+n-k}{n}s^i t^j=\frac{A_n^{(k)}(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}.
$$

The coefficient of $s^i t^j$ in $A_n^{(k)}(s,t)$ is the number of pairs of permutations (π,σ) such that $\pi\sigma=\tau$, $\operatorname{des}(\pi)=i$ and $\operatorname{des}(\sigma)=j$. Gessel [10] also pointed out that these generalized polynomials arise implicitly in [14]; compare (11.10) there with the above equation.

This suggests that Conjecture 1 holds in a more general form (this version of the conjecture appeared as Conjecture 10.2 in [2]).

Conjecture 12 (Gessel)**.** Let $\tau \in \mathfrak{S}_n$. Then

$$\sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi)+1} t^{\operatorname{des}(\pi^{-1}\tau)+1} = \sum_{i,j} \gamma^{\tau}_{n,i,j} (st)^i (s+t)^j (1+st)^{n+1-j-2i},$$

where $\gamma^{\tau}_{n,i,j}$ are nonnegative integers for all $i, j \in \mathbb{N}$. Furthermore, the coefficients $\gamma^{\tau}_{n,i,j}$ do not depend on the actual permutation τ, only on the number of descents in τ.

In the special case when $\tau = n(n-1)\cdots 21$ (and hence $\operatorname{des}(\tau) = n-1$) the roles of descents and ascents interchange.

Theorem 13. *For $n \geqslant 2$,*

$$\begin{aligned} nA^{(n)}_n(s,t;x,y) =& (n-1)(x-s)(t-y)A^{(n-1)}_{n-1}(s,t;x,y) \\ & + stxy \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right) A^{(n-1)}_{n-1}(s,t;x,y) \end{aligned} \tag{11}$$

with initial value $A^{(1)}_1(s,t;x,y) = stxy$.

In particular, we have the following identity.

Corollary 14.

$$A^{(n)}_n(s,t;x,y) = A_n(s,y;x,t).$$

4.1 A type B analog

Gessel [10] also noted that there is an analogous definition for the hyperoctahedral group $\mathfrak{B}_n$. The elements of $\mathfrak{B}_n$ can be thought of as signed permutations of $\{1,\dots,n\}$, and the type B descent statistic is defined as $\operatorname{des}_B(\sigma) = |\{i \in \{0,1,\dots,n\} : \sigma_i > \sigma_{i+1}\}|$ with $\sigma_0 := 0$ for $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{B}_n$. Letting

$$B^{(k)}_n(s,t) = \sum_{\sigma \in \mathfrak{B}_n} s^{\operatorname{des}_B(\sigma)} t^{\operatorname{des}_B(\sigma^{-1}\tau)},$$

with $\tau \in \mathfrak{B}_n$ such that $\operatorname{des}_B(\tau) = k-1$, we have that

$$\sum_{i,j=0}^{\infty} \binom{2ij+i+j+1+n-k}{n} s^i t^j = \frac{B^{(k)}_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}.$$

Therefore, mimicking the proof of Theorem 4 given by Petersen, we get an analog of Theorem 4 for the type B two-sided Eulerian polynomials, $B_n(s,t) = B^{(1)}_n(s,t)$.

Theorem 15. *For $n \geqslant 2$,*

$$\begin{aligned} nB_n(s,t) =& (2n^2st - nst + n)B_{n-1}(s,t) \\ & + (2nst(1-s) + s(1-s)(1-t))\frac{\partial}{\partial s}B_{n-1}(s,t) \\ & + (2nst(1-t) + t(1-s)(1-t))\frac{\partial}{\partial t}B_{n-1}(s,t) \\ & + 2st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}B_{n-1}(s,t). \end{aligned} \tag{12}$$

with initial value $B_1(s,t) = 1 + st$*.*

Proof. As in the case of the symmetric group in [15, eq. (9)], we start with the corresponding identity of binomial coefficients:

$$n\binom{2ij+i+j+n}{n} = (2ij+i+j)\binom{2ij+i+j+n-1}{n-1} + n\binom{2ij+i+j+n-1}{n-1}.$$

Multiplying both sides by the monomial $s^i t^j$ and summing over all integers i,j we get

$$\sum_{i,j=0}^{\infty} n\binom{2ij+i+j+n}{n} s^i t^j =$$
$$\sum_{i,j=0}^{\infty} (2ij+i+j)\binom{2ij+i+j+n-1}{n-1} s^i t^j + \sum_{i,j=0}^{\infty} n\binom{2ij+i+j+n-1}{n-1} s^i t^j\,,$$

from which we obtain the following recurrence for $F_n(s,t) = B_n(s,t)/(1-s)^{n+1}(1-t)^{n+1}$:

$$nF_n(s,t) = 2st\frac{\partial^2}{\partial s \partial t}F_{n-1}(s,t) + s\frac{\partial}{\partial s}F_{n-1}(s,t) + t\frac{\partial}{\partial t}F_{n-1}(s,t) + nF_{n-1}(s,t)\,.$$

Substitute back the expression for $F_n(s,t)$, multiply both sides with $(1-s)^{n+1}(1-t)^{n+1}$ and with a little work we get that

$$\begin{aligned} nB_n(s,t) =& (2n^2 st + nt(1-s) + ns(1-t) + n(1-s)(1-t))B_{n-1}(s,t) \\ &+ (2nst(1-s) + s(1-s)(1-t))\frac{\partial}{\partial s}B_{n-1}(s,t) \\ &+ (2nst(1-t) + t(1-s)(1-t))\frac{\partial}{\partial t}B_{n-1}(s,t) \\ &+ 2st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}B_{n-1}(s,t)\,. \end{aligned}$$

□

It would be of interest to find a homogeneous version of this theorem (an analogue of Theorem 6) and a recurrence for the corresponding $\gamma_{n,i,j}$ coefficients for type B.

4.2 Cyclic descents

One can also consider two-sided Eulerian-like polynomials using cyclic descents. A *cyclic descent* of a permutation π in $\mathfrak{S}_n$ is defined as

$$\operatorname{cdes}(\pi) = |\{i : \pi_i > \pi_{(i+1) \bmod n}\}| = \operatorname{des}(\pi) + \chi(\pi_n > \pi_1)\,,$$

where

$$\chi(a > b) = \begin{cases} 1, \text{ if } a > b, \text{ and} \\ 0, \text{ otherwise.} \end{cases}$$

The following theorem refines a (univariate) result of Fulman [8, Corollary 1].

Theorem 16. *For $n \geqslant 1$,*

$$(n+1)A_n(s,t) = \sum_{\pi \in \mathfrak{S}_{n+1}} s^{\mathrm{cdes}(\pi^{-1})} t^{\mathrm{cdes}(\pi)}.$$

Lemma 17. *Let $\sigma = 23 \cdots n1$ denote the cyclic rotation in $\mathfrak{S}_n$ (for $n \geqslant 2$). Then*

$$(\mathrm{cdes}(\pi), \mathrm{cdes}(\pi^{-1})) = (\mathrm{cdes}(\pi\sigma), \mathrm{cdes}((\pi\sigma)^{-1})).$$

In other words, the cyclic rotation simultaneously preserves the cyclic descent and the cyclic inverse descent stastics.

Remark 18. Lemma 17 is essentially the same as Theorem 6.5 in [11]. We give an elementary proof of it, for the sake of completeness.

Proof. The part that $\mathrm{cdes}(\pi) = \mathrm{cdes}(\pi\sigma)$ is obvious since cyclical rotation does not effect the cyclic descent set. For the other part, it is equivalent to show that $\mathrm{cdes}(\pi) = \mathrm{cdes}(\sigma^{-1}\pi)$. In other words, the cyclic descent statistic is invariant under the operation when we cyclically shift the values of a permutation, i.e., add 1 to each entry modulo n. For $\pi = \pi_1 \cdots \pi_n$ an arbitrary permutation in $\mathfrak{S}_n$ denote the entry preceding n and following n by a and b, respectively. Then $\pi = \pi_1 \cdots anb \cdots \pi_n$ and $\sigma^{-1}\pi = (\pi_1 + 1) \cdots (a+1)1(b+1) \cdots (\pi_n + 1)$. Clearly, in all but one position the cyclic descents are preserved, and the same is true for the cyclic ascents. The $a \nearrow n$ cyclic ascent is replaced by the $(a+1) \searrow 1$ cyclic descent and similarly, $n \searrow b$ gets replaced by $1 \nearrow (b+1)$. Thus, the total number of cyclic descents remains the same. $\square$

Proof of Theorem 16. Using Lemma 17 we can apply the cyclic rotation to any permutation in $\mathfrak{S}_{n+1}$ until $\pi_{n+1} = n+1$. This will map exactly $n+1$ permutations in $\mathfrak{S}_{n+1}$ to the same permutation $\pi_1 \cdots \pi_n(n+1)$. Clearly, $\mathrm{cdes}(\pi_1 \cdots \pi_n(n+1)) = \mathrm{des}(\pi_1 \cdots \pi_n) + 1$ and $\mathrm{cdes}((\pi_1 \cdots \pi_n(n+1))^{-1}) = \mathrm{des}((\pi_1 \cdots \pi_n)^{-1}) + 1$ and the theorem follows. $\square$

5 Connection to inversion sequences

We conclude by proposing a combinatorial model for the joint distribution of descents and inverse descents.

A permutation $\pi \in \mathfrak{S}_n$ can be encoded as its inversion sequence $e = (e_1, \ldots, e_n)$, where

$$e_j = |\{i : i < j, \pi_i > \pi_j\}|.$$

Let $I_n = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n : 0 \leqslant e_i \leqslant i-1\}$ denote the set of inversion sequences for $\mathfrak{S}_n$.

Recently, Savage and Schuster [16] studied the *ascent* statistic

$$\mathrm{asc}_I(e) = |\{i : e_i < e_{i+1}\}|$$

for inversion sequences (and their generalizations) and showed that this statistic is *Eulerian*, i.e., it is equidistributed with the descent statistic over permutations. We use the subscript I to emphasize that this is a statistic for inversion sequences which is different from the ascent statistic for permutations used earlier in the paper.

Mantaci and Rakotondrajao [12] also studied this representation of permutations under the name "subexceedant functions". They considered the statistic

$$\mathrm{dst}(e) = |\{e_i : 1 \leqslant i \leqslant n\}|$$

that counts the distinct entries in $e \in I_n$, and gave multiple proofs of the following observation (which they attributed to Dumont) that this statistic is also Eulerian.

Proposition 19 (Dumont)**.**

$$A_n(x) = \sum_{e \in I_n} x^{\mathrm{dst}(e)}\,.$$

In fact, the joint distribution $(\mathrm{asc}_I, \mathrm{dst} - 1)$ over inversion sequences seems to agree with the joint distribution $(\mathrm{des}, \mathrm{ides})$ of descents and inverse descents over permutations.

Conjecture 20.

$$A_n(s,t) = \sum_{e \in I_n} s^{\mathrm{dst}(e)} t^{\mathrm{asc}_I(e)+1}\,.$$

This observation clearly deserves a bijective proof. Such a proof might shed light on a combinatorial proof of recurrence (5). Note that it is not even clear to begin with why the right-hand side should be a symmetric polynomial in variables s and t.

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