A family of half-transitive graphs

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Abstract

We construct an infinite family of half-transitive graphs, which contains infinitely many Cayley graphs, and infinitely many non-Cayley graphs.

Keywords: half-transitive graphs; quotient graphs; automorphism groups.

1 Introduction

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. A permutation of V which preserves the adjacency of Γ is an automorphism of the graph, and all automorphisms form the automorphism group $\mathsf{Aut}\Gamma$. If a subgroup $G \leqslant \mathsf{Aut}\Gamma$ is transitive on V or E, then Γ is called *G-vertex-transitive* or *G-edge-transitive*, respectively. An ordered pair of adjacent vertices is called an *arc*, and Γ is called *arc-transitive* if $\mathsf{Aut}\Gamma$ is transitive on the set of arcs. An arc-transitive graph is vertex-transitive and edge-transitive, but the converse statement is not true. A graph which is vertex-transitive and edge-transitive but not arc-transitive is called *half-transitive*.

The study of half-transitive graphs was initiated with a question of Tutte [20, p. 60] regarding their existence, and he proved that a vertex-transitive and edge-transitive graph

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with odd valency must be arc-transitive. Bouwer [4] in 1970 constructed the first family of half-transitive graphs. Since then, constructing and characterizing half-transitive graphs has been an active topic in algebraic graph theory, refer to [1, 2, 10, 13, 16, 22] and a survey [12] for the work during 1990's, and [14, 16, 17, 18, 19] for more recent work.

In this paper, we present an infinite family of half-transitive graphs. These graphs were originated from *Johnson graphs* $\mathbf{J}(n,i)$ where $i = 1$, 2 or 3, the vertex set of which consists of the i-element subsets of an n-element set such that two vertices are adjacent when they meet in $(i-1)$-elements. It is known that the automorphism group of $\mathbf{J}(n,i)$ is S_n, see [9] or [15, Theorem 1].

As usual, we denote by $[n]$ the set $\{1,2,3,\ldots,n\}$. Let

$$V_n = \{\{\{i,j\},k\} \mid i,j,k \in [n]\}.$$

For convenience, we simply write the vertex $\{\{i,j\},k\}$ as (ij,k). Then $(ij,k) = (ji,k)$, and a 3-subset $\{i,j,k\}$ corresponds to exactly three vertices (ij,k), (ik,j) and (jk,i). Thus, the cardinality is

$$|V_n| = 3\binom{n}{3} = n(n-1)(n-2)/2.$$

Definition 1. For an integer $n > 3$, let $\mathit{\Gamma}_n$ be the graph with vertex set V_n such that two vertices (ij,k) and $(i'j',k') \in V_n$ are adjacent if and only if

$$\{i,j\} = \{i',k'\} \text{ or } \{j',k'\}$$

and $\{i,j,k\} \neq \{i',j',k'\}$.

The graph $\mathit{\Gamma}_n$ is regular and has valency $4(n-3)$. For example, the vertex $(12,3)$ has neighborhood

$$\{(1i,2),(2i,1),(13,i),(23,i) \mid i > 3\},$$

which has size $4(n-3)$.

Let $\mathit{\Gamma} = (V,E)$ be a graph, and let $\mathcal{B}$ be a partition of the vertex set V. Then the *quotient graph* $\mathit{\Gamma}_{\mathcal{B}}$ induced on $\mathcal{B}$ is the graph with vertex set $\mathcal{B}$ such that B, B' are adjacent if and only if there is an edge which lies between B and B'. In this case, $\mathit{\Gamma}$ is also said to be *homomorphic* to $\mathit{\Gamma}_{\mathcal{B}}$.

A graph $\mathit{\Gamma} = (V,E)$ is called a *Cayley graph* if there is a group R and a self-inversed subset $S \subset R$ such that $V = R$ and $u,v \in S$ are adjacent if and only if $vu^{-1} \in S$. Cayley graphs are vertex-transitive, but a vertex-transitive graph is not necessarily a Cayley graph. For example, the Petersen graph is the smallest vertex-transitive graph which is not a Cayley graph. The family of graphs $\mathit{\Gamma}_n$ contains infinitely many non-Cayley graphs.

Theorem 2. *Let $\mathit{\Gamma}_n$ be a graph defined above. Then the following statements hold:*

(i) $\mathit{\Gamma}_n$ is of order $\frac{n(n-1)(n-2)}{2}$, valency $4(n-3)$, girth 3, and diameter 3;

(ii) $\mathit{\Gamma}_n$ is homomorphic to $\mathbf{K}_n$, $\mathbf{J}(n,2)$ and $\mathbf{J}(n,3)$;

(iii) *Γ_4 and Γ_5 are arc-transitive, and for $n \geqslant 6$, $\mathsf{Aut}\Gamma_n = \mathrm{Sym}([n])$, and Γ_n is half-transitive;*

(iv) *Γ_n is a Cayley graph if and only if $n = 8$, or $n = q + 1$ with q a prime-power, and $q \equiv 3 \pmod 4$.*

2 Edge-transitivity

Let $\sigma \in \mathrm{Sym}([n])$. For convenience, we simply denote V_n by V, and denote Γ_n by Γ. Then σ induces a permutation on the vertex set V. Since $G = \mathrm{Sym}([n])$ is 3-transitive on $[n]$, G is transitive on V.

Take an edge $\{(i_1 j_1, k_1), (i_2 j_2, k_2)\}$ of Γ. Then $\{i_1, j_1\} = \{i_2, k_2\}$ or $\{j_2, k_2\}$, and hence $\{i_1^\sigma, j_1^\sigma\} = \{i_2^\sigma, k_2^\sigma\}$ or $\{j_2^\sigma, k_2^\sigma\}$. Thus, $(i_1^\sigma j_1^\sigma, k_1^\sigma)$ and $(i_2^\sigma j_2^\sigma, k_2^\sigma)$ are adjacent, that is, σ maps edges to edges. Similarly, σ maps non-edges to non-edges. So σ is an automorphism of Γ, and $G = \mathrm{Sym}([n])$ is a vertex-transitive automorphism group of Γ.

Lemma 3. *The graph $\Gamma = \Gamma_n$ is G-vertex-transitive and G-edge-transitive, but not G-arc-transitive.*

Proof. We consider the edges incident with the vertex $\alpha = (12, 3)$. The stabilizer $G_\alpha = \mathrm{Sym}(\{1, 2\}) \times \mathrm{Sym}(\{4, 5, \ldots, n\}) \cong \mathrm{S}_2 \times \mathrm{S}_{n-3}$, and G_α acting on the neighborhood $\Gamma(\alpha)$ has two orbits $\{(1i, 2), (2i, 1) \mid i > 3\}$ and $\{(13, i), (23, i) \mid i > 3\}$. Thus, G is not transitive on the arcs of Γ. Further, the element $g = (23i)$ maps $(1i, 2)$ to $(12, 3)$, and $(12, 3)$ to $(13, i)$, so g maps the edge $\{(1i, 2), (12, 3)\}$ to the edge $\{(12, 3), (13, i)\}$. Since Γ is G-vertex-transitive, we conclude that Γ is G-edge-transitive. □

Let H be a subgroup of G, and S be a subset of G. Define the *coset graph* of G with respect to H and S to be the directed graph with vertex set $[G : H]$ and such that, for any Hx, $Hy \in V$, Hx is connected to Hy if and only if $yx^{-1} \in HSH$ and denote the digraph by $Cos(G, H, HSH)$. With the vertex $\alpha = (12, 3) \in V_n$ and element $g = (234) \in G$, the graph $\Gamma = \Gamma_n$ can be described as a *coset graph* $\mathsf{Cos}(G, G_\alpha, G_\alpha\{g, g^{-1}\}G_\alpha)$, which has vertex set $[G : G_\alpha] = \{G_\alpha x \mid x \in G\}$ such that $G_\alpha x$ and $G_\alpha y$ are adjacent if and only if $yx^{-1} \in G_\alpha\{g, g^{-1}\}G_\alpha$. The right multiplication of each element $g \in G$

$$g : \quad G_\alpha x \mapsto G_\alpha x g, \text{ for all } x \in G$$

induces an automorphism of Γ.

Lemma 4. *If an automorphism $\tau \in \mathsf{Aut}(G)$ normalizes G_α and $(G_\alpha\{g, g^{-1}\}G_\alpha)^\tau = G_\alpha\{g, g^{-1}\}G_\alpha$, then τ is an automorphism of Γ.*

Proof. Since τ normalizes G_α, it induces a permutation on the vertex set $[G : G_\alpha]$.

For any two vertices $G_\alpha x$ and $G_\alpha y$, we have

$$\begin{aligned} G_\alpha x \sim G_\alpha y &\iff yx^{-1} \in G_\alpha\{g, g^{-1}\}G_\alpha \\ &\iff (yx^{-1})^\tau \in (G_\alpha\{g, g^{-1}\}G_\alpha)^\tau \\ &\iff y^\tau (x^\tau)^{-1} \in G_\alpha\{g, g^{-1}\}G_\alpha \\ &\iff (G_\alpha x)^\tau = G_\alpha x^\tau \sim G_\alpha y^\tau = (G_\alpha y)^\tau. \end{aligned}$$

Thus, τ is an automorphism of the graph Γ. $\square$

Lemma 5. *Both Γ_4 and Γ_5 are arc-transitive.*

Proof. Let $\alpha = (12, 3)$, and $g = (234)$. Consider first the case $G = \mathrm{S}_4$. Then $G_\alpha = \mathrm{Sym}(\{1, 2\})$, and $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha\{g, g^{-1}\}G_\alpha)$. Let τ be the inner-automorphism induced by the element $(34) \in G$. Then τ normalizes G_α and $g^\tau = g^{-1}$. By Lemma 4, τ is an automorphism of the graph. Further, for the edge $\{G_\alpha, G_\alpha g\}$, we have

$$(G_\alpha, G_\alpha g)^{\tau g} = (G_\alpha, G_\alpha g^{-1})^g = (G_\alpha g, G_\alpha),$$

and so Γ is arc-transitive.

Next, consider the case $G = \mathrm{S}_5$. Again let $\alpha = (12, 3)$ and $g = (234)$. Then $G_\alpha = \mathrm{Sym}(\{1, 2\}) \times \mathrm{Sym}(\{4, 5\})$, and $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha\{g, g^{-1}\}G_\alpha)$. Let τ be the inner-automorphism of G induced by the element $(15)(24) \in G$. Then τ normalizes G_α and reverses g. Arguing as above shows that Γ is arc-transitive. $\square$

However, we will show that Γ_n for $n \geqslant 6$ are all half-transitive.

3 The parameters

Denote the graph Γ_n simply by Γ in the following. For vertices $\alpha = (i_1 j_1, k_1)$ and $\beta = (i_2 j_2, k_2)$, we denote $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\}$ by $\alpha \cap \beta$. Then $|\alpha \cap \beta| = 2$ if α and β are adjacent.

Lemma 6. *The graph $\Gamma = \Gamma_n$ is of order $\frac{n(n-1)(n-2)}{2}$, valency $4(n-3)$, girth 3, and diameter 3.*

Proof. As noticed above, each 3-subset $\{i, j, k\}$ corresponds to exactly three vertices (ij, k), (jk, i) and (ik, j). Thus, the order $|V|$ of Γ equals $3\binom{n}{3} = \frac{n(n-1)(n-2)}{2}$. By the definition of the graph Γ, the neighborhood of the vertex $\alpha = (ij, k)$ is $\Gamma(\alpha) = \{(ik, m), (jk, m), (im, j), (jm, i) | m \neq i, j, k\}$. Thus, Γ is of valency $4(n-3)$. Moreover, since (jk, m) and (jm, i) are adjacent, Γ is of girth 3.

We next compute the distance $d(\alpha, \beta)$ between two vertices α and β. As Γ is a vertex-transitive graph, we take $\alpha = (12, 3)$. We first consider a small case that $n = 4$. Then $|V| = 12$, and the neighborhood $\Gamma(\alpha) = \{(14, 2), (24, 1), (13, 4), (23, 4)\}$. For other vertices except $(12, 4)$, we have

$$\Gamma(\alpha) \cap \Gamma(\beta) = \begin{cases} \{(23, 4)\}, & \text{if } \beta = (13, 2), \\ \{(13, 4)\}, & \text{if } \beta = (23, 1), \\ \{(24, 1)\}, & \text{if } \beta = (14, 3), \\ \{(14, 2), (23, 4)\}, & \text{if } \beta = (34, 1), \\ \{(14, 2)\}, & \text{if } \beta = (24, 3), \\ \{(24, 1), (13, 4)\}, & \text{if } \beta = (34, 2), \end{cases}$$

and so $d(\alpha, \beta) = 2$. For $\beta = (12, 4)$, $\Gamma(\alpha) \cap \Gamma(\beta) = 0$. Furthermore, the sequence

$$\alpha = (12, 3),\ (14, 2),\ (24, 3),\ \beta = (12, 4)$$

is a path between α and β of length 3. Hence, $d(\alpha, \beta) = 3$, and thus, the graph $\varGamma$ is of diameter 3.

Now we treat the cases where $n \geqslant 5$. Take $\alpha = (12, 3)$, and let $\beta = (ij, k)$.

Case 1. Assume first that $i, j, k \geqslant 4$. If a vertex $\gamma = (i'j', k')$ is adjacent to both α and β, then $|\gamma \cap \alpha| = 2$ and $|\gamma \cap \beta| = 2$, which is not possible. Thus, $d(\alpha, \beta)$ is at least 3. On the other hand, the sequence

$$\alpha = (12, 3),\ (1i, 2),\ (ik, 1),\ \beta = (ij, k)$$

is a path between α and β of length 3. Hence, $d(\alpha, \beta) = 3$.

Case 2. Next, consider the case where $|\alpha \cap \beta| = 1$.

Suppose that $\beta = (ij, 3)$, where $i, j \geqslant 4$. Then the sequence α, $(1i, 2)$, $(i3, 1)$, β is a path between α and β, and hence $d(\alpha, \beta) \leqslant 3$. As mentioned above,

$$\varGamma(\alpha) = \{(1i', 2), (2i', 1), (13, i'), (23, i') \mid i' > 3\},$$

and similarly,

$$\varGamma(\beta) = \{(ij', j), (jj', i), (3i, j'), (3j, j') \mid j' \notin \{3, i, j\}\}.$$

Thus, $\varGamma(\alpha) \cap \varGamma(\beta) = \emptyset$, and so $d(\alpha, \beta) = 3$.

Assume now that $k \neq 3$. Then $\beta = (1i, k)$, $(2i, k)$, $(3i, k)$, $(ij, 1)$ or $(ij, 2)$, where $i, j, k \geqslant 4$. In each of these cases, $d(\alpha, \beta) = 2$ because

$$\varGamma(\alpha) \cap \varGamma(\beta) = \begin{cases} \{(1k, 2), (13, i), (2i, 1)\}, & \text{if } \beta = (1i, k), \\ \{(2k, 1), (23, i), (1i, 2)\}, & \text{if } \beta = (2i, k), \\ \{(31, i), (32, i)\}, & \text{if } \beta = (3i, k), \\ \{(1i, 2), (1j, 2)\}, & \text{if } \beta = (ij, 1), \\ \{(2i, 1), (2j, 1)\}, & \text{if } \beta = (ij, 2). \end{cases}$$

Case 3. We then treat the case where $|\alpha \cap \beta| = 2$.

Assume that $k = 3$. Then $\beta = (1i, 3)$ or $(2i, 3)$, where $i \geqslant 4$. If $\beta = (1i, 3)$, then $\varGamma(\alpha) \cap \varGamma(\beta) = \{(13, m), (2i, 1) \mid 4 \leqslant m \leqslant n, m \neq i\}$, and thus $d(\alpha, \beta) = 2$. Similarly, if $\beta = (2i, 3)$, there exists $n - 3$ paths of length 2 between α and β, and so $d(\alpha, \beta) = 2$.

Suppose that $k \neq 3$. Then $\beta = (12, k)$, $(1i, 2)$, $(2i, 1)$, $(13, k)$, $(23, k)$, $(3i, 1)$, $(3i, 2)$, where $i, k \geqslant 4$. If $\beta = (12, k)$, then $\varGamma(\alpha) \cap \varGamma(\beta) = \{(1m, 2), (2m, 1) \mid 4 \leqslant m \leqslant n, m \neq k\}$, and so $d(\alpha, \beta) = 2$. For these vertices $\beta = (1i, 2)$, $(2i, 1)$, $(13, k)$, or $(23, k)$, we have $\beta \in \varGamma(\alpha)$, and thus $d(\alpha, \beta) = 1$. If $\beta = (3i, 1)$, $\varGamma(\alpha) \cap \varGamma(\beta) = \{(13, m), (23, i), (1i, 2) | 4 \leqslant m \leqslant n, m \neq i\}$, and hence $d(\alpha, \beta) = 2$. Similarly, if $\beta = (3i, 2)$, $\varGamma(\alpha) \cap \varGamma(\beta) = \{(23, m), (13, i), (2i, 1) | 4 \leqslant m \leqslant n, m \neq i\}$, and so $d(\alpha, \beta) = 2$.

Case 4. Let $|\alpha \cap \beta| = 3$. Then $\beta = (23, 1)$ or $(13, 2)$.

For $\beta = (23, 1)$, we have $d(\alpha, \beta) = 2$ as $(13, m)$ is adjacent to both α and β, and thus, there are exactly $n - 3$ paths of length 2 between α and β, where $m \neq 1, 2, 3$. Similarly, if $\beta = (13, 2)$, $d(\alpha, \beta) = 2$ because there exist exactly $n - 3$ paths α, $(23, m)$, β of length 2 between α and β, where $m \neq 1, 2, 3$.

Thus, $\varGamma = \varGamma_n$ is of diameter 3. □

4 Quotients

The action of $G = \mathrm{Sym}([n])$ on V_n has three types of blocks as below

$$\begin{aligned} B_k &= \{(ij,k) \mid i,j \in [n] \setminus \{k\}\}, \text{ size } \tbinom{n-1}{2}, \\ B_{ij} &= \{(ij,k) \mid k \in [n] \setminus \{i,j\}\}, \text{ size } n-2, \\ B_{ijk} &= \{(ij,k),(jk,i),(ki,j)\}, \text{ size } 3. \end{aligned}$$

Let $\mathcal{B}_1 = B_k^G$, $\mathcal{B}_2 = B_{ij}^G$, and $\mathcal{B}_3 = B_{ijk}^G$. Then $|\mathcal{B}_1| = n$, $|\mathcal{B}_2| = \binom{n}{2}$, and $|\mathcal{B}_3| = \binom{n}{3}$.

Lemma 7. *If $n \geqslant 7$, then $V = V_n$ has exactly three non-trivial G-invariant partitions: $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$.*

Proof. For the vertex $\beta = (23,1)$, the stabilizer $G_\beta = \mathrm{Sym}(\{2,3\}) \times \mathrm{Sym}([n] \setminus \{1,2,3\})$, and G_β is contained in G_{B_1}, $G_{B_{23}}$ and $G_{B_{123}}$. Moreover, these three subgroups are maximal in G and are the only proper subgroups of G which properly contain G_β. Thus, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are the only block systems of G acting on V_n. □

By Lemma 7, we have three block systems $\mathcal{B}_i$ with $i = 1$, 2 or 3, and we have three quotient graph $\Gamma_{\mathcal{B}_i}$. Clearly, the induced action of G on $\mathcal{B}_i$ is equivalent to the action of G on $[n]^{\{i\}}$, where $i = 1$, 2 or 3. We thus identify $\mathcal{B}_1$ with $[n]$, $\mathcal{B}_2$ with $[n]^{\{2\}}$, and $\mathcal{B}_3$ with $[n]^{\{3\}}$. The quotient graph $\Gamma_{\mathcal{B}_1} = \mathbf{K}_n = \mathbf{J}(n,1)$. For $\Gamma_{\mathcal{B}_2}$, two vertices B_{ij} and $B_{i'j'}$ are adjacent if and only if $|\{i,j\} \cap \{i',j'\}| = 1$, and so $\Gamma_{\mathcal{B}_1} = \mathbf{J}(n,2)$. For $\Gamma_{\mathcal{B}_3}$, two vertices B_{ijk} and $B_{i'j'k'}$ are adjacent if and only if $|\{i,j,k\} \cap \{i',j',k'\}| = 2$. Thus, we have the following lemma.

Lemma 8. *The quotient graph $\Gamma_{\mathcal{B}_i}$ is the Johnson graph $\mathbf{J}(n,i)$, where $i = 1$, 2 or 3.*

For a quotient graph $\Gamma_{\mathcal{B}}$, let $B, B' \in \mathcal{B}$ be adjacent in $\Gamma_{\mathcal{B}}$. The *induced subgraph* $[B \cup B']$ of Γ over $B \cup B'$ is the graph with vertex set $B \cup B'$ and edge set $E_0 = \{\{u,v\} \in E \mid u,v \in B \cup B'\}$. Then $[B \cup B']$ is a bipartite graph with biparts B and B'; denoted by $[B,B']$. We next determine the induced subgraph $[B,B']$ for the quotient graph $\Gamma_{\mathcal{B}_i}$.

For a graph $\Sigma = (V,E)$, the *vertex-edge incidence graph* is the bipartite graph with biparts V and E such that two vertices $v \in V$ and $w \in E$ are adjacent if and only if v,w are incident in Σ. This incidence graph is also called the *subdivision* of Σ.

Lemma 9. *Let $B, B' \in \mathcal{B}_1$ be adjacent in $\Gamma_{\mathcal{B}_1}$. Then the induced subgraph $[B,B']$ consists of 2 copies of the subdivision of $\mathbf{K}_{n-2}$.*

Proof. Since $(23,1) \in B_1$ is adjacent to $(34,2) \in B_2$, the vertices B_1 and B_2 are adjacent in the quotient $\Gamma_{\mathcal{B}_1}$. The edges of the induced subgraph $[B_1,B_2]$ are

$$\{\{(2i,1),(ij,2)\} \mid 3 \leqslant i \leqslant n,\ j \neq i,1,2\} \cup \{\{(1i,2),(ij,1)\} \mid 3 \leqslant i \leqslant n,\ j \neq i,1,2\},$$

which form two copies of the subdivision of $\mathbf{K}_{n-2}$. □

A *star* $\mathbf{K}_{1,m}$ is a bipartite graph with $m+1$ vertices, in which there is one vertex that is adjacent to all other m vertices. In particular, $\mathbf{K}_{1,2}$ is a path of length 2.

Lemma 10. *Let $B, B' \in \mathcal{B}_2$ be adjacent in $\Gamma_{\mathcal{B}_2}$. Then the induced subgraph $[B, B']$ consists of 2 copies of the star $\mathbf{K}_{1,n-3}$.*

Proof. Since $(12, 3) \in B_{12}$ is adjacent to $(23, 4) \in B_{23}$, the vertices B_{12} and B_{23} are adjacent in the quotient $\Gamma_{\mathcal{B}_2}$. The edges of the induced subgraph $[B_{12}, B_{23}]$ are $\{\{(12, 3), (23, i)\} \mid i \geqslant 4\} \cup \{\{(23, 1), (12, j)\} \mid j \geqslant 4\}$, which form two stars $\mathbf{K}_{1,n-3}$. □

Lemma 11. *Let $B, B' \in \mathcal{B}_3$ be adjacent in $\Gamma_{\mathcal{B}_3}$. Then the induced subgraph $[B, B']$ consists of 2 paths of length 2.*

Proof. For the blocks $B = \{(12, 3), (23, 1), (31, 2)\}$ and $B' = \{(12, 4), (24, 1), (41, 2)\}$, the induced subgraph $[B, B']$ has 4 edges

$$\{(12, 3), (14, 2)\},\ \{(12, 3), (24, 1)\},\ \{(12, 4), (13, 2)\},\ \{(12, 4), (32, 1)\}.$$

These edges form two paths of length 2:

$$(23, 1), (12, 4), (13, 2), \quad \text{and } (24, 1), (12, 3), (14, 2).$$

Thus, $[B, B'] = 2\mathbf{K}_{1,2}$. □

5 The automorphism group

In this section, we determine the automorphism group $\mathsf{Aut}\Gamma$.

Lemma 12. *Let $n \geqslant 7$, and let X be a subgroup such that $G \leqslant X \leqslant \mathsf{Aut}\Gamma$. Then X is almost simple, and if $X \neq G$, then X is primitive on V.*

Proof. Let M be a minimal normal subgroup of X. Suppose that M is intransitive on V. Let $\mathcal{B}$ be the set of M-orbits on V. Then $\mathcal{B}$ is X-invariant and G-invariant. By Lemma 7, $\mathcal{B} = \mathcal{B}_i$ with $i = 1$, 2 or 3. For $B_i \in \mathcal{B}_i$, we have that $G_{B_1}^{B_1} = \mathrm{S}_{n-1}$, $G_{B_2}^{B_2} = \mathrm{S}_{n-2}$, and $G_{B_3}^{B_3} = \mathrm{S}_3$. Thus, $G_{B_i}^{B_i}$ is primitive, and so is $X_{B_i}^{B_i}$.

Let $K = X_{(\mathcal{B})}$, the kernel of X acting on $\mathcal{B}$. Suppose that $K \neq 1$. Then $1 \neq K^B \lhd X_B^B$, and so K^B is transitive as X_B^B is primitive. Let $B' \in \mathcal{B}$ be adjacent in $\Gamma_{\mathcal{B}}$ to B. Then K is transitive on B', and since $|B| = |B'|$, we conclude that the induced subgraph $[B, B']$ is regular, which is a contradiction by Lemmas 9-11. Thus, $K = 1$, and so $M = 1$, which is a contradiction. So M is transitive on V, and X is quasiprimitive on V. Further, M is non-abelian since $|V| = 3\binom{n}{3}$.

Now let $M = T_1 \times T_2 \times \ldots \times T_l$, where $l \geqslant 1$, and $T_1 \cong T_2 \cong \ldots \cong T_l$ are non-abelian simple groups. Then $M \cap G \lhd G$, and so $M \cap G = 1$, A_n or S_n.

Suppose that $M \cap G = 1$. Let $Z = MG = M{:}G$. If Z is imprimitive on V, then a block system $\mathcal{B} = \mathcal{B}_1$, $\mathcal{B}_2$ or $\mathcal{B}_3$. Hence $Z \cong Z^{\mathcal{B}} \leqslant G^{\mathcal{B}} \cong \mathrm{S}_n$, which is not possible. Thus, Z is primitive on V, and G does not centralize M. By O'Nan-Scott's theorem (see [7]), we have that $l \geqslant n$, and $\frac{n(n-1)(n-2)}{2} = |V| = m^l$ or m^{l-1} where $m \geqslant 5$, which is not possible.

Therefore, $M \cap G = \mathrm{A}_n$ or S_n, and letting $L = \mathsf{soc}(G) = \mathrm{A}_n$, we have $L \leqslant M$. Since L is non-abelian simple, L is contained in a simple group T_i, say T_1. Hence, T_1 is transitive on V. If $l \geqslant 2$, then as T_2 centralizes T_1, we have that T_2 is semiregular on V. Then $|V|$ divides $|T_1|$, and $|T_2| = |T_1|$ divides $|V|$. So T_1 is regular on V, and since G is transitive on V, $L \leqslant T_1$ is semiregular with at most 2 orbits. Since T_1 has no subgroup of index 2 and $G/L \cong \mathbb{Z}_2$, we have that $L = T_1$ is regular on V, which is a contradiction. Thus, $M = T_1$ is simple and $L \leqslant M$. Assume that there exists another minimal normal subgroup N of X such that $N \neq M$. Then $M \cap N = 1$; however, the above argument with N in the place of M shows that $L \leqslant N$. So $M \cap N \geqslant L$, which is a contradiction. Therefore, M is simple and the unique minimal normal subgroup of X, and hence X is almost simple.

Suppose that $X > G$ and X is imprimitive on V. Let $\mathcal{B}$ be a block system for X on V. Then $\mathcal{B}$ is a block system for G on V. By Lemma 7, $\mathcal{B} = \mathcal{B}_i$ where $i = 1$, 2 or 3, and by Lemma 8, $\Gamma_{\mathcal{B}} = \mathbf{J}(n, i)$. As noticed in the Introduction, $\mathsf{Aut}\Gamma_{\mathcal{B}} = \mathrm{S}_n$, and so $\mathrm{S}_n \cong G < X \cong X^{\mathcal{B}} \leqslant \mathsf{Aut}\Gamma_{\mathcal{B}} \cong \mathrm{S}_n$, which is not possible. Hence either $X = G$, or X is primitive on V, as claimed. □

A transitive permutation group G on Ω is called *k-homogeneous* if G is transitive on the set of k-subsets of Ω, where k is a positive integer.

Lemma 13. *If $n \geqslant 8$, then $\mathsf{Aut}\Gamma = G = \mathrm{Sym}([n])$.*

Proof. Let $n \geqslant 8$. Suppose that $G < \mathsf{Aut}\Gamma$. Let $L \leqslant \mathsf{Aut}\Gamma$ be such that G is a maximal subgroup of L. Since G is transitive on V, the almost simple group L has a factorization $L = GL_\alpha$. Further, since L is primitive on V, the factorization $L = GL_\alpha$ is a maximal factorization. Thus, the triple (L, G, L_α) is classified in [11], see the MAIN THEOREM on page 1. An inspection of the candidates with one factor being $G = \mathrm{S}_n$, we conclude that one of the following holds:

(i) $n \leqslant 12$, or

(ii) $L = \mathrm{S}_{n+1}$, or

(iii) $L = \mathrm{S}_m$ or A_m, and G is k-homogenous of degree m, where $1 \leqslant k \leqslant 5$.

Consider the small groups where $n \leqslant 12$. We note that as Γ is not a complete group, $L < \mathrm{Sym}(V)$.

Let $n = 12$ first. Then $|V| = \frac{n(n-1)(n-2)}{2} = 660$, and L is a primitive group of degree 660. Hence L lies in Appendix B of [7], which shows that $\mathsf{soc}(L) = \mathrm{PSL}(2, 659)$ or $\mathrm{PSL}(2, 11) \times \mathrm{PSL}(2, 11)$. So L does not contains S_{12}, which is a contradiction. Similarly, the cases where $n = 8$, 9 and 10 are excluded.

Suppose that $n = 11$. Then $|V| = 495$, and Γ is of valency 32. By Appendix B of [7], as $\mathrm{S}_{11} < L$, we conclude that $(L, L_\alpha) = (\mathrm{S}_{12}, \mathrm{S}_8 \times \mathrm{S}_4)$ or $(\mathrm{O}_{10}^-(2), 2^8{:}\mathrm{O}_8^-(2))$. The former is not possible since $\mathrm{S}_{12} \neq \mathrm{S}_{11}(\mathrm{S}_8 \times \mathrm{S}_4)$, and the latter is not possible since $\mathrm{O}_8^-(2)$ does not have a transitive representation of degree at most 32.

Next, let $L = \mathrm{S}_{n+1}$, with $n \geqslant 13$. Since $L = GL_\alpha$, the stabilizer L_α is a transitive permutation group of degree $n+1$ such that $|L : L_\alpha| = |V| = \frac{n(n-1)(n-2)}{2}$. Assume that L_α is primitive of degree $n+1$. By Bochert's theorem (see [21, Theorem 14.2]),

$|L:L_\alpha| \geqslant [\frac{n+2}{2}]!$. Computation shows that $n \leqslant 10$, which is a contradiction. Thus, L_α is imprimitive of degree $n+1$. Then $L_\alpha = \mathrm{S}_b \wr \mathrm{S}_m$, where $bm = n+1$, and thus,

$$\frac{n(n-1)(n-2)}{2} = |V| = |L:L_\alpha| = \frac{(n+1)!}{(b!)^l m!},$$

which is not possible.

Finally, assume that $G = \mathrm{S}_n$ is k-homogenous of degree m, and $L = \mathrm{S}_m$ or A_m, and $L_\alpha = \mathrm{S}_k \times \mathrm{S}_{m-k}$, where $k \leqslant 5$. Since L is not 2-transitive on V, we have $k \geqslant 2$. Thus, by the classification of 2-homogeneous groups, we conclude that $n \leqslant 8$, which is a contradiction.

Therefore, $\mathsf{Aut}\Gamma = G = \mathrm{Sym}([n])$, as claimed. □

6 Proof of Theorem 2

In this section, we prove the main theorem.

By Lemma 6, part (i) of Theorem 2 is true. By Lemma 8, Theorem 2 (ii) holds.

For $n = 4$ or 5, Theorem 2 is proved by Lemma 5. Thus, we next assume $n \geqslant 6$.

For $n = 6$ or 7, a computation using Gap shows that $\mathsf{Aut}\Gamma_n = \mathrm{S}_n$, and for $n \geqslant 8$, Lemma 13 shows that $\mathsf{Aut}\Gamma = G$. Then, by Lemma 3, Γ is half-transitive, as in part (iii).

Finally, assume that Γ is a Cayley graph of a group R. Then R is regular on V (see [3, Proposition 16.3]), and hence R is 3-homogeneous but not 3-transitive on $[n]$. Further, as $|R| = |V| = 3\binom{n}{3}$, R is not sharply 3-homogeneous on $[n]$. Inspecting 3-homogeneous groups which are not 3-transitive, refer to [7, Theorem 9.4B], we conclude that $R = \mathrm{A\Gamma L}(1,8)$ or $\mathrm{PSL}(2,q)$ where $q \equiv 3 \pmod 4$. So $n = 8$ or $q+1$, respectively. This proves part (iv) of Theorem 2. □

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