On perfect packings in dense graphs

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Abstract

We say that a graph G has a perfect H-packing if there exists a set of vertex-disjoint copies of H which cover all the vertices in G. We consider various problems concerning perfect H-packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect K_r-packing in any graph G on n vertices and with minimum degree $\delta(G) \geqslant D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect H-packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning K_r-free graphs that satisfy a certain degree sequence condition.

1 Introduction

Given two graphs H and G, a *perfect H-packing* in G is a collection of vertex-disjoint copies of H which cover all the vertices in G. Perfect H-packings are also referred to as *H-factors* or *perfect H-tilings*. Hell and Kirkpatrick [8] showed that the decision problem whether a graph G has a perfect H-packing is NP-complete precisely when H has a component consisting of at least 3 vertices. So for such graphs H, it is unlikely that there is a complete characterisation of those graphs containing a perfect H-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph G contains a perfect H-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [7] which states that a graph G whose order n is divisible by r has a perfect K_r-packing provided that $\delta(G) \geqslant (r-1)n/r$. Kühn and Osthus [12, 13] characterised, up to an additive constant, the

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minimum degree which ensures a graph G contains a perfect H-packing for an arbitrary graph H.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. *Ore-type* degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [10] implies the Hajnal-Szemerédi theorem.

Theorem 1 (Kierstead and Kostochka [10])**.** *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices such that for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geqslant 2(1 - 1/r)n - 1.$$

Then G contains a perfect K_r-packing.

Kühn, Osthus and Treglown [14] characterised, asymptotically, the Ore-type degree condition which ensures a graph G contains a perfect H-packing for an arbitrary graph H.

1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph G to contain a perfect K_r-packing when there are no other restrictions. Indeed, given $n, r \in \mathbb{N}$ such that $r \geqslant 2$ divides n, if G is a graph on n vertices and $e(G) \geqslant \binom{n}{2} - n + r$ then G contains a perfect K_r-packing. Moreover, if G is a copy K of K_{n-1} together with a vertex which sends precisely $r-2$ edges to K, then $e(G) = \binom{n}{2} - n + r - 1$ and G does not contain a perfect K_r-packing. The following result of Akiyama and Frankl [1] refines this observation.

Theorem 2 (Akiyama and Frankl [1])**.** *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose G is a graph on n vertices and $e(\overline{G}) \leqslant \min\{\binom{n/r+1}{2}, n - r + 1\}$. Then G has a perfect K_r-packing unless $\overline{G}$ is isomorphic to one of the following graphs:*

(i) A copy of $K_{n/r+1}$ together with $(1 - 1/r)n - 1$ isolated vertices;

(ii) The disjoint union of $K_{1,n-r-j+1}$, j edges and $r - j - 2$ isolated vertices, for some $1 \leqslant j \leqslant r - 2$.

When (for example) $n \geqslant r^3$, $\binom{n/r+1}{2} > n - r + 1$. Hence, in this case Theorem 2 is equivalent to the following: If G is a graph on n vertices and $e(G) \geqslant \binom{n}{2} - n + r - 1$ then either G contains a perfect K_r-packing or $\overline{G}$ is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let $n, r \in \mathbb{N}$ such that r divides n. Given some $D \in \mathbb{N}$, what edge density condition ensures that any graph G on n vertices and of minimum degree $\delta(G) \geqslant D$ contains a perfect K_r-packing?

We fully resolve the problem, and our answers for $r = 2$ and $r \geqslant 3$ differ.

Theorem 3. *For an even positive n and integer $1 \leqslant d < n/2$, let $h(n,d) := \binom{n-d-1}{2} + d(d+1)$ and let $f(2,n,d)$ denote the maximum integer c such that some n-vertex graph with minimum degree at least d and at least c edges has no perfect matching. Then*

$$f(2,n,d) = \max\{h(n,d), h(n,0.5n-1)\}.$$

Theorem 4. *Let $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and r divides n. Given any $D \in \mathbb{N}$ such that $r-1 \leqslant D \leqslant (r-1)n/r - 1$ define*

$$g(n,r,D) := \max\left\{\binom{n}{2} - \binom{n/r+1}{2}, D(n-D) + \binom{n-1-D}{2} + e(T(D,r-2))\right\}.$$

Suppose that G is a graph on n vertices with $\delta(G) \geqslant D$ and $e(G) > g(n,r,D)$. Then G contains a perfect K_r-packing. Moreover, there exists a graph G' on n vertices with $\delta(G') \geqslant D$ and $e(G') = g(n,r,D)$ but such that G' does not contain a perfect K_r-packing.

Clearly a graph G of minimum degree $\delta(G) < r-1$ cannot contain a perfect K_r-packing. Further, regardless of edge density, every graph G whose order n is divisible by r and with $\delta(G) \geqslant (r-1)n/r$ contains a perfect K_r-packing. Thus, Theorem 4 covers all values of D where our problem was not solved previously.

An *equitable k-colouring* of a graph G is a proper k-colouring of G such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that r divides n. Notice that a graph G on n vertices has a perfect K_r-packing if and only if the complement $\overline{G}$ of G has an equitable n/r-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let G be a graph on n vertices such that r divides n. If $\Delta(G) \leqslant n/r - 1$ then G has an equitable n/r-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

Theorem 5. *Let $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and r divides n. Recall that $T(n,r)$ denotes the Turán graph. Given any $D \in \mathbb{N}$ such that $n/r \leqslant D \leqslant n - r$ define*

$$f(n,r,D) := \min\left\{\binom{n/r+1}{2}, D + e(\overline{T}(n-D-1, r-2))\right\}.$$

Suppose that G is a graph on n vertices with $\Delta(G) \leqslant D$ and $e(G) < f(n,r,D)$. Then G has an equitable n/r-colouring. Moreover, there exists a graph G' on n vertices with $\Delta(G') \leqslant D$ and $e(G') = f(n,r,D)$ but such that G' does not have an equitable n/r-colouring.

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of D such that $r-1 \leqslant D \leqslant (r-1)n/r - 1$. Section 3 contains a proof of Theorem 5.

1.2 Degree sequence conditions forcing a perfect packing

Chvátal [3] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that G is a graph on n vertices and that the degrees of the graph are $d_1 \leqslant \ldots \leqslant d_n$. If $n \geqslant 3$ and $d_i \geqslant i+1$ or $d_{n-i} \geqslant n-i$ for all $i < n/2$ then G is Hamiltonian. The following is a simple consequence of Chvátal's theorem.

Theorem 6 (Chvátal [3])**.** *Suppose that G is a graph on $n \geqslant 2$ vertices and the degrees of the graph are $d_1 \leqslant \ldots \leqslant d_n$. If*

$$d_i \geqslant i \quad \text{or} \quad d_{n-i+1} \geqslant n-i \quad \text{for all} \quad 1 \leqslant i \leqslant n/2$$

then G contains a Hamilton path.

We propose the following conjecture on the degree sequence of a graph which forces a perfect K_r-packing.

Conjecture 7. *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leqslant \ldots \leqslant d_n$ such that:*

(α) $d_i \geqslant (r-2)n/r + i$ for all $i < n/r$;

(β) $d_{n/r+1} \geqslant (r-1)n/r$.

Then G contains a perfect K_r-packing.

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for n/r vertices to have degree less than $(r-1)n/r$. Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially "best possible". It is easy to see that Theorem 6 implies Conjecture 7 in the case when $r = 2$. We prove the conjecture in the case when G is additionally K_{r+1}-free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

Conjecture 8. *Suppose $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geqslant n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leqslant \ldots \leqslant d_n$ satisfies*

- *$d_i \geqslant (r-2)n/r + i + \gamma n$ for all $i < n/r$,*

then G contains a perfect H-packing.

Since first submitting this paper, the third author and Knox [11] have proven Conjecture 8 in the case when $r = 2$. (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [6] characterises those degree sequences which force a copy of K_r in a graph G.

Theorem 9 (Erdős [6]). *Let G be a graph on n vertices with degree sequence $d_1 \leqslant \ldots \leqslant d_n$. If G is K_{r+1}-free then there is an r-partite graph G' on n vertices whose degree sequence $d'_1 \leqslant \ldots \leqslant d'_n$ satisfies*

$$d_i \leqslant d'_i \quad \text{for all} \quad i \leqslant n.$$

In Section 6 we prove the following related structural theorem.

Theorem 10. *Suppose that $n, r \in \mathbb{N}$ such that $n \geqslant r$ and so that r divides n. Let G be a K_{r+1}-free graph on n vertices whose degree sequence $d_1 \leqslant \ldots \leqslant d_n$ is such that $d_{n/r} \geqslant (r-1)n/r$. Then $G \subseteq T(n,r)$, where $T(n,r)$ is the complete r-partite* Turán graph *on n vertices; so each vertex class has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.*

2 The case $r = 2$ and extremal examples for $r \geqslant 3$

2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph G on an even number n of vertices and of minimum degree $\delta(G) \geqslant d$ contains a perfect matching. Note that we only consider values of d such that $1 \leqslant d < n/2$, since if $\delta(G) \geqslant n/2$ then G has a perfect matching, regardless of the edge density.

Recall that $h(n,d) := \binom{n-d-1}{2} + d(d+1)$. Note that for a fixed even n, $h(n,d)$ decreases with d in the interval $[0, n/3 - 5/6]$ and increases with d in $[n/3 - 5/6, 0.5n - 1]$.

For a positive even n and an integer $0 \leqslant d < n/2$, let A, B and C be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n,d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in A is adjacent to each vertex in B and to no vertex in C. So H does not contain a perfect matching and has exactly $h(n,d)$ edges.

The examples of $H(n,d)$ show that $f(2,n,d) \geqslant \max\{h(n,d), h(n,0.5n-1)\}$. Thus to derive Theorem 3, it suffices to prove that an n-vertex graph G with $\delta(G) \geqslant d$ and $e(G) > \max\{h(n,d),\, h(n,0.5n-1)\}$ has a perfect matching.

Consider such a graph G. Let $d_1 \leqslant \ldots \leqslant d_n$ denote the degree sequence of G. If $d_i \geqslant i$ for all $1 \leqslant i \leqslant n/2$ then Theorem 6 implies that G contains a perfect matching. Suppose for a contradiction that $d_{i'} \leqslant i' - 1$ for some $1 \leqslant i' \leqslant n/2$. Note that $i' > d$ as $\delta(G) \geqslant d$.

Let A denote the set of i' vertices in G that correspond to the first i' terms $d_1, \ldots, d_{i'}$ of the degree sequence. Set $B := V(G) \backslash A$. Then

$$e(G[B]) \geqslant e(G) - i'(i'-1) > \max\{h(n,d), h(n,0.5n-1)\} - i'(i'-1)$$

since $d(x) \leqslant i' - 1$ for all $x \in A$. Note that $\max\{h(n,d), h(n,0.5n-1)\} \geqslant h(n,i'-1)$ since $d < i' \leqslant n/2$. Therefore,

$$e(G[B]) > \max\{h(n,d), h(n,0.5n-1)\} - i'(i'-1) \geqslant h(n,i'-1) - i'(i'-1) = \binom{n-i'}{2},$$

a contradiction as $|B| = n - i'$. Thus, $d_i \geqslant i$ for all $1 \leqslant i \leqslant n/2$, as desired.

2.2 Examples for $r \geqslant 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

Proposition 11. *Suppose that $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and r divides n. Then there exists a graph G_1 on n vertices such that $\Delta(G_1) = n/r$,*

$$e(G_1) = \binom{n/r+1}{2},$$

but such that G_1 does not have an equitable n/r-colouring.

Proof. Let G_1 denote the disjoint union of a clique V on $n/r+1$ vertices and an independent set W of $(1-1/r)n-1$ vertices. So every independent set in G_1 contains at most one vertex from V. But since $|V| = n/r+1$, G_1 does not have an equitable n/r-colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \binom{n/r+1}{2}$. □

Proposition 12. *Suppose that $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and $n = kr$ for some $k \geqslant 2$. Further, let $D \in \mathbb{N}$ such that $n/(r-1) \leqslant D \leqslant n-r$. Then there exists a graph G_2 on n vertices such that $\Delta(G_2) = D$,*

$$e(G_2) = D + e(\overline{T}(n-D-1, r-2)),$$

but such that G_2 does not have an equitable n/r-colouring.

Proof. Let G_2 denote the disjoint union of a copy K of $K_{1,D}$ and a copy of $\overline{T}(n-D-1, r-2)$. So $|G| = n$. Let v denote the vertex of degree D in K. The largest independent set in G_2 that contains v is of size $r-1$. Thus, G_2 does not have an equitable n/r-colouring. Further, $e(G_2) = D + e(\overline{T}(n-D-1, r-2))$.

Since $n/(r-1) \leqslant D$ we have that $n-1 \leqslant (r-1)D$. Thus, every vertex in the copy of $\overline{T}(n-D-1, r-2)$ has degree at most

$$\left\lceil \frac{n-D-1}{r-2} \right\rceil - 1 \leqslant \frac{n-D-1}{r-2} \leqslant D.$$

This implies that $\Delta(G_2) = D$. □

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when $n/(r-1) \leqslant D \leqslant n-r$. The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when $n/r \leqslant D \leqslant n/(r-1)$.

Proposition 13. *Let $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and r divides $n \geqslant 2r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leqslant D \leqslant n/(r-1)$. Then*

$$f(n, r, D) = \binom{n/r+1}{2}.$$

The following simple consequence of Turán's theorem will be used in the proof of Theorem 5.

Fact 14. *Let $n, r \in \mathbb{N}$ such that $r \leqslant n$. Then*

$$e(T(n,r)) \leqslant \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \quad \textit{and thus} \quad e(\overline{T}(n,r)) \geqslant \frac{n^2}{2r} - \frac{n}{2}.$$

We will also require the following easy result.

Lemma 15. *Let $n, r \in \mathbb{N}$ such that $r \geqslant 4$ and r divides $n \geqslant 3r$. Suppose that $D \in \mathbb{N}$ such that $n/r \leqslant D < (n+r)/(r-1)$. Then*

$$f(n,r,D) = \binom{n/r+1}{2}.$$

3 Proof of Theorem 5

3.1 Preliminaries

Suppose for a contradiction that the result is false. Let G be a counterexample with the fewest vertices. That is, $n = |V(G)| = rk$ for some $k \in \mathbb{N}$, $\Delta(G) \leqslant D$ for some $D \in \mathbb{N}$ such that $n/r \leqslant D \leqslant n - r$, $e(G) < f(n,r,D)$ and G has no equitable n/r-colouring. By the Hajnal-Szemerédi theorem, $\Delta(G) \geqslant n/r$. Notice that given fixed n and r, $f(n,r,D)$ is non-increasing with respect to D. Thus, we may assume that $\Delta(G) = D$.

We first show that $k \geqslant 4$. Indeed, if $n = 2r$ then $f(n,r,D) \leqslant \binom{3}{2} = 3$. But it is easy to see that every graph G_1 on $2r$ vertices and with $e(G_1) \leqslant 2$ has an equitable 2-colouring. If $n = 3r$ then $f(n,r,D) \leqslant \binom{4}{2} = 6$. Consider any graph G_1 on $3r$ vertices with $e(G_1) \leqslant 5$ and $3 \leqslant \Delta(G_1) \leqslant 5$. Let x denote the vertex in G_1 where $d_{G_1}(x) = \Delta(G_1)$. Since $3 \leqslant d_{G_1}(x) \leqslant 5$, x lies in an independent set I in G_1 of size r. But then $G_1 - I$ contains $2r$ vertices and at most 2 edges. So $G_1 - I$ has an equitable 2-colouring and hence G_1 has an equitable 3-colouring.

Let $v \in V(G)$ such that $d_G(v) = D$. Set $G^* := G - (N_G(v) \cup \{v\})$. Since $f(n,r,D) \leqslant D + e(\overline{T}(n-D-1, r-2))$ we have that $e(G^*) < e(\overline{T}(n-D-1, r-2))$. Thus, by Turán's theorem, G^* contains an independent set of size $r-1$. Hence, v lies in an independent set in G of size r. Amongst all such independent sets of size r that contain v, choose a set $I = \{v, x_1, \ldots, x_{r-1}\}$ such that $d_G(x_1) + \cdots + d_G(x_{r-1})$ is maximised.

Set $G' := G - I$, $n' := |V(G')| = n - r$ and $D' := \Delta(G') \leqslant D$. Notice that $D' \geqslant n'/r$. (Indeed, if not, then by the Hajnal-Szemerédi theorem G' contains an equitable n'/r-colouring. Thus, as I is an independent set in G this gives us an equitable n/r-colouring of G, a contradiction.) Furthermore, $D' \leqslant n' - r$. If not then

$$e(G) \geqslant D + D' \geqslant 2D' \geqslant 2(n' - r + 1) = 2n - 4r + 2$$

and further,

$$\begin{aligned} e(G) < f(n,r,D) &\leqslant f(n,r,n-2r+1) \leqslant (n-2r+1) + e(\overline{T}(2r-2,r-2)) \\ &\leqslant (n-2r+1)+(r+3) = n-r+4. \end{aligned}$$

Therefore, $2n-4r+2 < n-r+4$ and so $n < 3r+2$ a contradiction since $n = kr \geqslant 4r$.

Since $n'/r \leqslant D' \leqslant n'-r$, if $e(G') < f(n',r,D')$ then the minimality of G implies that G' has an equitable n'/r-colouring. This then implies that G has an equitable n/r-colouring, a contradiction. Thus,

$$e(G') \geqslant f(n',r,D'). \tag{1}$$

We now split our argument into three cases.

3.2 Case 1: $\mathbf{f(n',r,D') = \binom{n'/r+1}{2}}$.

By (1), $e(G') \geqslant \binom{n'/r+1}{2} = \binom{n/r}{2}$. Since $d_G(v) = D \geqslant n/r$,

$$e(G) \geqslant \frac{n}{r} + \binom{n/r}{2} = \binom{n/r+1}{2} \geqslant f(n,r,D),$$

a contradiction, as desired.

3.3 Case 2: $\mathbf{D' \leqslant D-1}$ and $\mathbf{f(n',r,D') = D' + e(\overline{T}(n'-D'-1, r-2))}$.

The following claim will be useful.

Claim 16. $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$.

Proof. Note that

$$D + D' + e(\overline{T}(n'-D'-1,r-2)) \overset{(1)}{\leqslant} e(G) < f(n,r,D) \leqslant D + e(\overline{T}(n-D-1,r-2)). \tag{2}$$

Since $D' \leqslant D-1$, clearly $e(\overline{T}(n'-D,r-2)) \leqslant e(\overline{T}(n'-D'-1,r-2))$. Thus, (2) implies that

$$D' + e(\overline{T}(n'-D,r-2)) < e(\overline{T}(n-D-1,r-2)). \tag{3}$$

One can obtain $\overline{T}(n-D-1,r-2)$ from $\overline{T}(n'-D,r-2)$ by adding $r-1$ vertices and at most

$$(n'-D) + \frac{n-D-2}{r-2} \text{ edges.} \tag{4}$$

Hence (3) and (4) give

$$D' < n' - D + \frac{n - D - 2}{r - 2}.$$

Rearranging, and using that $D' \leqslant D - 1$ and $n' = n - r$ we get that

$$\left(2 + \frac{1}{r-2}\right) D' < \left(1 + \frac{1}{r-2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.$$

Thus,

$$D' < \frac{r-1}{2r-3} n - \frac{(r^2 - r + 1)}{2r - 3},$$

as desired. □

Since we are in Case 2 we have that

$$D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leqslant \binom{n'/r + 1}{2} = \binom{n/r}{2}. \tag{5}$$

Notice that for fixed n and r, $D' + e(\overline{T}(n - r - D' - 1, r - 2))$ is non-increasing as D' increases. Hence, (5) and Claim 16 imply that

$$D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leqslant \frac{n^2}{2r^2} - \frac{n}{2r} \tag{6}$$

where $D'' := \lfloor (r-1)n/(2r-3) - (r^2 - r + 1)/(2r - 3) \rfloor$. Note that

$$n - r - \frac{r-1}{2r-3} n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r-2}{2r-3} n + \frac{4 - r^2}{2r - 3}.$$

So Fact 14 and (6) imply that

$$\left(\frac{r-1}{2r-3} n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r-4)}{2r-3}\right) + \frac{1}{2(r-2)} \left(\frac{r-2}{2r-3} n + \frac{4 - r^2}{2r - 3}\right)^2$$
$$- \frac{1}{2} \left(\frac{r-2}{2r-3} n + \frac{4 - r^2}{2r - 3}\right) \leqslant \frac{n^2}{2r^2} - \frac{n}{2r}.$$

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of n^2 is

$$\frac{r-2}{2(2r-3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r-3)^2}. \tag{7}$$

The coefficient of n is

$$\frac{r-1}{2r-3} - \frac{(r-2)}{2(2r-3)} + \frac{1}{2r} + \frac{(4 - r^2)}{(2r-3)^2} = \frac{r^2 - 4r + 9}{2r(2r-3)^2}. \tag{8}$$

The constant term is

$$-\frac{(r^2+r-3)}{2r-3}+\frac{(r^2-4)^2}{2(r-2)(2r-3)^2}+\frac{(r^2-4)}{2(2r-3)}=\frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2}. \quad (9)$$

Since $n \geqslant 4r$, (7)–(9) imply that

$$\frac{8(r^3-6r^2+12r-9)}{(2r-3)^2}+\frac{2(r^2-4r+9)}{(2r-3)^2}+\frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2} \leqslant 0. \quad (10)$$

Multiplying (10) by $2(r-2)(2r-3)^2$ we get

$$15r^4-121r^3+364r^2-486r+244 \leqslant 0$$

This yields a contradiction, since it is easy to check that

$$15r^4-121r^3+364r^2-486r+244 > 0$$

for all $r \in \mathbb{N}$ such that $r \geqslant 3$.

3.4 Case 3: $\mathbf{D'=D}$ and $\mathbf{f(n',r,D')=D'+e(\overline{T}(n'-D'-1,r-2))}$.

By (1) we have that

$$e(G') \geqslant f(n',r,D')=D'+e(\overline{T}(n'-D'-1,r-2)). \quad (11)$$

Consider any vertex $x \in V(G')$ such that $d_{G'}(x)=D'=D$. Since $\Delta(G)=D$, x is not adjacent to any vertex in $I=\{v,x_1,\ldots,x_{r-1}\}$. Further, I was chosen such that $d_G(x_1)+\cdots+d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1)=\cdots=d_G(x_{r-1})=D$. Together with (11) this implies that

$$e(G) \geqslant (r+1)D+e(\overline{T}(n'-D-1,r-2)). \quad (12)$$

Since $e(G)<f(n,r,D) \leqslant D+e(\overline{T}(n-D-1,r-2))$, (12) implies that

$$rD+e(\overline{T}(n'-D-1,r-2))<e(\overline{T}(n-D-1,r-2)). \quad (13)$$

One can obtain $\overline{T}(n-D-1,r-2)$ from $\overline{T}(n'-D-1,r-2)$ by adding r vertices and at most

$$(n'-D-1)+\frac{2(n-D-3)}{r-2}+1 \text{ edges.} \quad (14)$$

Thus, (13) and (14) imply that

$$rD<n-r-D+\frac{2(n-D-3)}{r-2}$$

and so

$$\left(r+1+\frac{2}{r-2}\right)D < \left(1+\frac{2}{r-2}\right)n + \frac{(-r^2+2r-6)}{r-2} < \left(1+\frac{2}{r-2}\right)n. \qquad (15)$$

If $r=3$ then (15) implies that

$$D < \frac{n}{2}.$$

Since $f(n',3,D) = \min\{\binom{n'/3+1}{2}, D + \binom{n'-D-1}{2}\}$ it is easy to see that if $f(n',3,D) = D+\binom{n'-D-1}{2}$ then $D \geqslant 2n'/3+1 = 2n/3-1$. Thus, $2n/3-1 \leqslant D < n/2$, a contradiction since $n \geqslant 4r = 12$.

If $r \geqslant 4$ then (15) implies that

$$D < \frac{n}{r-1} = \frac{n'}{r-1} + \frac{r}{r-1}.$$

Since $n' \geqslant 3r$, Lemma 15 implies that $f(n',r,D') = \binom{n'/r+1}{2}$ and so we are in Case 1, which we have already dealt with.

4 The extremal examples for Conjecture 7

Proposition 17. *Suppose that $n,r,k \in \mathbb{N}$ such that $r \geqslant 2$ divides n and $1 \leqslant k < n/r$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leqslant \ldots \leqslant d_n$ satisfies*

- *$d_i = (r-2)n/r + k - 1$ for all $1 \leqslant i \leqslant k$;*
- *$d_i = (r-1)n/r$ for all $k+1 \leqslant i \leqslant (r-2)n/r + k$;*
- *$d_i = n-k-1$ for all $(r-2)n/r + k + 1 \leqslant i \leqslant n-k+1$;*
- *$d_i = n-1$ for all $n-k+2 \leqslant i \leqslant n$,*

but such that G does not contain a perfect K_r-packing.

Proof. Let G' denote the complete $(r-2)$-partite graph whose vertex classes $V_1,\ldots,V_{r-2}$ each have size n/r. Obtain G from G' by adding the following vertices and edges: Add a set V_{r-1} of $2n/r - 2k + 1$ vertices to G', a set V_r of $k-1$ vertices and a set V_0 of k vertices. Add all edges from $V_0 \cup V_{r-1} \cup V_r$ to $V_1 \cup \cdots \cup V_{r-2}$. Further, add all edges with both endpoints in $V_{r-1} \cup V_r$. Add all possible edges between V_0 and V_r.

So V_0 is an independent set, and there are no edges between V_0 and V_{r-1}. This implies that any copy of K_r in G containing a vertex from V_0 must also contain at least one vertex from V_r. But since $|V_0| > |V_r|$ this implies that G does not contain a perfect K_r-packing. Furthermore, G has our desired degree sequence. □

Notice that the graphs G considered in Proposition 17 satisfy (β) from Conjecture 7 and only fail to satisfy (α) in the case when $i = k$ (and in this case $d_k = (r-2)n/r+k-1$).

Let $n, r \in \mathbb{N}$ such that r divides n. Denote by $T^*(n,r)$ the complete r-partite graph on n vertices with $r-2$ vertex classes of size n/r, one vertex class of size $n/r-1$ and one vertex class of size $n/r+1$. Then $T^*(n,r)$ does not contain a perfect K_r-packing. Furthermore, $T^*(n,r)$ satisfies (α) but condition (β) fails; we have that $d_{n/r+1} = (r-1)n/r - 1$ here. Thus, together $T^*(n,r)$ and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.

5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when G is K_{r+1}-free.

Theorem 18. *Let $n, r \in \mathbb{N}$ such that $r \geqslant 2$ divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leqslant \ldots \leqslant d_n$ such that:*

- *$d_i \geqslant (r-2)n/r + i$ for all $i < n/r$;*
- *$d_{n/r+1} \geqslant (r-1)n/r$.*

Further suppose that no vertex $x \in V(G)$ of degree less than $(r-1)n/r$ lies in a copy of K_{r+1}. Then G contains a perfect K_r-packing.

Proof. We prove the theorem by induction on n. In the case when $n = r$ then $d_{n/r+1} = d_2 \geqslant (r-1)r/r = r-1$. This implies that every vertex in G has degree $r-1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of n. Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geqslant (r-2)n/r + 1$. If $d_G(x_1) \geqslant (r-1)n/r$ then $\delta(G) \geqslant (r-1)n/r$. Thus G contains a perfect K_r-packing by the Hajnal-Szemerédi theorem. So we may assume that $(r-2)n/r + 1 \leqslant d_G(x_1) < (r-1)n/r$. In particular, x_1 does not lie in a copy of K_{r+1}. We first find a copy of K_r containing x_1. If $r = 2$, x_1 has a neighbour and so we have our desired copy of K_2. So assume that $r \geqslant 3$. Certainly $N_G(x_1)$ contains a vertex x_2 such that $d_G(x_2) \geqslant (r-1)n/r$. Thus, $|N_G(x_1) \cap N_G(x_2)| \geqslant (r-3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of K_r. Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geqslant (r-1)n/r$. We can repeat this argument until we have obtained vertices $x_1, \ldots, x_r$ that together form a copy K'_r of K_r.

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since G does not contain a copy of K_{r+1} containing x_1, every vertex $x \in V(G) \backslash V(K'_r)$ sends at most $r-1$ edges to K'_r in G. Thus, $d_{G'}(x) \geqslant d_G(x) - (r-1)$ for all $x \in V(G')$. So if $d_G(x) \geqslant (r-1)n/r$ then $d_{G'}(x) \geqslant (r-1)n/r - (r-1) = (r-1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of K_{r+1} in G then clearly y does not lie in a copy of K_{r+1} in G'. This means that no vertex $y \in V(G')$ of degree less than $(r-1)n'/r$ lies in a copy of K_{r+1}.

Let $d'_1 \leqslant \ldots \leqslant d'_{n'}$ denote the degree sequence of G'. It is easy to check that $d'_i \geqslant (r-2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geqslant (r-1)n'/r$. Indeed, since $x_1 \in V(K'_r)$

where $d_G(x_1) = d_1$, we have that $d'_i \geqslant d_{i+1} - (r-1)$ for all $1 \leqslant i \leqslant n'$. Thus, for all $1 \leqslant i < n'/r = n/r - 1$, $d'_i \geqslant d_{i+1} - (r-1) \geqslant (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geqslant d_{n/r+1} - (r-1) \geqslant (r-1)n/r - (r-1) = (r-1)n'/r$. Hence, by induction G' contains a perfect K_r-packing. Together with K'_r this gives us our desired perfect K_r-packing in G. □

6 Proof of Theorem 10

Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geqslant (r-1)n/r$. Since $d_{n/r} \geqslant (r-1)n/r$ we can greedily select vertices $x_2, \ldots, x_{r-1}$ such that

- $x_1, \ldots, x_{r-1}$ induce a copy of K_{r-1} in G;
- $d_G(x_i) \geqslant (r-1)n/r$ for all $1 \leqslant i \leqslant r-1$.

Note that since G is K_{r+1}-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \ldots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geqslant n/r$. Let V_1 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size n/r. Thus V_1 contains a vertex x_1^1 of degree at least $(r-1)n/r$.

As before we can find vertices $x_2^1, \ldots, x_{r-1}^1$ such that

- $x_1^1, \ldots, x_{r-1}^1$ induce a copy of K_{r-1} in G;
- $d_G(x_i^1) \geqslant (r-1)n/r$ for all $1 \leqslant i \leqslant r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^1)$ is an independent set of size at least n/r. Let V_2 denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^1)$ of size n/r. Note that $N_G(x_1^1) \cap V_1 = \emptyset$ since $x_1^1 \in V_1$ and V_1 is an independent set. Thus as $V_2 \subseteq N_G(x_1^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \ldots, V_r \subseteq V(G)$ of size n/r and vertices $x_1^1, \ldots, x_{r-1}^1$, $x_1^2, \ldots, x_{r-1}^2, \ldots, x_1^{r-1}, \ldots, x_{r-1}^{r-1}$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leqslant j \leqslant r$;
- Given any $1 \leqslant j \leqslant r-1$, $x_k^j \in V_k$ for each $1 \leqslant k \leqslant j$;
- $d_G(x_k^j) \geqslant (r-1)n/r$ for all $1 \leqslant j \leqslant r-1$ and $1 \leqslant k \leqslant r-1$;
- $x_1^j, \ldots, x_{r-1}^j$ induce a copy of K_{r-1} in G for all $1 \leqslant j \leqslant r-1$.

Clearly finding such a partition $V_1, \ldots, V_r$ of $V(G)$ implies that $G \subseteq T(n,r)$.

Suppose that for some $1 < j < r$ we have defined sets $V_1, \ldots, V_j$ and vertices $x_1^1, \ldots, x_{r-1}^1, \ldots, x_1^{j-1}, \ldots, x_{r-1}^{j-1}$ with our desired properties. Since $d_{n/r} \geqslant (r-1)n/r$ and $V_1, \ldots, V_j$ are independent sets of size n/r we can choose vertices $x_1^j, \ldots, x_j^j$ such that for all $1 \leqslant k \leqslant j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geqslant (r-1)n/r$.

This degree condition, together with the fact that $x_1^j, \dots, x_j^j$ lie in different vertex classes, implies that these vertices form a copy of K_j in G. We now greedily select further vertices $x_{j+1}^j, \dots, x_{r-1}^j$ such that

- $x_1^j, \dots, x_{r-1}^j$ induce a copy of K_{r-1} in G;
- $d_G(x_k^j) \geqslant (r-1)n/r$ for all $j+1 \leqslant k \leqslant r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least n/r. Let V_{j+1} denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size n/r. Note that, for each $1 \leqslant k \leqslant j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \in V_k$ and V_k is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leqslant k \leqslant j$, V_{j+1} is disjoint from $V_1 \cup \dots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \dots, V_r \subseteq V(G)$ and vertices $x_1^1, \dots, x_{r-1}^1$, $x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$.

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Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

Fact 19. *Fix $n, r \in \mathbb{N}$ such that $r \geqslant 3$ and r divides $n \geqslant 2r$. Define*

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

Then $h(x)$ is a decreasing function for $x \in [0, n/(r-1)]$. Moreover, if $n \geqslant 3r$ then $h(x)$ is a decreasing function for $x \in [0, (n+r)/(r-1)]$.

Proof. Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$

So for $x \leqslant n/(r-1)$,

$$h'(x) \leqslant \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.$$

Note that $3(r-1)/2 + (r-1)/(r-2) < n$ since $n \geqslant 2r$ and $r \geqslant 3$. Thus,

$$h'(x) \leqslant -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.$$

If $x \leqslant (n+r)/(r-1)$ then

$$h'(x) \leqslant \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.$$

If $n \geqslant 3r$ then $n > 3r/2 + 4$. So $n > 3(r-1)/2 + (2r-1)/(r-2)$. Thus,

$$h'(x) \leqslant -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,$$

as desired. □

Proof of Proposition 13. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leqslant D \leqslant n/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leqslant D + e(\overline{T}(n-D-1, r-2)).$$

Since $D \leqslant n/(r-1)$, Facts 14 and 19 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geqslant D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geqslant \frac{n}{r-1} + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 1\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 1\right] \\ &\geqslant \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{r-2}{2(r-1)} \geqslant \frac{n}{2r^2} + \frac{1}{2r}. \tag{16}$$

Notice that

$$\frac{r-2}{2(r-1)^2} - \frac{1}{2r^2} = \frac{(r-2)r^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2} \tag{17}$$

and

$$\frac{r-2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.$$

Since $n \geqslant 2r$, (16) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \geqslant 0. \tag{18}$$

Note that $r^3 \geqslant 4r^2 - 4r + 3$ as $r \geqslant 3$. Thus, $2(r^3 - 3r^2 + 2r - 1) \geqslant (r^2 - r - 1)(r-1)$. So indeed (18) is satisfied, as desired.

Proof of Lemma 15. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leqslant D < (n+r)/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leqslant D + e(\overline{T}(n-D-1, r-2)).$$

Since $D < (n+r)/(r-1)$ we have that $D \leqslant n/(r-1)+1$. So Facts 14 and 19 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geqslant D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geqslant \frac{n}{r-1} + 1 + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 2\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 2\right] \\ &\geqslant \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n - \frac{n}{r-1}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{(r-2)}{2(r-1)} - \frac{1}{r-1} \geqslant \frac{n}{2r^2} + \frac{1}{2r}. \tag{19}$$

Notice that

$$\frac{r-2}{2(r-1)} + \frac{1}{r-1} + \frac{1}{2r} = \frac{r^2+r-1}{2r(r-1)}.$$

Since $n \geqslant 3r$, (17) and (19) imply that it suffices to show that

$$\frac{3(r^3-3r^2+2r-1)}{2r(r-1)^2} - \frac{r^2+r-1}{2r(r-1)} \geqslant 0. \tag{20}$$

Note that $2r^3-9r^2+8r-4 \geqslant 0$ as $r \geqslant 4$. Thus, $3(r^3-3r^2+2r-1) \geqslant (r^2+r-1)(r-1)$. So indeed (20) is satisfied, as desired.