An Iterative Approach to the Traceability Conjecture for Oriented Graphs

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Submitted: Jun 10, 2011; Accepted: Mar 4, 2013; Published: Mar 24, 2013
Mathematics Subject Classifications: 05C20, 05C38

Abstract

A digraph is k-traceable if its order is at least k and each of its subdigraphs of order k is traceable. The Traceability Conjecture (TC) states that for $k \geqslant 2$ every k-traceable oriented graph of order at least $2k-1$ is traceable. It has been shown that for $2 \leqslant k \leqslant 6$, every k-traceable oriented graph is traceable. We develop an iterative procedure to extend previous results regarding the TC. In particular, we prove that every 7-traceable oriented graph of order at least 9 is traceable and every 8-traceable graph of order at least 14 is traceable.

Keywords: Traceability Conjecture, Path Partition Conjecture, oriented graph, generalized tournament, traceable, k-traceable, longest path.

*Supported by NRF grant 71308.

1 Introduction and Background

A digraph is *hamiltonian* if it contains a cycle that visits every vertex, *traceable* if it contains a path that visits every vertex, and *strong* (or *strongly connected*) if it contains a closed walk that visits every vertex. A digraph is k*-traceable* if its order is at least k and each of its subdigraphs of order k is traceable.

This paper contributes to a body of work to establish the validity of the following traceability conjecture, called the TC (see [2, 3, 4]).

Conjecture 1. (TC) For $k \geqslant 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable.

It is obvious that an oriented graph is 2-traceable if and only if it is a nontrivial tournament. Thus we can think of a k-traceable oriented graph as a generalized tournament. It is well-known that every nontrivial strong tournament is hamiltonian and every tournament is traceable. The following theorem shows that these properties are retained by k-traceable oriented graphs for small values of k.

Theorem 2. [2, 4]

1. *For $k = 2, 3, 4$, every strong k-traceable oriented graph of order at least $k + 1$ is hamiltonian.*
2. *For $k = 2, 3, 4, 5, 6$, every k-traceable oriented graph is traceable.*

However, for $k = 7$ and for every $k \geqslant 9$ there exist k-traceable oriented graphs of order $k + 1$ that are nontraceable, as shown in [6]. There also exist nontraceable k-traceable oriented graphs of order $k + 2$ for infinitely many k, as shown in [5], but the following theorem shows that the order of nontraceable k-traceable oriented graphs is bounded above by a function of k.

Theorem 3. [2, 4] *Let $k \geqslant 7$ and suppose D is a k-traceable oriented graph of order n and independence number α.*

1. *If $n \geqslant 6k - 20$, then $\alpha \leqslant 2$.*
2. *If $\alpha \leqslant 2$ and $n \geqslant 2k^2 - 20k + 59$, then D is traceable.*

It is therefore natural to ask: what is the smallest integer $t(k)$ such that $t(k) \geqslant k$ and every k-traceable oriented graph of order at least $t(k)$ is traceable? The TC asserts that $t(k) \leqslant 2k - 1$ for all $k \geqslant 2$. It seems likely that $t(k)$ is considerably less than $2k - 1$ for all $k \geqslant 2$, but proving the TC would suffice for proving the Path Partition Conjecture for 1-deficient oriented graphs, as shown in [4]. The latter conjecture is an important special case of the Path Partition Conjecture for Digraphs, which is discussed in [1, 7, 8].

In this paper we present results that suggest an iterative method for proving the TC. The method enables us to prove the special cases $k = 7, 8$ and to substantially improve Theorem 3 (2) in the cases $k = 9, 10$.

We shall use the following notation and terminology.

The set of vertices and the set of arcs of a digraph D are denoted by $V(D)$ and $A(D)$, respectively, and the order of D is denoted by $n(D)$. If $X \subset V(D)$, then $\langle X \rangle$ denotes the subdigraph induced by X in D. If $v \in V(D)$, we denote the sets of *out-neighbours* and *in-neighbours* of v in D by $N^+(v)$ and $N^-(v)$ and the cardinalities of these sets by $d^+(v)$ and $d^-(v)$, respectively. The *degree* $d(v)$ of v is defined as $d(v) = d^+(v) + d^-(v)$ and the *minimum degree* of D is denoted by $\delta(D)$. The *independence number* of D, denoted by $\alpha(D)$, is the cardinality of a largest set $X \subset V(D)$ such that $\langle X \rangle$ has no arcs.

Let $P = u_1 \dots u_p$ be a p-path and $Q = v_1 \dots v_q$ a q-path in a digraph D. If $u_p = v_1$, then the $(p+q-1)$-path $u_1 \dots u_{p-1} v_1 \dots v_q$ is called the *concatenation* of P and Q.

A maximal strong subdigraph of a digraph D is called a *strong component* of D. We say that a strong component is *trivial* if it has only one vertex. If D is a digraph with h strong components, then its strong components have an acyclic ordering $D_1, D_2, \dots, D_h$ such that if there is an arc from D_i to D_j, then $i \leqslant j$. If D is k-traceable for some $k \geqslant 2$, this acyclic ordering is unique since there is at least one arc from D_i to D_{i+1} for $i = 1, 2, \dots, h-1$. Throughout this paper we label the strong components of a k-traceable digraph in accordance with this acyclic ordering and if $1 \leqslant r \leqslant s \leqslant h$, we denote by D_r^s the subdigraph of D induced by the vertex set $\bigcup_{i=r}^{s} V(D_i)$.

2 Auxiliary results

The lemmas in this section extend results proven in [2, 4, 11]. For easy reference, we repeat some of the proofs from these papers.

The following result will be used frequently. It follows immediately from the fact that every strong tournament is hamiltonian.

Lemma 4. *Let D be a tournament with strong components $D_1, \dots, D_h$. Then every vertex in D_1 is the initial vertex of some Hamilton path of D and every vertex in D_h is the terminal vertex of some Hamilton path of D.*

Lemma 5. *Let $k \geqslant 2$ and suppose D is a k-traceable oriented graph of order n. Then the following hold.*

1. *$d(x) \geqslant n - k + 1$ for every $x \in V(D)$.*

2. *$|N^+(x) \cup N^+(y)| \geqslant n - k + 1$ and $|N^-(x) \cup N^-(y)| \geqslant n - k + 1$ for every pair of distinct nonadjacent vertices $x, y \in V(D)$.*

Proof.

1. If D has a vertex x with $d(x) \leqslant n - k$, then we can choose an induced subdigraph H of order k in D such that $x \in V(H) \subseteq V(D) - N(x)$. But then x is an isolated vertex in H, so H is nontraceable, contradicting the k-traceability of D.

2. Suppose $|N^+(x) \cup N^+(y)| \leqslant n-k$. Let H be an induced subdigraph of order k in D such that $\{x, y\} \subseteq V(H) \subseteq V(D) - (N^+(x) \cup N^+(y))$. Then H is nontraceable, since neither x nor y has an out-neighbour in H. This contradiction proves that $|N^+(x) \cup N^+(y)| \geqslant n-k+1$. A symmetric argument shows that $|N^-(x) \cup N^-(y)| \geqslant n-k+1$. □

Lemma 6. *Let $k \geqslant 2$ and let x and y be distinct nonadjacent vertices in a k-traceable oriented graph D of order n. Let*

$$S \in \{N^+(x),\ N^-(x),\ N^+(x) \cup N^+(y),\ N^-(x) \cup N^-(y)\}.$$

Now suppose $|S| = n-k+1$ and $\langle S \rangle$ is traceable. Then D is traceable.

Proof. Let $v_1 \dots v_{n-k+1}$ be a Hamilton path of $\langle S \rangle$. First, let $S = N^+(x) \cup N^+(y)$. Let $H = D - \{v_1, \dots, v_{n-k}\}$. Then $n(H) = k$, so H has a k-path P. We consider two cases.
Case 1. $\{v_1, v_{n-k+1}\} \subseteq N^+(x)$.

Since v_{n-k+1} is the only out-neighbour of x in H, it follows that either x is the terminal vertex of P, or P contains the arc xv_{n-k+1}. If the former, then $Pv_1 \cdots v_{n-k}$ is a Hamilton path of D. If the latter, then the path obtained from P by replacing the arc xv_{n-k+1} with the path $xv_1 \dots v_{n-k+1}$ is a Hamilton path of D.
Case 2. $v_1 \in N^+(x) - N^+(y)$ and $v_{n-k+1} \in N^+(y) - N^+(x)$.

In this case x has no out-neighbour in H, so x is the terminal vertex of P and hence $Pv_1 \dots v_{n-k}$ is a Hamilton path of D.

The argument in Case 1 above also proves the case when $S = N^+(x)$ and the remaining cases follow by symmetric arguments. □

Our next result is an immediate consequence of the fact that the strong components of an oriented graph have an acyclic ordering.

Lemma 7. *If P is a path in an oriented graph D, then the intersection of P with any strong component of D is either empty or a path.*

The following two lemmas are consequences of Lemma 7.

Lemma 8. *Let D be a k-traceable oriented graph with strong components $D_1, \dots, D_h$. Let $1 < t < h$ and let $p = n(D_1^{t-1})$, $q = n(D_t)$ and $r = n(D_{t+1}^h)$. Then the following hold.*

1. *If $0 \leqslant i \leqslant r$ and $1 \leqslant k-i \leqslant p+q$, then D_1^t is $(k-i)$-traceable.*

2. *If $0 \leqslant i \leqslant p$ and $1 \leqslant k-i \leqslant q+r$, then D_t^h is $(k-i)$-traceable.*

3. *If $0 \leqslant i \leqslant p+r$ and $1 \leqslant k-i \leqslant q$, then D_t is $(k-i)$-traceable.*

Lemma 9. *Let $k \geqslant 2$ and let D be a k-traceable oriented graph of order $n \geqslant 2k-5$ with strong components $D_1, \dots, D_h$. Then for every positive integer $i \leqslant h-1$ at least one of D_1^i and D_{i+1}^h is a tournament.*

Proof. Suppose, to the contrary, that for some $i \leqslant h-1$ neither D_1^i nor D_{i+1}^h is a tournament. Since $n \geqslant 2k-5$, one of D_1^i and D_{i+1}^h, say D_1^i, has at least $k-2$ vertices. Let H be an induced subdigraph of D such that H contains $k-2$ vertices of D_1^i together with any two nonadjacent vertices of D_{i+1}^h. Then it follows from Lemma 7 that H is nontraceable, contrary to the hypothesis. □

Lemma 10. *Let $k \geqslant 7$ and suppose D is a nontraceable k-traceable oriented graph of order $n \geqslant 2k-3$. Then D has a nonhamiltonian strong component D_t of order at least $n-k+5$ and D_1^{t-1} and D_{t+1}^h are tournaments, whenever they are defined.*

Proof. Let t be the smallest integer such that D_1^t is not a tournament. If $t < h$, then D_{t+1}^h is a tournament by Lemma 9. Furthermore, if $t > 1$, then D_1^{t-1} is a tournament by the minimality of t.

Now suppose D_t is hamiltonian. Then, since D is nontraceable, it follows from Lemma 4 that $1 < t < h$. Let C be a Hamilton cycle of D_t. Then, for every in-neighbour of D_{t+1} on C, its successor is not an out-neighbour of D_{t-1}. Hence $V(C) - N_C^+(D_{t-1}) \neq \emptyset$. Now suppose $|N_C^+(D_{t-1})| \leqslant n-k$. Then let H be a subdigraph of order k in $V(D) - N_C^+(D_{t-1})$ such that H contains at least one vertex in $V(C) - N_C^+(D_{t-1})$ and at least one vertex in D_{t-1}. Then H is nontraceable, contradicting that D is k-traceable. Hence $|N_C^+(D_{t-1})| \geqslant n-k+1 \geqslant (2k-3)-k+1 = k-2$, which implies that at least $k-2$ vertices in C are not in $N_C^-(D_{t+1})$. Let H be a subdigraph of order k that has $k-2$ vertices in $V(C) - N_C^-(D_{t+1})$, together with one vertex from D_{t-1} and one from D_{t+1}. Then H has order k but is nontraceable. This contradiction shows that D_t is not hamiltonian.

Since D_1^{t-1} and D_{t+1}^h are tournaments but D is nontraceable, it follows from Lemma 4 that $n(D_t) \neq 1$. Thus D_t is a nonhamiltonian, strong oriented graph of order at least 4. Now suppose $n(D_t) \leqslant n-k+4$. Then $n(D - V(D_t)) \geqslant k-4$ and hence Lemma 8(3) implies that D is 4-traceable. If $n(D_t) \geqslant 5$, this contradicts Theorem 2. If $n(D_t) = 4$, then $n(D - D_t) \geqslant 2k-7 > k-3$ and then Lemma 8 implies that D_t is 3-traceable, which again contradicts Theorem 2. □

Chen and Manalastas [10] proved that every strong digraph with independence number two is traceable. Havet [12] strengthened their result as follows.

Theorem 11. [12] *If D is a strong digraph with $\alpha(D) = 2$, then D has two nonadjacent vertices that are terminal vertices of Hamilton paths in D and two nonadjacent vertices that are initial vertices of Hamilton paths in D.*

The following corollary of Havet's result appears in [2].

Corollary 12. *If D is a connected digraph with $\alpha(D) = 2$ and at most two strong components, then D is traceable.*

Proof. By the Chen-Manalastas Theorem we may assume that D has exactly two strong components, D_1 and D_2, and both are traceable. Since every strong tournament is hamiltonian, we may assume that at least one of the two strong components, say D_1, is not a tournament. By Theorem 11, D_1 has two nonadjacent vertices y and z, each of which is a terminal vertex of a Hamilton path of D_1. Let a be the initial vertex of a Hamilton path of D_2. Then at least one of y and z is adjacent to a and hence D has a Hamilton path. □

Thus every nontraceable oriented graph D with $\alpha(D) = 2$ has at least three strong components. The following lemma provides useful information on the strong component structure of nontraceable k-traceable oriented graphs of order at least $2k-3$. Item 5 provides additional information in the case when the independence number equals 2.

Lemma 13. *Let $k \geqslant 7$ and suppose D is a nontraceable k-traceable oriented graph of order $n \geqslant 2k-3$, with strong components $D_1, \ldots, D_h$. Let D_t be the nonhamiltonian strong component of D of order at least $n-k+5$. Then the following hold.*

1. *If $t > 1$, then $|N^+_{D_t}(D_{t-1})| \geqslant n-k+1$ and if $|N^+_{D_t}(D_{t-1})| = n-k+1$, then $\langle N^+_{D_t}(D_{t-1})\rangle$ is nontraceable.*
2. *If $t < h$, then $|N^-_{D_t}(D_{t+1})| \geqslant n-k+1$ and if $|N^-_{D_t}(D_{t+1})| = n-k+1$, then $\langle N^-_{D_t}(D_{t+1})\rangle$ is nontraceable.*
3. *If $t > 1$ and $v \in V(D_t) - N^+_{D_t}(D_{t-1})$, then $|N^-_{D_t}(v)| \geqslant n-k+1$ and if $|N^-_{D_t}(v)| = n-k+1$, then $\langle N^-_{D_t}(v)\rangle$ is nontraceable.*
4. *If $t < h$ and $v \in V(D_t) - N^-_{D_t}(D_{t+1})$, then $|N^+_{D_t}(v)| \geqslant n-k+1$ and if $|N^+_{D_t}(v)| = n-k+1$, then $\langle N^+_{D_t}(v)\rangle$ is nontraceable.*
5. *If $\alpha(D) = 2$, then $1 < t < h$ and D_t^h has a Hamilton path such that the successor of the last in-neighbour of its initial vertex has an in-neighbour in D_{t-1}.*

Proof. We shall prove (1), (3) and (5). The proofs of (2) and (4) are similar to those of (1) and (3), respectively. First we note from Lemma 10 that D_1^{t-1} and D_{t+1}^h are tournaments.

1. Suppose $|N^+_{D_t}(D_{t-1})| \leqslant n-k$. Let S consist of k vertices from the set $V(D) - N^+_{D_t}(D_{t-1})$ such that S contains at least one vertex in D_{t-1} and at least one in D_t. Then, since there are no arcs from $S \cap V(D_{t-1})$ to $S \cap V(D_t)$, the induced subdigraph $\langle S\rangle$ is nontraceable.

   Next suppose $N^+_{D_t}(D_{t-1}) = n-k+1$ and $\langle N^+_{D_t}(D_{t-1})\rangle$ contains an $(n-k+1)$-path $u_1 \ldots u_{n-k+1}$. Let H be the subdigraph of D induced by the vertex set $V(D) - \{u_1, \ldots, u_{n-k}\}$. Then $n(H) = k$, so H has a k-path P. Since u_{n-k+1} is the only out-neighbour of D_{t-1} in H, the intersection of the path P with D_t^h is a path R that has u_{n-k+1} as its initial vertex. Let x be a vertex in D_{t-1} such that $u_1 \in N^+(x)$. Since D_1^{t-1} is a tournament, D_1^{t-1} has a Hamilton path Q ending in x. Thus the path $Qu_1 \ldots u_{n-k}R$ is an n-path of D.

3. Let $v \in V(D_t) - N^+_{D_t}(D_{t-1})$ and suppose $|N^-_{D_t}(v)| \leqslant n-k$. Then let S consist of k vertices in $V(D) - N^-_{D_t}(v)$ such that S contains v and at least one vertex y in D_{t-1}. Since v has no in-neighbours in $S \cap D_{t-1}^t$, no path in $\langle S\rangle$ contains both v and y, so $\langle S\rangle$ is nontraceable.

Now suppose $|N^-_{D_t}(v)| = n - k + 1$ and $\langle N^-_{D_t}(v)\rangle$ contains an $(n - k + 1)$-path $u_1 \dots u_{n-k+1}$. Let H be the subdigraph of D induced by the vertex set $V(D) - \{u_2, \dots, u_{n-k+1}\}$. Then $n(H) = k$, so H has a k-path P. Since $t > 1$, v is not the initial vertex of P, and since u_1 is the only in-neighbour of v in $H \cap D^t_{t-1}$, the arc u_1v is in P. But then the path obtained from P by replacing the arc u_1v with the path $u_1 \dots u_{n-k+1}v$ is an n-path of D.

5. By Corollary 12, $h \geqslant 3$. First suppose $t = 1$. Then D^h_2 is a tournament and hence by Lemma 4 every vertex in the strong component D_2 is an initial vertex of a Hamilton path of D^h_2. But, by Theorem 11, there exist two nonadjacent vertices u and v in $V(D_1)$ such that both are end vertices of Hamilton paths in D_1. Since we are assuming D is nontraceable, this implies that both u and v are nonadjacent with every vertex in D_2, contradicting that $\alpha(D) = 2$. Hence $t \neq 1$ and we can prove similarly that $t \neq h$.

   Let $n(D_t) = q$. Then $q \leqslant n-2$. Since $\alpha(D) = 2$ and D^h_{t+1} is a tournament it follows from Lemma 4 and Theorem 11 that D^h_t is traceable. Among all the Hamilton paths in D^h_t, choose one such that the subpath from its initial vertex to the last in-neighbour of that initial vertex on the Hamilton path has maximum order. Let the intersection of this Hamilton path of D^h_t with D_t be $Q = v_1 \dots v_q$ and its intersection with D^h_{t+1} be R. Let v_ℓ be the last in-neighbour of v_1 on Q and let $S = v_{\ell+1} \dots v_q R$.

   Suppose $v_{\ell+1}$ has no in-neighbour in D_{t-1}. Then v_1 and $v_{\ell+1}$ are adjacent since $\alpha(D) = 2$ and so $v_{l+1} \in N^+(v_1)$ by the choice of l. But now $Q' = v_2v_3 \dots v_\ell v_1 S$ is a Hamilton path of D^h_t and therefore $v_2 \notin N^+(D_{t-1})$. But by the maximality of ℓ, v_1 is the last in-neighbour of v_2 on Q', so $v_{l+1} \notin N^-(v_2)$. Since $\alpha(D) = 2$, this implies that $v_{l+1} \in N^+(v_2)$. But then $v_3v_4 \dots v_\ell v_1 v_2 S$ is a Hamilton path in D^h_t. Repeating this process we can show that $v_i \notin N^+(D_{t-1})$ for $i = 1, \dots, \ell$. Since $N^-_{D_t}(v_1) \subseteq \{v_3, \dots, v_\ell\}$, it follows by Lemma 13(3) that $\ell \geqslant n - k + 4$ and hence $|N^+_{D_t}(D_{t-1})| \leqslant q - (\ell + 1) \leqslant n - 2 - (n - k + 4) - 1 = k - 7$. But by Lemma 13(1), $|N^+_{D_t}(D_{t-1})| \geqslant n - k + 1 \geqslant 2k - 3 - k + 1 = k - 2$, a contradiction. □

3 Main Results

Lemma 14. *Let D be an oriented graph of order n and suppose there exist integers n_1, n_2 such that D is n_1-traceable as well as n_2-traceable and $n = n_1 + n_2 - j$; $j = 1$ or 2. Suppose D has a vertex v such that*

$$d^-(v) \leqslant n_1 \text{ and } d^+(v) \leqslant n_2 \text{ if } j = 1$$

and

$$d^-(v) < n_1 \text{ and } d^+(v) < n_2 \text{ if } j = 2.$$

Then D is traceable.

Proof.
Case 1. $j = 1$.

Suppose $d^-(v) = n_1$. Then $|N^-(v)| = n_1 = n - n_2 + 1$. Since D is n_1-traceable, $\langle N^-(v)\rangle$ is traceable and hence, since D is also n_2-traceable, it follows from Lemma 6 that D is traceable. Similarly, if $d^+(v) = n_2$, then D is traceable.

We therefore assume that $d^-(v) \leqslant n_1 - 1$ and $d^+(v) \leqslant n_2 - 1$. Then we can partition $V(D) - \{v\}$ into two sets U and W such that $|U| = n_1 - 1$, $|W| = n_2 - 1$ and $N^-(v) \subseteq U$, $N^+(v) \subseteq W$. By the n_1-traceability of D and the fact that v has no out-neighbours in U, the subdigraph $\langle U \cup \{v\}\rangle$ has an n_1-path P with v as terminal vertex. Similarly, by the n_2-traceability of D and the fact that v has no in-neighbours in W, the subdigraph $\langle \{v\} \cup W\rangle$ has an n_2-path Q with v as initial vertex. The concatenation of P and Q is an n-path of D.
Case 2. $j = 2$.

Since $d^-(v) + d^+(v) \leqslant n - 1 = n_1 + n_2 - 3$, we cannot have both $d^-(v) = n_1 - 1$ and $d^+(v) = n_2 - 1$. By symmetry, we may assume $d^+(v) \leqslant n_2 - 2$. Then we can partition $V(D) - \{v\}$ into two sets U, W such that $|U| = n_1 - 1$, $|W| = n_2 - 2$ and $N^-(v) \subseteq U$, $N^+(v) \subseteq W$. Then $\langle U \cup \{v\}\rangle$ has an n_1-path $u_1u_2 \dots u_{n_1-1}v$ and $\langle W \cup \{u_1, v\}\rangle$ has an n_2-path Q. If $u_1v \in A(Q)$, then the path obtained from Q by replacing the arc u_1v with the path $u_1 \dots u_{n_1-1}v$ is an n-path of D. If $u_1v \notin A(Q)$, then v is the initial vertex of Q and then $u_2 \dots u_{n_1-1}Q$ is an n-path of D. □

In order to be able to apply Lemma 14, we need a vertex with sufficiently small in- and out-degree. For k-traceable oriented graphs with independence number greater than 2 we have the following result.

Lemma 15. *Let $k \geqslant 5$ and suppose D is a k-traceable oriented graph of order n and $\{v_1, v_2, v_3\}$ is an independent set of vertices in D. Then the following hold.*

1. $\min\{d^-(v_i), d^+(v_i)\} \leqslant (n-3)/2$ *for each* $i \in \{1, 2, 3\}$.
2. $\max\{d^-(v_i), d^+(v_i)\} \leqslant (n+k-7)/2$ *for at least one* $i \in \{1, 2, 3\}$.

Proof.

1. For each $i \in \{1, 2, 3\}$, the three vertices v_1, v_2, v_3 are not in $N(v_i)$, so $d^-(v_i) + d^+(v_i) \leqslant n - 3$.
2. Suppose $\max\{d^-(v_i), d^+(v_i)\} \geqslant (n+k-6)/2$ for each $i \in \{1, 2, 3\}$. Then we may assume without loss of generality that $d^+(v_i) \geqslant (n+k-6)/2$ for $i = 1, 2$. Then $d^-(v_i) \leqslant n-3-(n+k-6)/2 = (n-k)/2$ for $i = 1, 2$. But then $d^-(v_1)+d^-(v_2) \leqslant n-k$, contradicting Lemma 5(2). □

By combining Lemmas 14 and 15 we obtain the following iteration theorem for k-traceable oriented graphs with independence number greater than 2.

Theorem 16. *Let $k \geqslant 5$ and suppose n_1 and n_2 are integers such that $k \leqslant n_1 \leqslant n_2$ and every k-traceable oriented graph of order n_i is traceable for $i = 1, 2$. If $n = n_1 + n_2 - j$; $j = 1$ or 2, and*

$$k - 9 \leqslant n_2 - n_1 \leqslant 5 \text{ if } j = 1,$$

$$k - 9 < n_2 - n_1 < 5 \text{ if } j = 2,$$

then every k-traceable oriented graph of order n with independence number at least 3 is traceable.

Proof. Let D be a k-traceable oriented graph with independence number at least 3 and order $n = n_1 + n_2 - j$; $j = 1$ or 2. Then our assumption implies that D is n_1-traceable as well as n_2-traceable. Hence, by Lemma 15, D has a vertex v such that

$$\min\{d^-(v), d^+(v)\} \leqslant \lfloor (n-3)/2 \rfloor \text{ and } \max\{d^-(v), d^+(v)\} \leqslant \lfloor (n+k-7)/2 \rfloor.$$

Now let $n = n_1 + n_2 - 1$. Then, since $n_2 \leqslant n_1 + 5$,

$$\lfloor (n-3)/2 \rfloor = \lfloor (n_1 + n_2 - 4)/2 \rfloor \leqslant \lfloor (2n_1 + 1)/2 \rfloor = n_1,$$

and, since $n_1 \leqslant n_2 - k + 9$,

$$\lfloor (n+k-7)/2 \rfloor = \lfloor (n_1 + n_2 + k - 8)/2 \rfloor \leqslant \lfloor (2n_2 + 1)/2 \rfloor = n_2.$$

Thus $d^-(v) \leqslant n_1$ and $d^+(v) \leqslant n_2$, or $d^+(v) \leqslant n_1$ and $d^-(v) \leqslant n_2$. In either case, it follows from Lemma 14 that D is traceable. (In the second case we interchange the labels of n_1 and n_2 before applying Lemma 14.)

If $n = n_1 + n_2 - 2$, then, since $n_2 \leqslant n_1 + 4$ and $n_1 \leqslant n_2 - k + 8$, it follows that

$$(n-3)/2 \leqslant (2n_1 - 1)/2 < n_1 \text{ and } (n+k-7)/2 \leqslant (2n_2 - 1)/2 < n_2$$

so in this case it also follows from Lemma 14 that D is traceable. □

By Corollary 12, a nontraceable digraph with independence number 2 has at least three strong components. For such digraphs it is convenient to consider the in- and out-degrees of a vertex in its own component only (in stead of in the whole digraph). The following lemma is useful in this respect.

Lemma 17. *Let D be an oriented graph of order n with strong components $D_1, \ldots, D_h$; $h \geqslant 3$. Suppose n_1, n_2 are integers such that D is n_1-traceable as well as n_2-traceable and $n = n_1 + n_2 - j$; $j = 1$, 2 or 3. If for some $t \in \{2, \ldots, h-1\}$ there is a vertex $v \in V(D_t)$ such that*

$$d^-_{D_t}(v) \leqslant n_1 - p \text{ and } d^+_{D_t}(v) \leqslant n_2 - r \text{ if } j = 1 \text{ or } 2,$$

and

$$d^-_{D_t}(v) < n_1 - p \text{ and } d^+_{D_t}(v) < n_2 - r \text{ if } j = 3,$$

where $p = n(D_1^{t-1})$ and $r = n(D_{t+1}^h)$, then D is traceable.

Proof. Let
$$X = V(D_1^{t-1}),\ Z = V(D_{t+1}^h).$$

Case 1. $j = 1$.

We note that $d^-(v) \leqslant d^-_{D_t}(v) + p \leqslant n_1$ and $d^+(v) \leqslant d^+_{D_t}(v) + r \leqslant n_2$ and hence Lemma 14 implies that D is traceable.

Case 2. $j = 2$.

If $d^-_{D_t}(v) < n_1 - p$ and $d^+_{D_t}(v) < n_2 - r$, it follows from Lemma 14 that D is traceable.

If $d^-_{D_t}(v) = n_1 - p$, let $U = X \cup N^-_{D_t}(v)$. Then $|U| = n_1$ and $|V(D) - U| = n_2 - 2$. Hence $\langle U \rangle$ has an n_1-path $P = u_1 \dots u_{n_1}$. Let u_l be the last vertex of P in D_{t-1}. Then $\langle V(D-U) \cup \{u_l, u_{l+1}\}\rangle$ has order n_2 and hence has an n_2-path Q with u_l as initial vertex. If the arc $u_{l+1}v$ is in Q, let Q^* be the path obtained from Q by replacing the arc $u_{l+1}v$ with the path $u_{l+1} \dots u_{n_1} v$. Then the concatenation of the paths $u_1 \dots u_l$ and Q^* is a Hamilton path of D. If $u_{l+1}v$ is not in Q, then $u_l v$ is the first arc of Q. Then the path $Q' = Q - u_l$ is an $(n_2 - 1)$-path of $\langle V(D - U) \cup \{u_{l+1}\}\rangle$, with v as initial vertex. But $\langle (U - \{u_{l+1}\}) \cup \{v\}\rangle$ has order n_1 and hence has an n_1-path R with v as terminal vertex. The concatenation of R and Q' is an $(n_1 + n_2 - 2)$-path of D.

A symmetric argument proves that D is traceable if $d^+_{D_t}(v) = n_2 - r$.

Case 3. $j = 3$.

If $d^-_{D_t}(v) = n_1 - p - 1$, let $U = X \cup N^-(v)$. Then $|U| = n_1 - 1$ and $n(D-U) = n_2 - 2$. Hence $U \cup \{v\}$ has an n_1-path $P = u_1 \dots u_{n_1 - 1} v$. Let u_l be the last vertex of P in D_{t-1}. Then $\langle V(D-U) \cup \{u_l, u_{l+1}\}\rangle$ has order n_2 and hence has an n_2-path Q, with u_l as initial vertex.

If the arc $u_{l+1}v$ is in Q, let Q^* be the path obtained from Q by replacing the arc $u_{l+1}v$ with the path $u_{l+1} \dots u_{n_1 - 1} v$. Then the concatenation of the paths $u_1 \dots u_l$ and Q^* is a Hamilton path of D.

If $u_{l+1}v$ is not in Q, then $u_l v$ is the first arc of Q. Then the path $Q' = Q - u_l$ is an $(n_2 - 1)$-path of $\langle V(D-U) \cup \{u_{l+1}\}\rangle$, with v as initial vertex. By Lemma 8, D_1^t is $(n_1 - 1)$-traceable, and hence $\langle (U - \{v_{l+1}\}) \cup \{v\}\rangle$ has an $(n_1 - 1)$-path R with v as terminal vertex. The concatenation of R and Q' is an $(n_1 + n_2 - 3)$-path in D. A symmetric argument shows that D is traceable if $d^+_{D_t}(v) = n_2 - r - 1$.

Thus we may assume $d^-_{D_t}(v) \leqslant n_1 - p - 2$ and $d^+_{D_t}(v) \leqslant n_2 - r - 2$. Then we can partition $D - v$ into two sets U, W such that $|U| = n_1 - 2$, $|W| = n_2 - 2$ and $X \cup N^-(v) \subseteq U$, $N^+(v) \cup Z \subseteq W$. Since D_1^t is $(n_1 - 1)$-traceable and D_t^h is $(n_2 - 1)$-traceable, $\langle U \cup \{v\}\rangle$ has an $(n_1 - 1)$-path P with v as terminal vertex, and $\langle \{v\} \cup W\rangle$ has an $(n_2 - 1)$-path with v as initial vertex. The concatenation of P and Q is an $(n_1 + n_2 - 3)$-path in D. □

By using Lemma 17, together with results on the structure of k-traceable oriented graphs with independence number 2, we obtain the following iteration theorem in the case when $k \leqslant 10$.

Theorem 18. *Let $7 \leqslant k \leqslant 10$ and suppose there exist integers n_1, n_2, such that $k \leqslant n_1 \leqslant n_2$ and every k-traceable oriented graph with independence number 2 and order n_i is traceable for $i = 1, 2$. Then every k-traceable oriented graph with independence number 2 and order $n_1 + n_2 - j$ is traceable, for $j = 1, 2, 3$.*

Proof. Suppose, to the contrary, that there exists a nontraceable k-traceable oriented graph D with $\alpha(D) = 2$ and order $n_1 + n_2 - j$; $j = 1, 2$ or 3. Then, by our assumption, D is also n_1-traceable and n_2-traceable. Let $D_1, \ldots, D_h$ be the strong components of D. Since $n(D) \geqslant n_1 + n_2 - 3 \geqslant 2k - 3$, Lemma 10 implies that D has a nonhamiltonian strong component D_t of order at least $n - k + 5$ and that D_1^{t-1} and D_{t+1}^h are tournaments. Let

$$n(D_1^{t-1}) = p,\ n(D_t) = q,\ n(D_{t+1}^h) = r.$$

Lemma 13(5) implies that $p \geqslant 1$, $r \geqslant 1$ and D_t^h has a Hamilton path $v_1 \ldots v_{q+r}$ such that if v_l is the last in-neighbour of v_1 on this path, then v_{l+1} has an in-neighbour x in D_{t-1}. Let

$$L = v_1 \ldots v_l \text{ and } Q = v_1 \ldots v_q.$$

The structure of D is depicted in Figure 1.

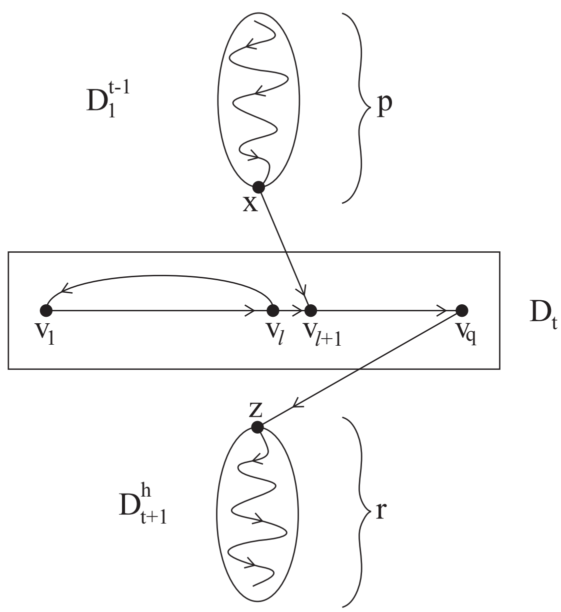

Figure 1: Structure of D

The following three claims will be used repeatedly. They are all easy consequences of Lemma 4 and our assumption that D is nontraceable.

Claim (i) If $v_i \in N^+_{D_t}(D_{t-1})$ for some $i \in \{2, 3, \ldots, l\}$, then $v_{i-1} \notin N^-_{D_t}(v_{l+1})$.

Claim (ii) If $v_i \in N^+(v_j)$ for some $i \in \{2, 3, \ldots, l\}$ and $j \in \{l+1, \ldots, q-1\}$, then $v_{i-1} \notin N^-(v_{j+1})$.

Claim (iii) If $v_i \in N^+(v_q)$ for some $i \in \{2, 3, \ldots, l\}$, then $v_{i-1} \notin N^-_{D_t}(D_{t+1})$.

Now we show that $|N^+_{D_t}(D_{t-1})| \geqslant n - n_1 + 2$.

If $n_1 > k$, then it follows from Lemma 13(1) that $|N^+_{D_t}(D_{t-1})| \geqslant n - n_1 + 2$.

If $n_1 = k$, then $n - k + 1 = n_1 + n_2 - j - n_1 + 1 = n_2 - j + 1$. Since $p + r \geqslant 2$, and $q \geqslant n - k + 5 > n_2$, Lemma 8(3) implies that D_t is $(n_2 - i)$-traceable for $i = 0, 1, 2$. Since we are assuming that $j = 1$, 2 or 3, this implies that D_t is $(n_2 - j + 1)$-traceable, i.e., D_t is $(n - k + 1)$-traceable. Hence, if $|N^+_{D_t}(D_{t-1})| = n - k + 1$, then $\langle N^+_{D_t}(D_{t-1})\rangle$ is traceable, contradicting the second part of Lemma 13(1). Thus $|N^+_{D_t}(D_{t-1})| \neq n - k + 1$ and hence, by the first part of Lemma 13(1), $|N^+_{D_t}(D_{t-1})| \geqslant n - k + 2$.

Thus, in either case $|N^+_{D_t}(D_{t-1})| \geqslant n - n_1 + 2$. Hence

$$|N^+_L(D_{t-1})| \geqslant n - n_1 + 2 - (q - l). \tag{I}$$

Since D_t is nonhamiltonian, $l < q$ and since $N^-_{D_t}(v_1) \subseteq \{v_3, \ldots, v_{q-1}\}$, it follows from Lemma 13(3) that $l \geqslant n - k + 4$. But $q \leqslant n - 2$, so

$$1 \leqslant q - l \leqslant k - 6. \tag{II}$$

We consider two cases, depending on the difference between q and l.

Case 1. $q - l = 1$.

In this case (I) becomes

$$|N^+_L(D_{t-1})| \geqslant n - n_1 + 1.$$

Hence, by Claim (i), at least $n - n_1 + 1$ vertices in L are not in $N^-(v_q)$. Since v_q is also not in $N^-(v_q)$, it follows that

$$d^-_{D_t}(v_q) \leqslant q - (n - n_1 + 2) < n_1 - p$$

since $n = p + q + r$. Now, suppose $d^+_{D_t}(v_q) \geqslant n_2 - r$. Then, by Claim (iii),

$$|N^-_{D_t}(D_{t+1})| \leqslant q - (n_2 - r) = q + r - n_2 = n - p - n_2 < n - k,$$

contradicting Lemma 13(2). Hence $d^+_{D_t}(v_q) < n_2 - r$. Thus we have shown that $d^-(v_q) < n_1 - p$ and $d^+(v_q) < n_2 - r$, contradicting Lemma 17.

Case 2. $q - l \geqslant 2$.

It follows from (I) and (II) that

$$|N^+_L(D_{t-1})| \geqslant (n - n_1 + 2) - (k - 6) = n - n_1 - k + 8.$$

Hence, by Claim (i) and the fact that v_{l+1} as well as v_{l+2} are not in $N^-_{D_t}(v_{l+1})$,

$$d^-_{D_t}(v_{l+1}) \leqslant q - n + n_1 + k - 10.$$

Since $q = n - p - r$, $r \geqslant 1$, and $k \leqslant 10$, it follows that

$$d^-_{D_t}(v_{l+1}) < n_1 - p \tag{III}$$

Now we consider two subcases.

Case 2.1. $n = n_1 + n_2 - 1$ or $n_1 + n_2 - 2$.

Since we are assuming that D is nontraceable, it follows from (III) and Lemma 17 that

$$d^+_{D_t}(v_{l+1}) \geqslant n_2 - r + 1.$$

We now show, by means of induction, that

$$d^+_{D_t}(v_i) \geqslant n_2 - r + 1 \text{ for } i = l+1, \ldots, q.$$

Suppose $d^+_{D_t}(v_i) \geqslant n_2 - r + 1$ for some $i \in \{l+1, \ldots, q-1\}$. Then

$$d^+_L(v_i) \geqslant n_2 - r + 1 - (q - l - 1) \geqslant n_2 - r - k + 8$$

since $q - l \leqslant k - 6$. Hence, by Claim (ii),

$$d^-_{D_t}(v_{i+1}) \leqslant q - (n_2 - r - k + 8) - 1 \leqslant n_1 - p,$$

since $q+r = n-p \leqslant n_1+n_2-1-p$ and $k \leqslant 10$. Hence, by Lemma 17, $d^+_{D_t}(v_{i+1}) \geqslant n_2-r+1$.

This completes the induction and proves that $d^+_{D_t}(v_q) \geqslant n_2 - r + 1$.

Since $q \geqslant l+2$, and $v_{q-1}, v_q \notin N^+(v_q)$, at most $q - l - 2$ out-neighbours of v_q are in $D_t - L$. Hence

$$d^+_L(v_q) \geqslant n_2 - r + 1 - (q - l - 2) \geqslant n_2 - r - k + 9.$$

Hence, by Claim (iii),

$$|N^-_{D_t}(D_{t+1})| \leqslant q - (n_2 - r - k + 9) \leqslant n - p - n_2 + k - 9 \leqslant n - n_2 \leqslant n - k,$$

contradicting Lemma 13(2).

Case 2.2. $n = n_1 + n_2 - 3$.

In this case it follows from (III) and Lemma 17 that $d^+_{D_t}(v_{l+1}) \geqslant n_2 - r$. Now suppose we have shown that $d^+_{D_t}(v_i) \geqslant n_2 - r$ for some $i \in \{l+1, \ldots, q-1\}$. Then

$$d^+_L(v_i) \geqslant n_2 - r - (q - l - 1) \geqslant n_2 - r - k + 7.$$

Hence, by Claim (ii),

$$d^-_{D_t}(v_{i+1}) \leqslant q - (n_2 - r - k + 7) - 1 < n_1 - p,$$

since $q+r = n-p = n_1+n_2-3-p$ and $k \leqslant 10$. Hence, by Lemma 17, $d^+_{D_t}(v_{i+1}) \geqslant n_2-r$. Thus we have shown by induction that $d^+_{D_t}(v_q) \geqslant n_2 - r$. Hence

$$d^+_L(v_q) \geqslant n_2 - r - (q - l - 2) \geqslant n_2 - r - k + 8.$$

Hence, by Claim (iii),

$$|N^-_{D_t}(D_{t+1})| \leqslant q - (n_2 - r - k + 8) \leqslant n - n_2 + 1.$$

If $n_2 > k$, this contradicts Lemma 13(2). If $n_2 = k$, then $n - k + 1 = n - n_2 + 1 = n_1 - 2$. Since D_t is $(n_1 - 2)$-traceable, this also contradicts Lemma 13(2). □

In order to apply our two iteration theorems effectively, we need some initial values for n_1 and n_2. For $k = 7$, 8, 9, these are provided by the following theorem, which Burger [9] derived by means of computer search.

Theorem 19. [9]

1. *Every* 7*-traceable oriented graph of order* 9*,* 10 *or* 11 *is traceable.*

2. *Every* 8*-traceable oriented graph of order* 9*,* 10 *or* 11 *is traceable.*

3. *Every* 9*-traceable oriented graph of order* 11 *is traceable.*

Theorems 16, 18 and 19 now enables us to prove the following four theorems.

Theorem 20. *Every* 7*-traceable oriented graph of order at least* 9 *is traceable.*

Proof. By Theorem 19, every 7-traceable oriented graph of order 9, 10 or 11 is traceable. Hence every 7-traceable oriented graph of order at least 12 is also 9-, 10- and 11-traceable.

Let D be a 7-traceable oriented graph of order n.

First, suppose $\alpha(D) = 2$. If $n = 12$ or 13, we apply Theorem 18 with $n_1 = n_2 = 7$ and $j = 1, 2$ to prove that D is traceable. For $n = 14$, we take $n_1 = 7$, $n_2 = 9$, $j = 2$. We conclude that every 7-traceable oriented graph with independence number 2 and order 12, 13 or 14 is traceable. Then we show that every 7-traceable oriented graph with independence number 2 and order $n \geqslant 15$ is traceable, by applying Theorem 18 iteratively with $n_1 = 7$, $n_2 = n - 6$ and $j = 1$.

Now suppose $\alpha(D) \geqslant 3$. Then $n < 22$ by Theorem 3(1). If $n_1, n_2 \in \{7, 9, 10, 11\}$ and $n_1 \leqslant n_2$, then $n_2 - n_1 < 5$, so it follows from Theorem 16 that D is traceable if $12 \leqslant n \leqslant 21$. □

Theorem 21.

1. *Every* 8*-traceable oriented graph with independence number* 2 *and order at least* 13 *is traceable.*

2. *Every* 8*-traceable oriented graph of order at least* 14 *is traceable.*

Proof. Let D be an 8-traceable oriented graph of order $n \geqslant 13$. By Theorem 19, D is also 9-, 10- and 11- traceable.

1. If $\alpha(D) = 2$, we use $n_1, n_2 \in \{8, 9, 10\}$ in Theorem 18 to prove that D is traceable if $13 \leqslant n \leqslant 19$. Then we show that D is traceable if $n = 20,\ 21,\ 22,\ \ldots$ by putting $n_1 = 8$ and $n_2 = 13,\ 14,\ 15,\ \ldots$ in successive applications of Theorem 18.

2. If $\alpha(D) \geqslant 3$, then $n < 27$ by Theorem 3(1). If $14 \leqslant n \leqslant 21$, we use Theorem 16 with $n_1, n_2 \in \{8, 9, 10, 11\}$ to prove that D is traceable. Then we use $n_1, n_2 \in \{10, 11, 14, 15\}$ to prove it for $21 < n < 27$. □

We have not yet succeeded in settling the case $k = 9$ of the TC, but our next result shows that if there exists a 9-traceable counterexample D to the TC, then $\alpha(D) \geqslant 3$ and $21 < n(D) < 33$.

Theorem 22.

1. *Every* 9*-traceable oriented graph with independence number* 2 *and order* $n \geqslant 15$ *is traceable.*

2. *Every* 9*-traceable oriented graph of order* n *is traceable if* $n \in \{11, 17, 18, 19, 21\}$ *or* $n \geqslant 33$.

Proof. Let D be a 9-traceable oriented graph of order $n \geqslant 15$. By Theorem 19(3), D is also 11-traceable.

1. Suppose $\alpha(D) \leqslant 2$. We use Theorem 18 with $n_1, n_2 \in \{9, 11\}$ to prove that D is traceable if $15 \leqslant n \leqslant 21$. Then we prove it for $n = 22,\ 23,\ 24, \ldots$ by putting $n_1 = 9$ and $n_2 = 16,\ 17,\ 18, \ldots$ in successive applications of Theorem 18.

2. Suppose $\alpha(D) \geqslant 3$. Then $n < 34$ by Theorem 3(1). We use Theorem 16 with $n_1, n_2 \in \{9, 11\}$ to prove that D is traceable if $n \in \{17, 18, 19, 21\}$. Then we use $n_1 = n_2 = 17$ to prove it for $n = 33$. □

We do not have a result similar to Theorem 19 for 10-traceable oriented graphs. (Theorem 19 already required a lot of computer time.) However, in the case $\alpha = 2$, we can apply Theorem 18 iteratively, starting with n_1 and n_2 both equal to 10. This procedure, together with Theorem 3(1), yields the following result.

Theorem 23.

1. *Every* 10*-traceable oriented graph with independence number* 2 *and order* n *is traceable for every* $n \in \{17, 18, 19, 24, 25, 26, 27, 28\}$ *and every* $n \geqslant 31$.

2. *Every* 10*-traceable oriented graph of order at least* 40 *is traceable.*

Implications of Theorems 20 - 23 with regard to the TC are as follows.

Corollary 24. *If* D *is a* k*-traceable oriented graph of order at least* $2k - 1$, *then* D *is traceable in each of the following cases.*

1. $k \leqslant 8$.

2. $k = 9$ *and* $\alpha(D) = 2$.

3. $k = 10$, $\alpha(D) = 2$, $n \notin \{20, 21, 22, 23, 29, 30\}$.

Acknowledgments

The authors wish to thank the University of South Africa and the National Research Foundation of South Africa for their sponsorship of "Detour Workshop 2010" (Salt Rock, 6-20 March, 2010) and "Detour Workshop 2011" (Salt Rock, 1-15 March, 2011) where joint research for this paper was conducted and the main ideas were generated.

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