An involution proof of the Alladi-Gordon key identity for Schur's partition theorem

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Abstract

The Alladi-Gordon identity $\sum_{k=0}^{j}(q^{i-k+1};q)_k \begin{bmatrix} j \\ k \end{bmatrix} q^{(i-k)(j-k)} = 1$ plays an important role for the Alladi-Gordon generalization of Schur's partition theorem. By using Joichi-Stanton's insertion algorithm, we present an overpartition interpretation for the Alladi-Gordon key identity. Based on this interpretation, we further obtain a combinatorial proof of the Alladi-Gordon key identity by establishing an involution on the underlying set of overpartitions.

Keywords: the Alladi-Gordon key identity; Joichi-Stanton's insertion algorithm; Schur's celebrated partition theorem; overpartitions

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Let

$$(a)_k = (a;q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k>0, \\ 1, & \text{if } k=0, \end{cases}$$

denote the common notation of q-shifted factorials [14]. Given $j,k \in \mathbb{N}$, let

$$\begin{bmatrix} j \\ k \end{bmatrix} = \frac{(q;q)_j}{(q;q)_k (q;q)_{j-k}},$$

denote the Gaussian coefficients, which are also called the q-binomial coefficients, or the Gaussian polynomials [6]. The main objective of this paper is to give a combinatorial proof of the following identity:

$$\sum_{k=0}^{j}(q^{i-k+1};q)_k \begin{bmatrix} j \\ k \end{bmatrix} q^{(i-k)(j-k)} = 1, \tag{1}$$

which we call *the Alladi-Gordon key identity*, since it was first introduced by Alladi and Gordon [4] in an equivalent form for the study of some generalization of Schur's celebrated partition theorem of 1926.

Schur [18] proved that the number of partitions of m into parts with minimal difference 3 and with no consecutive multiples of 3 is equal to the number of partitions of m into distinct parts $\equiv 1, 2 \pmod 3$. This significant result is now known as *Schur's celebrated partition theorem of 1926*. There are many proofs, refinements, and generalizations of Schur's partition theorem, see [5, 7, 12, 15] and references therein.

From the viewpoint of generating functions, each partition theorem implies a corresponding q-identity. The Alladi-Gordon key identity (1) is essentially equivalent to the following q-identity [4, Lemma 2] corresponding to Alladi and Gordon's notable generalization of Schur's partition theorem,

$$\sum_{0 \leqslant m \leqslant min\{i,j\}} \frac{q^{T_{i+j-m}+T_m}}{(q)_{i-m}(q)_{j-m}(q)_m} = \frac{q^{T_i+T_j}}{(q)_i(q)_j}, \tag{2}$$

where i and j are given nonnegative integers and $T_i = i(i+1)/2$ is the i-th triangle number.

The Alladi-Gordon key identity turned out to have many interesting applications in the theory of partitions. Alladi and Berkovich [3, Eq. (2.1)]) obtained a double bounded version of Schur's partition theorem by generalizing an equivalent form of (2). Alladi, Andrews and Gordon [2, Lemma 2] introduced a three parameter generalization of (2) and obtained a generalization of the Göllnitz theorem [15], a higher level extension of Schur's partition theorem. Alladi, Andrews and Berkovich [1, Eq. (1.7)] further obtained a remarkable four parameter extension of the identity (2), which implies a four parameter partition theorem and thereby extends the Göllnitz theorem.

Due to its significance, the Alladi-Gordon key identity certainly deserves to be further studied. Alladi and Gordon [4] gave two proofs of (2), one combinatorial and the other algebraic. Alladi, Andrews and Berkovich [1] also pointed out that (2) is a special case of q-Chu-Vandermonde summation formula. In this paper we will present an overpartition interpretation of the left-hand side of (1) and then give another combinatorial proof of the Alladi-Gordon key identity.

This paper is organized as follows. In Section 2 we will review Joichi-Stanton's insertion algorithm for partitions and then give an overpartition interpretation of the Alladi-Gordon key identity. In Section 3, based on this interpretation, we will give an involution proof of the Alladi-Gordon key identity.

2 An overpartition interpretation of the key identity

The aim of this section is to give a combinatorial interpretation of the left-hand side of (1) in terms of overpartitions. This is achieved by applying Joichi-Stanton's insertion algorithm for partitions.

Let us first review some definitions and notations about partitions. Recall that a *partition* λ of $n \in \mathbb{N}$ with k parts is denoted by a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$ and $\sum_{i=1}^{k} \lambda_i = n$. The number n is called the *size* of λ, denoted by $|\lambda|$. For convenience, the *length* of λ is defined to be the number k of nonnegative parts of λ, denoted by $\ell(\lambda)$. (Note that $\ell(\lambda)$ usually enumerates the number of positive parts.) An *overpartition* is a partition in which the first occurrence of a number may be overlined. For example, $\lambda = (9, \overline{7}, 6, 5, 5, \overline{2}, 2, \overline{1})$ is an overpartition with three overlined parts. An ordinary partition can also be treated as an overpartition with no overlined parts. The concept of overpartition was first proposed by Corteel and Lovejoy [11] while studying basic hypergeometric series. For deeper research on overpartitons, see for instance [8, 9, 13, 17].

An overpartition can also be understood as a pair of partitions (α, β), where α is a partition with distinct parts and β is an ordinary partition. Joichi and Stanton [16] found the following fundamental bijection which can be restated in terms of overpartitions.

Theorem 1. *There is a one-to-one correspondence between overpartitions with n nonnegative parts, and pairs of partitions (α, β), where α is a partition with distinct parts from the set $\{0, 1, 2, \ldots, n-1\}$ and β is a partition with n nonnegative parts.*

The above correspondence can be described as an insertion algorithm [16, Algorithm Φ]. Given an ordinary partition β, we may insert a part m into β, by adding 1 to the first m parts of β, and putting an overline above the $(m+1)$-th part. Moreover, we can add other distinct parts in the same way.

Example 2. If $\alpha = (5, 3, 0)$ and $\beta = (9, 6, 5, 2, 2, 0)$, then we get an overpartition $(\overline{11}, 8, 7, \overline{3}, 3, \overline{0})$.

To give a combinatorial interpretation of (1), we shall assign a weight to each overlined part of an overpartition. As in [10], each overlined part of an overpartition has the same weight. For example, $\lambda = (9, \overline{7}, 6, 5, 5, \overline{2}, 2, \overline{1}, 0)$ with a weight 3 endowed in each overline is displayed in Figure 1, where each overline of λ is represented by a row of three hollow dots, and the part zero is represented by the symbol $\varnothing$.

Given $0 \leqslant k \leqslant j \leqslant i$, let $A(i, k)$ denote the set of partitions into distinct parts from the set $\{i-k+1, i-k+2, \ldots, i\}$ plus the empty partition, and let $B(j, k)$ denote the set of partitions into k nonnegative parts with each part not exceeding $j-k$. It is easy to see

$$\sum_{\lambda \in A(i,k)} (-1)^{\ell(\lambda)} q^{|\lambda|} = (q^{i-k+1}; q)_k, \tag{3}$$

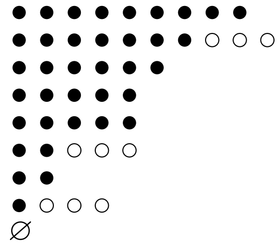

Figure 1: An overpartition $\lambda = (9, \overline{7}, 6, 5, 5, \overline{2}, 2, \overline{1}, 0)$ with a weight 3 in each overline.

and it is well known [6, Theorem 3.1] that

$$\sum_{\lambda \in B(j,k)} q^{|\lambda|} = \begin{bmatrix} j \\ k \end{bmatrix}. \tag{4}$$

We now come to the main result of this section. For a weighted overpartition λ, let $ol(\lambda)$ denote the number of overlined parts of λ, and let $w(\lambda)$ denote the weight assigned to each overlined part of λ. Given $0 \leqslant k \leqslant j \leqslant i$, let $O(i,j,k)$ denote the set of weighted overpartitions into k nonnegative parts, which satisfy the following three conditions:

(1) each part is less than or equal to $j-1$;

(2) for $k \geqslant 2$ and each $1 \leqslant s \leqslant k-1$ there are at least $k-s$ overlined parts to the right of $j-s$ if it occurs as a part;

(3) and, each overline is endowed with a weight $i-k+1$.

Note that the empty partition is the sole element of $O(i,j,0)$. For fixed i,j satisfying $0 \leqslant j \leqslant i$, let

$$O(i,j) = \biguplus_{k=0}^{j} O(i,j,k). \tag{5}$$

The main result of this section is as follows.

Theorem 3. *Given $0 \leqslant j \leqslant i$, we have*

$$\sum_{k=0}^{j} (q^{i-k+1};q)_k \begin{bmatrix} j \\ k \end{bmatrix} q^{(i-k)(j-k)} = \sum_{\lambda \in O(i,j)} (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda) w(\lambda)} q^{(i-\ell(\lambda))(j-\ell(\lambda))}. \tag{6}$$

Proof. By (5), it suffices to prove that

$$(q^{i-k+1};q)_k \begin{bmatrix} j \\ k \end{bmatrix} = \sum_{\lambda \in O(i,j,k)} (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda)(i-k+1)},$$

since for each $\lambda \in O(i,j,k)$ we have $\ell(\lambda) = k$ and $w(\lambda) = i-k+1$. It is clearly true for $k=0$. In the following we may assume that $k \geqslant 1$. In view of (3) and (4), we only need to give a weight-preserving bijection $\overline{\Phi}$ between the set $A(i,k) \times B(j,k)$ and the set $O(i,j,k)$. Actually, we can obtain $\overline{\Phi}$ by using Joichi-Stanton's insertion algorithm.

For any given pair $(\gamma,\beta) \in A(i,k) \times B(j,k)$, define $\overline{\Phi}(\gamma,\beta)$ to be the partition λ obtained as follows.

(i) If γ is the empty partition, then let $\lambda = \beta$. By Property (2) of the definition of $O(i,j,k)$, it is clear that $B(j,k) \subseteq O(i,j,k)$. Therefore, in this case we have $\lambda \in O(i,j,k)$.

(ii) If γ is not the empty partition, then let $\overline{\gamma}$ denote the partition obtained from γ by decreasing each part by $i-k+1$. Therefore, $\overline{\gamma}$ is a partition into distinct parts from the set $\{0,1,\ldots,k-1\}$. Now we insert $\overline{\gamma}$ into β by applying Joichi-Stanton's insertion algorithm, and obtain an overpartition λ with at least one overlined part. If each overline is endowed with a weight $i-k+1$, then it is routine to verify that the weighted overpartition λ lies in $O(i,j,k)$. Note that the number of parts of γ is equal to the number of overlined parts of λ. Thus

$$(-1)^{\ell(\gamma)} q^{|\gamma|} q^{|\beta|} = (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda)(i-k+1)}.$$

It remains to show that $\overline{\Phi}$ is reversible. There are two cases to consider.

(i') If $\lambda \in O(i,j,k)$ and there are no overlined parts in λ, then again by Property (2) of the definition of $O(i,j,k)$, we must have $\lambda \in B(j,k)$. In this case, let $\overline{\Phi}^{-1}(\lambda) = (\gamma,\beta)$, where γ is the empty partition and $\beta = \lambda$.

(ii') If $\lambda \in O(i,j,k)$ and there is at least one overlined part in λ, then by reversing the insertion algorithm, we will obtain a pair of partitions $(\overline{\gamma},\beta)$. Clearly, $\overline{\gamma}$ is a partition into distinct parts from the set $\{0,1,\ldots,k-1\}$ since there are k parts in λ. It is also clear that β has only k parts. We further need to show that each part of β is not exceeding $j-k$. Suppose that there are t overlined parts to the right of λ_1, then $\beta_1 = \lambda_1 - t$. Assume that $\lambda_1 = j-s$ for some $1 \leqslant s \leqslant k-1$. By Property (2) of the definition of $O(i,j,k)$ we have $t \geqslant k-s$. Therefore, $\beta_1 = \lambda_1 - t = j-s-t \leqslant j-k$.

This completes the proof. □

The following example gives an illustration of the map $\overline{\Phi}$ of the above proof.

Example 4. For $i=9,\ j=6,\ k=3,\ \gamma=(8,7) \in A(9,3),\ \beta=(3,3,2) \in B(6,3)$. We shall transform (γ,β) into $\lambda \in O(9,6,3)$ in two steps.

(1) Change $\gamma=(8,7)$ into $\overline{\gamma}=(1,0)$ by decreasing each part by 7.

(2) Insert $\overline{\gamma}=(1,0)$ into $\beta=(3,3,2)$ to obtain an overpartition $\lambda=(\overline{4},\overline{3},2) \in O(9,6,3)$, where each overline contains a weight 7. See Figure 2.

By reversing the procedure it is easy to obtain (γ,β) from λ.

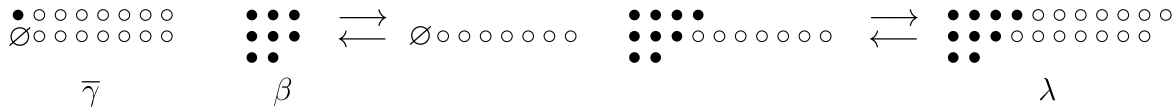

Figure 2: Insertion of $\overline{\gamma} = (1, 0)$ into $\beta = (3, 3, 2)$ leads to $\lambda = (\overline{4}, \overline{3}, 2)$, where $\overline{\gamma}$ is represented as the overpartition $(\overline{1}, \overline{0})$ and each overline has a weight 7 endowed.

3 Combinatorial proof of the Alladi-Gordon key identity

The aim of this section is to prove the following result by constructing an involution on the set $O(i, j)$.

Theorem 5. *Given $0 \leqslant j \leqslant i$, we have*

$$\sum_{\lambda \in O(i,j)} (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda) w(\lambda)} q^{(i-\ell(\lambda))(j-\ell(\lambda))} = 1, \tag{7}$$

where $O(i, j)$ is as defined in (5).

Combining Theorem 3, this provides a combinatorial proof of the the Alladi-Gordon key identity.

To prove Theorem 5, we first give a decomposition of $O(i, j)$. For $\lambda \in O(i, j)$ let λ_t denote the largest overlined part of λ. Let

$$\begin{aligned} O_1(i, j) &= \{ \lambda \in O(i, j) \mid ol(\lambda) \geqslant 1, \lambda_t = j - \ell(\lambda) + ol(\lambda) - 1 \}, \\ O_2(i, j) &= \{ \lambda \in O(i, j) \mid ol(\lambda) \geqslant 1, \lambda_t < j - \ell(\lambda) + ol(\lambda) - 1 \}, \\ O_3(i, j) &= \{ \lambda \in O(i, j) \mid ol(\lambda) = 0 \}. \end{aligned}$$

For the convenience, let the empty partition belong to $O_3(i, j)$.

Lemma 6. *For $0 \leqslant j \leqslant i$, we have*

$$O(i, j) = O_1(i, j) \uplus O_2(i, j) \uplus O_3(i, j).$$

Proof. It is clear that $O_1(i, j), O_2(i, j)$ and $O_3(i, j)$ are disjoint from each other. It suffices to show that for each $\lambda \in O(i, j)$ with $ol(\lambda) \geqslant 1$, we have $\lambda_t \leqslant j - \ell(\lambda) + ol(\lambda) - 1$. Otherwise, suppose that $\lambda_t = j - s$ and $s < \ell(\lambda) - ol(\lambda) + 1$. By Property (2) of the definition of $O(i, j, k)$, there are at least $\ell(\lambda) - s \geqslant ol(\lambda)$ overlined parts to the right of λ_t, contradicting the definition of $ol(\lambda)$. This completes the proof. $\square$

With the above decomposition of $O(i, j)$, we can now give a bijective proof of Theorem 5.

Proof of Theorem 5. For $\lambda \in O(i,j)$, let

$$f(\lambda) = (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda)w(\lambda)} q^{(i-\ell(\lambda))(j-\ell(\lambda))}.$$

To give a bijective proof, we define an involution, denoted Ψ, acting on $O(i,j)$ as follows:

(1) If $\lambda \in O_1(i,j)$, then let $\Psi(\lambda)$ denote the overpartition obtained from λ by removing the largest overlined part λ_t. In this case, we have

$$\begin{aligned} \ell(\Psi(\lambda)) &= \ell(\lambda) - 1 \\ ol(\Psi(\lambda)) &= ol(\lambda) - 1 \\ |\Psi(\lambda)| &= |\lambda| - \lambda_t \\ &= |\lambda| - (j - \ell(\lambda) + ol(\lambda) - 1). \end{aligned}$$

By Property 3 of the definition of $O(i,j,k)$, we have

$$w(\Psi(\lambda)) = i - \ell(\Psi(\lambda)) + 1 = (i - \ell(\lambda) + 1) + 1 = w(\lambda) + 1.$$

It is routine to verify that $f(\Psi(\lambda)) + f(\lambda) = 0$. Note that if $ol(\lambda) = 1$, then clearly $\Psi(\lambda) \in O_3(i,j)$. If $ol(\lambda) > 1$, then we must have $\Psi(\lambda) \in O_2(i,j)$ since

$$\Psi(\lambda)_t < \lambda_t = j - \ell(\lambda) + ol(\lambda) - 1 = j - \ell(\Psi(\lambda)) + ol(\Psi(\lambda)) - 1.$$

(The inequality $\Psi(\lambda)_t < \lambda_t$ follows from the definition of overpartitions.) In both cases, we have $\ell(\Psi(\lambda)) = \ell(\lambda) - 1 \leqslant j - 1$.

(2) If $\lambda \in O_2(i,j)$, then we must have $j > \ell(\lambda)$. Otherwise, if $j = \ell(\lambda)$, then $\lambda_t < ol(\lambda) - 1$, contradicting the fact that the overlined parts of an overpartition are distinct nonnegative integers. Then let $\Psi(\lambda)$ denote the overpartition obtained from λ by inserting an overlined part $j - \ell(\lambda) + ol(\lambda) - 1$. Clearly, $\Psi(\lambda) \in O_1(i,j)$ and $\Psi(\Psi(\lambda)) = \lambda$.

(3) If $\lambda \in O_3(i,j)$, then we define $\Psi(\lambda)$ as follows according to whether $j > \ell(\lambda)$. If $j > \ell(\lambda)$, then let $\Psi(\lambda)$ denote the overpartition obtained from λ by inserting an overlined part $j - \ell(\lambda) - 1$. In this case, it is clear that $\Psi(\lambda) \in O_1(i,j)$ and $\Psi(\Psi(\lambda)) = \lambda$. If $j = \ell(\lambda)$, then λ must be the partition $(\underbrace{0, 0, \ldots, 0}_{j's})$. Otherwise, we will have $\lambda_1 > 0$, and by Property 2 of $O(i,j,j)$ there must be at least one overlined part in λ contradicting $ol(\lambda) = 0$. In this case let $\Psi(\lambda) = \lambda$.

By the involution Ψ of $O(i,j)$, we have

$$\sum_{\lambda \in O(i,j)} f(\lambda) = \sum_{\lambda = (\underbrace{0, 0, \ldots, 0}_{j's})} f(\lambda) = 1.$$

This completes the proof. $\square$

In fact, there is a graphical representation of the involution Ψ of $O(i,j)$ in the above proof, which seems more convenient and intuitive. Since each $\lambda \in O(i,j)$ contributes a term

$$f(\lambda) = (-1)^{ol(\lambda)} q^{|\lambda| + ol(\lambda)w(\lambda)} q^{(i-\ell(\lambda))(j-\ell(\lambda))},$$

we may consider λ as a pair of partitions $(\lambda, \widehat{\lambda})$, where $\widehat{\lambda}$ is the unique rectangular partition $(\underbrace{i-\ell(\lambda), \ldots, i-\ell(\lambda)}_{(j-\ell(\lambda))'s})$.

Example 7. Take $i = 9$, $j = 6$ and let $\lambda = (\overline{4}, \overline{3}, 2)$. In this case, we have $\widehat{\lambda} = (6,6,6)$, $ol(\lambda) = 2$, $\ell(\lambda) = 3$, $w(\lambda) = i - \ell(\lambda) + 1 = 7$, and hence $\lambda \in O_1(i,j)$. Thus, $\Psi(\lambda) = (\overline{3}, 2) \in O_2(i,j)$ and $\widehat{\Psi(\lambda)} = (7,7,7,7)$. Geometrically, Ψ acts on λ (or equivalently $(\lambda, \widehat{\lambda})$) as illustrated in Figure 3: remove a row of dots representing the largest overlined part, add a hollow dot to the rightmost of each overlined part, and append a hook to the top-left of the diagram of $\widehat{\lambda}$.

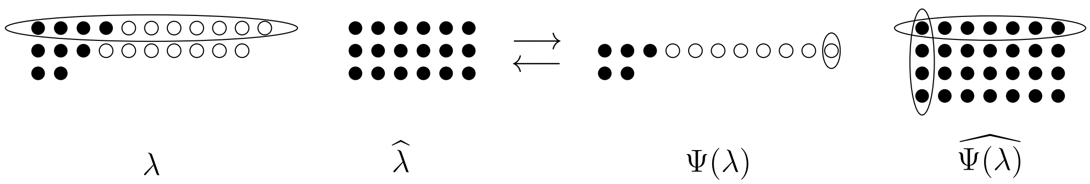

Figure 3: The involution Ψ.

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