Biregular cages of girth five

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Abstract

Let $2 \leqslant r < m$ and g be positive integers. An $(\{r,m\};g)$*–graph* (or biregular graph) is a graph with degree set $\{r,m\}$ and girth g, and an $(\{r,m\};g)$*–cage* (or biregular cage) is an $(\{r,m\};g)$–graph of minimum order $n(\{r,m\};g)$. If $m=r+1$, an $(\{r,m\};g)$–cage is said to be a *semiregular cage*.

In this paper we generalize the reduction and graph amalgam operations from [M. Abreu, G. Araujo–Pardo, C. Balbuena, D. Labbate. Families of Small Regular Graphs of Girth 5. *Discrete Math.* **312**(18) (2012) 2832–2842] on the incidence graphs of an affine and a biaffine plane obtaining two new infinite families of biregular cages and two new semiregular cages. The constructed new families are $(\{r,2r-3\};5)$–cages for all $r=q+1$ with q a prime power, and $(\{r,2r-5\};5)$–cages for all $r=q+1$ with q a prime. The new semiregular cages are constructed for $r=5$ and 6 with 31 and 43 vertices respectively.

Keywords: biregular, cage, girth.

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1 Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [15], [13] and [16].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called r*–regular* if all its vertices have the same degree r. A (r, g)*–graph* is a r–regular graph of girth g and a (r, g)*–cage* is a (r, g)–graph with the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [31] in 1947. Erdős and Sachs [20] proved the existence of a (r, g)–graph for any value of r and g. Biggs is the author of an impressive report on distinct methods for constructing cubic cages [14]. More details about constructions of cages can be found in the surveys by Wong [32], by Holton and Sheehan [27, Chapter 6], or the recent one by Exoo and Jajcay [21].

The cages theory has been generalized in many ways, one such is as follows: if $D = \{a_1, \ldots, a_k\}$ is a set of positive integers with $2 \leqslant a_1 < a_2 < \ldots < a_k$ then a $(D; g)$–graph is a graph with degree set D and girth g and a $(D; g)$–cage is a $(D; g)$–graph with minimum order $n(D; g) = n(a_1, \ldots, a_k; g)$. It is obvious that the $(r; g)$–cage is a special case of the $(D; g)$–cage when $D = \{r\}$.

Few values of $n(D; g)$ are known. In particular, Kapoor et al. [28] proved that $n(D; 3) = 1 + a_k$. Moreover, the following lower bound for $n(D; g)$ was given by Downs et al. [18]:

$$n(D;g) \geqslant \begin{cases} 1 + \displaystyle\sum_{i=1}^{t} a_k(a_1-1)^{i-1} & \text{if } g = 2t+1; \\ 1 + \displaystyle\sum_{i=1}^{t-1} a_k(a_1-1)^{i-1} + (a_1-1)^{t-1} & \text{if } g = 2t. \end{cases} \tag{1}$$

A *biregular* $(\{r, m\}; g)$*–graph* is a $(D; g)$–graph with degree set $D = \{r, m\}$ and girth g and a *bi-regular* $(\{r, m\}; g)$*–cage* is an $(\{r, m\}; g)$–graph of smallest possible order. Note that bouquets of r cycles of length g are $(\{2, 2r\}; g)$-cages and the complete bipartite graphs $K_{r,m}$ are $(\{r, m\}; 4)$–cages. If $m = r + 1$, an $(\{r, m\}; g)$–cage is said to be a *semiregular cage*.

The existence of biregular $(\{r, m\}; g)$–graphs has been proved by Chartrand, Gould, and Kapoor in [17] for all $2 \leqslant r < m$ and $g \geqslant 3$ (also proved by Füredi et al. in [23]). On the other hand, several contructions of biregular $(\{r, m\}; g)$–cages have been achieved for different values of r, m and g. In particular, Chartrand et al. proved in [17] that $n(\{r, m\}; 4) = r + m$, for $2 \leqslant r < m$, and $g = 4$ and they also proved in [17] that $n(\{2, m\}; g)$ attains the lower bound (1). Furthermore, Yuansheng and Liang in [33] proved that $n(\{r, m\}; 6) \geqslant 2(rm - m + 1)$ for any $2 \leqslant r < m$; that $n(\{r, m\}; 6) = 2(rm - m + 1)$ for $g = 6$ and $r < m$ when $2 \leqslant r \leqslant 5$ or $r \geqslant 2$ and $m - 1$ a prime power; and they conjectured that $n(\{r, m\}; 6) = 2(rm - m + 1)$, for any $2 \leqslant r < m$.

In this paper, we focus our attention on biregular graphs of girth exactly five, where $n(\{r,m\};5) = rm+1$. Some known results in this case, Downs et al. [18] have shown that $n(\{3;m\};5) = 3m+1$, for any $m \geqslant 4$, Hanson et al. [24] have shown that $n(\{4;m\};5) = 4m+1$, for any integer $m \geqslant 5$, and Araujo–Pardo et al. in [7] proved several results in this context. Moreover, it is worth to note that Yuansheng and Liang in [33], claim to have proved that $n(\{5,m\};5) = 5m+1$ for $m \geqslant 6$ in an unpublished manuscript. In Table 1, we summarize these and other known results on the exact values of $n(\{r,m\};g)$, for $g \geqslant 5$, together with the results obtained in this paper which we now proceed to describe.

r	m	$g=5$	$g=6$	$g=7$	$g=8$	$g=9$	$g=11$
$r=3$	$m \geqslant 4$	$3m+1$ [18]	$4m+2$ [24, 33]	$7m+1$ [18]	$\frac{25m}{3}+5$ $m=3k$ [7] $9m+3$ $m=4,5,7$ [10]	$15m+1$ $m \geqslant 6$ [18]	$31m+1$ $m=4k$ [7]
$r=4$	$m \geqslant 5$	$4m+1$ [24]	$6m+2$ [33]	$13m+1$ $m=6k$ [7]			$121m+1$ $m=6k$ [7]
$r \geqslant 5$ $r=p^h+1$ p prime	$m=2k(r-1)$	$1+rm$ [7]		$1+m(r^2-r+1)$ [7]			$1+m\frac{(r-1)^5-1}{r-2}$ [7]
	$m=k(r-1)+1$		$2(rm-m+1)$ [6, 33]				
	$m=kr$		[7]				
	$m=2r-3$	(*)					
$h=1$ $p \geqslant 7$	$m=2r-5$	(*)					

Table 1: Exact values of $n(\{r,m\};g)$. The symbol (*) means results obtained in this paper.

We generalize the reduction and graph amalgam operations from [1] on the incidence graphs A_q of an affine and B_q of a biaffine plane (elliptic semiplane of type $\mathcal{C}$) obtaining two new infinite families of biregular $(\{r,m\};5)$–cages and two new semiregular cages of girth 5. The constructed families are $(\{r,2r-3\};5)$–cages for all $r = q+1$ with q a prime power, and $(\{r,2r-5\};5)$–cages for all $r = q+1$ with q a prime. The new semiregular cages are constructed for $r = 5$ and 6 with 31 and 43 vertices respectively.

The paper is organized as follows: the graphs A_q and B_q are presented in Section 2, with a labelling which will be necessary for the construction of biregular cages. In Section 3, we construct previously unknown $(\{r,2r-3\};5)$–cages, for a prime power $q \geqslant 2$ and the integer $r = q+1$, in Theorem 3.3, adding edges to the graph A_q. Moreover if $r-1$ is even, we exhibit r non-isomorphic such $(\{r,2r-3\};5)$–cages in Theorem 3.5. In particular, we find a $(\{4,5\};5)$–semiregular cage with 21 vertices. In Section 4 we slightly generalize reduction and amalgam operations, described in [1] and [22], that, performed on the bipartite graph B_q, will allow us to construct new $(\{r,2r-5\};5)$–cages, for $r = q+1$ with $q \geqslant 7$ prime, in Section 5, Theorems 5.10 and 5.11. Finally, in Section 6 we construct two new semiregular cages, namely a $(\{5,6\};5)$-cage with 31 vertices and a $(\{6,7\};5)$-cage with 43 vertices. Note that the latter is a sporadic example in which we adapt and slightly generalize the techniques that we have used in Section 4.

2 Preliminaries

Let $q = p^n \geqslant 2$ be a prime power and α a primitive $(q-1)^{th}$–root of unity. Consider the finite field $GF(q) = \{0, 1, \alpha, \alpha^2, \dots, \alpha^{q-2}\}$ and denote $GF^*(q) = GF(q) \setminus \{0\}$.

The graphs constructed in this paper arise from the (bipartite) incidence graph B_q of an elliptic semiplane of type C (cf. [19, 5, 22]) together with the (bipartite) incidence graph A_q of the affine plane of order q. We fix a labelling on their vertices which will be central for our constructions since it allows us to keep track of the properties (such as regularity and girth) of the graphs obtained from B_q and A_q applying some operations such as reductions and amalgams (cf. Sections 3, 4).

Definition 2.1. Let $q \geqslant 2$ be a prime power, and consider the finite field $GF(q)$.

(i) Let B_q be a bipartite graph with vertex set (V_0, V_1) where $V_r = GF(q) \times GF(q)$, $r = 0, 1$; and the edge set defined as follows:

$$(x, y)_0 \in V_0 \text{ adjacent to } (m, b)_1 \in V_1 \text{ if and only if } y = mx + b. \tag{2}$$

(ii) Let A_q be the graph obtained from B_q by adding the following set $L_q := \{(q, x)_1 \,|\, x \in GF(q)\}$ of q vertices and the set $E_q := \{uv \,|\, u := (q, x)_1\,, v := (x, y)_0 \, and \, x, y \in GF(q)\}$ of q^2 edges.

The graph B_q is also known as the incidence graph of the biaffine plane [25] and the graph A_q is the incidence graph of an affine plane of order q. The graph B_q has been used in the problem of finding extremal graphs without short cycles (cf. e.g. [1, 3, 4, 8, 9, 11, 30]).

The following properties of the graph B_q are well known (see [1, 25, 30]) and they will be fundamental throughout the paper.

Proposition 2.2. *Let B_q be the (bipartite) incidence graph defined above. Let $P_x = \{(x, y)_0 |\ y \in GF(q)\}$, for $x \in GF(q)$, and $L_m = \{(m, b)_1 |\ b \in GF(q)\}$, for $m \in GF(q)$. Then the graph B_q has the following properties:*

(i) it is q–regular, vertex transitive, of order $2q^2$ and has girth 6 for $q \geqslant 3$;

(ii) it admits a partition $V_0 = \bigcup_{x \in GF(q)} P_x$ and $V_1 = \bigcup_{m \in GF(q)} L_m$ of its vertex set;

(iii) each block P_x is connected to each block L_m by a perfect matching, for $x, m \in GF(q)$;

(iv) each vertex in P_0 and L_0 is connected straight *to all its neighbours in B_q, meaning that $N((0, y)_0) = \{(i, y)_1 | i \in GF(q)\}$ and $N((0, b)_1) = \{(j, b)_0 | j \in GF(q)\}$;*

(v) the other matchings between P_x and L_m are twisted *and the rule is defined algebraically in $GF(q)$ according to (2).* □

For further information regarding these properties and for constructions of the adjacency matrix of B_q as a block $(0, 1)$–matrix please refer to [2, 5, 12].

3 Construction of a family of $(\{r, 2r-3\}; 5)$–cages.

In this section, for a prime power $q \geqslant 2$ and the integer $r = q+1$, we construct previously unknown $(\{r, 2r-3\}; 5)$–cages, adding edges to the graph A_q presented in Section 2.

Let $q \geqslant 2$ be a prime power and let $r = q+1$. We define R_q to be the graph with $V(R_q) := V(A_q)$ and $E(R_q) := E(A_q) \cup D$ where $D = \{(m,0)_1(m,b)_1 \mid b \in GF^*(q)$ and $m \in GF(q) \cup \{q\}\}$.

Theorem 3.3. *Let $q \geqslant 2$ be a prime power and let $r = q+1$. Then the graph R_q is a $(\{r, 2r-3\}; 5)$–cage satisfying Downs' bound, i.e. $n(\{r, 2r-3\}; 5) = r(2r-3)+1$.*

Proof. The vertices $M := \{(m,0)_1 | m \in GF(q) \cup \{q\}\} \subset V(R_q)$ have degree $q + (q-1) = 2q - 1 = 2r - 3$, and the remaining vertices of R_q have degree $q + 1 = r$.

By construction $B_q \subset A_q \subset R_q$. Let C be a cycle in R_q. If the edges of C are totally contained in A_q then the length of C is at least six, since A_q is the incidence graph of an affine plane. Otherwise, C contains at least an edge $e = xy \in D$, where $x, y \in L_m$ and $m \in GF(q) \cup \{q\}$. Then it follows from the bipartition and girth of A_q that the distance $d_{A_q}(x, y) = 4$. Thus the length of C in this case is at least five and exactly five if C contains exactly one edge of D. Hence, the graph R_q has girth 5 since $C = (q,0)_1(q,1)_1(1,0)_0(0,0)_1(0,0)_0(q,0)_1$ where $(q,0)_1(q,1)_1$ is the only edge of C in D.

Finally, $|V(R_q)| = |V(A_q)| = 2q^2 + q = 2r^2 - 3r + 1 = r(2r-3) + 1$. □

Corollary 3.4. *The graph R_q is a semi–regular cage if and only if $r = 4$.*

Proof. It follows immediately since $2r - 3 = r + 1$ if and only if $r = 4$. □

Note that an isomorphic graph to R_3 has been found also in [24].
In a similar way, for q even, we construct a family of non–isomorphic $(\{r, 2r-3\}; 5)$–cages.

Let q be an even prime power and let $D_m := \{(m,0)_1(m,b)_1 \mid b \in GF(q)\}$ and $F_m := \{(m,0)_1(m,1)_1\} \cup \{(m,\alpha^i)_1(m,\alpha^{i+1})_1 \mid 1 \leqslant i \leqslant q-3,\ i \,\mathrm{odd}\}$, for $m \in GF(q) \cup \{q\}$. Then $D_m \cong K_{1,q-1}$ is a star with vertex set L_m and F_m is a matching between the vertices of L_m.

Let $0 \leqslant t \leqslant q-1$ and let $I_t = \{\alpha^{q-t-1}, \ldots, \alpha^{q-2}\}$ be a set of indexes. We define G_t to be the graph with $V(G_t) := V(A_q)$ and $E(G_t) := E(A_q) \cup F_t \cup D_t$, where

$$F_t := \bigcup_{m \in GF(q) \setminus I_t} F_m \quad \text{and} \quad D_t := \bigcup_{m \in I_t \cup \{q\}} D_m.$$

Note that the graph G_t is obtained from the graph A_q adding $t+1$ stars and $q-t$ matchings within the sets L_m. In particular, for $t = 0$ the index set $I_t = \emptyset$ and the only star added to A_q is the one in L_q. Moreover, if we set $I_q := GF(q)$, then we can say that the graph $G_q := R_q$.

Theorem 3.5. *Let $q = 2^s$ be an even prime power, with $s > 1$. Then there are at least $q+1$ non–isomorphic $(\{r, 2r-3\}; 5)$–cages.*

Proof. Let G_t be the family of graphs defined as above, for $0 \leqslant t \leqslant q$. Reasoning as in Theorem 3.3, is follows that, for every $0 \leqslant t \leqslant q-1$, the graphs G_t have girth 5, order $r(2r-3)+1$ and are biregular with $t+1$ vertices of degree $2r-3$. Thus, G_t is a $(\{r, 2r-3\}; 5)$–cage for every $0 \leqslant t \leqslant q-1$. Moreover, $G_i \ncong G_j$, for $i, j \in GF(q) \cup \{q\}$ with $i \neq j$, since they have a different number of vertices of degree $2r-3$. $\square$

4 Operations on B_q

In this section we slightly generalize reduction and amalgam operations, described in [1] and [22], that, performed on the bipartite graph B_q, will allow us to construct new $(\{r, 2r-5\}; 5)$–cages, for $r = q+1$ with $q \geqslant 7$ prime, in Section 5.

4.1 Reductions

We describe two reduction operations on B_q already introduced in [1]. The first one is exactly the same while the second one is slightly generalized.

Reduction 1[1] Remove vertices from P_0 and L_0.

Let $T \subseteq S \subseteq GF(q)$, $S_0 = \{(0, y)_0 | y \in S\} \subseteq P_0$, $T_0 = \{(0, b)_1 | b \in T\} \subseteq L_0$ and $B_q(S, T) = B_q - S_0 - T_0$.

Lemma 4.6. *Let $T \subseteq S \subseteq GF(q)$. Then $B_q(S, T)$ is biregular with degrees $(q-1, q)$ of order $2q^2 - |S| - |T|$. Moreover, the vertices $(i, t)_0 \in V_0$ and $(j, s)_1 \in V_1$, for each $i, j \in GF(q) - \{0\}$, $s \in S$ and $t \in T$ are the only vertices of degree $q-1$ in $B_q(S, T)$, together with $(0, s)_1 \in V_1$ for $s \in S - T$ if $T \subsetneq S$.*

Proof. It is an immediate consequence of Proposition 2.2 (i), (v). $\square$

Reduction 2 Remove blocks P_i and L_j from B_q or from $B_q(S, T)$.

Let u_0, u_1 be non–negative integers such that $0 \leqslant u_0 \leqslant u_1 < q-1$. If $u_i > 0$, let $U_i := \{\alpha^{q-j} \in GF(q) : j = 2, \ldots, u_i + 1\}$ be an index set, for $i = 0, 1$. Let $\mathcal{U}_0$ and $\mathcal{U}_1$ be sets of blocks of B_q chosen as follows:

$$\mathcal{U}_0 := \{P_x \subset V_0 : x \in U_0\} \text{ if } 1 \leqslant u_0 \leqslant q-1 \qquad \text{or} \qquad \mathcal{U}_0 := \emptyset \text{ if } u_0 = 0$$

$$\mathcal{U}_1 := \{L_m \subset V_1 : m \in U_1\} \text{ if } 1 \leqslant u_1 \leqslant q-1 \qquad \text{or} \qquad \mathcal{U}_1 := \emptyset \text{ if } u_1 = 0.$$

Then, we define $B_q(u_0, u_1) := B_q - \mathcal{U}_0 - \mathcal{U}_1$ to be the graph obtained from B_q by deleting the last u_0 blocks of V_0, and the last u_1 blocks of V_1. Analogously, we define $B_q(S, T, u_0, u_1) := B_q - (S_0 \cup T_0) - (\mathcal{U}_0 \cup \mathcal{U}_1)$ to be the graph obtained from the graph $B_q(S, T)$ obtained with Reduction 1. Clearly, for $u_0 = u_1 = 0$, $B_q(0, 0) = B_q$ and $B_q(S, T, 0, 0) = B_q(S, T)$.

Lemma 4.7. *Let u_0, u_1 be non–negative integers, with $0 \leqslant u_0 \leqslant u_1 < q-1$. Then*

(i) the graph $B_q(u_0, u_1)$ is a biregular graph with degrees $\{q-u_0, q-u_1\}$ of order $2q^2 - q(u_0 + u_1)$ if $u_0 \neq u_1$;

(ii) *the graph* $B_q(u_0, u_1)$ *is a* $(q-u_0)$*–regular graph of order* $2q(q-u_0)$ *if* $u_0 = u_1$ *;*

(iii) *the graph* $B_q(S, T, u_0, u_1)$ *has degrees* $\{q-u_0, q-u_1, q-u_0-1, q-u_1-1\}$ *and order* $2q^2 - q(u_0+u_1) - |S| - |T|$*. Moreover, the vertices* $(i,t)_0 \in V_0$ *with* $i \in GF(q) - U_0$ *and* $t \in T$ *are the only vertices of degree* $q-u_1-1$ *in* $B_q(S,T,u_0,u_1)$ *and the vertices* $(j,s)_1 \in V_1$ *with* $j \in GF(q) - U_1$ *and* $s \in S$*, together with the vertices* $(0,s)_1 \in V_1$*, for* $s \in S - T$ *if* $T \subsetneq S$*, are the only vertices of degree* $q-u_0-1$ *in* $B_q(S,T,u_0,u_1)$*;*

Proof. It is an immediate consequence of Proposition 2.2 (i), (v) and Lemma 4.6. □

4.2 Amalgam

In this section we describe an *amalgam* operation inspired by Jørgensen [29], Funk [22] and Abreu et al. [1], where regular bipartite graphs were transformed into (no longer bipartite) regular graphs of higher degree adding *weighted* edges with different weights on opposite sides of the bipartition.

Since we apply Reduction 1 before increasing the degree of B_q, we describe the amalgam operation performed on the reduced graph $B_q(S,T,u_0,u_1)$ for $0 \leqslant u_0 \leqslant u_1 < q-1$. The labelling for B_q introduced in Section 2, will be essential, in the choice of the graphs used for the amalgam, to guarantee the biregularity and the girth 5 in the final graph.

Let Γ_1 and Γ_2 be two graphs of the same order and with the same labels on their vertices. In general, an *amalgam of* Γ_1 *into* Γ_2 is a graph obtained adding all the edges of Γ_1 to Γ_2.

Let P_i and L_i be defined as in Section 2. Consider the graph $B_q(S,T,u_0,u_1)$, for some $T \subseteq S \subseteq GF(q)$ and $0 \leqslant u_0 \leqslant u_1 < q-1$. Let $S_0 \subseteq P_0$, $T_0 \subseteq L_0$ as in Reduction 1, and let $P_0' := P_0 - S_0$ and $L_0' := L_0 - T_0$ be the blocks in $B_q(S,T,u_0,u_1)$ of order $q-|S|$ and $q-|T|$, respectively.

Let H_i, G_i, for $i = 0,1$, be graphs of girth at least 5 and order $q-|S|$, $q-|T|$ and q, respectively. To simplify notation in our results, we label P_i and L_i as in Section 2, but assume that the labellings of H_0, H_1, G_0 and G_1 correspond to the second coordinates of P_0', L_0', P_i and L_j respectively for $i \in GF^*(q) - U_0$ and $j \in GF^*(q) - U_1$.

We define $B_q^*(S,T,u_0,u_1)$ to be the *amalgam* of H_0 into P_0', H_1 into L_0', G_0 into P_i, for $i \in GF^*(q) - U_0$, and G_1 into L_j, for $j \in GF^*(q) - U_1$. Note that $|V(B_q^*(S,T,u_0,u_1))| = |V(B_q(S,T,u_0,u_1))|$.

The next lemma is immediate and it shows the behavior of the degree set of the graph $B_q^*(S,T,u_0,u_1)$.

Lemma 4.8. *Let* $G := B_q^*(S,T,u_0,u_1)$*. Then the degrees of the vertices of* G *are:*

1. $d_G((0,y)_0) = q - u_1 + d_{H_0}(y)$,

2. $d_G((0,b)_1) = q - u_0 + d_{H_1}(b)$ *if* $b \notin S - T$,

3. $d_G((0,b)_1) = q - u_0 - 1 + d_{H_1}(b)$ *if* $b \in S - T$,

4. $d_G((x,y)_0) = q - u_1 + d_{G_0}(y)$ *if* $y \notin T$,

5. $d_G((x,y)_0) = q - u_1 - 1 + d_{G_0}(y)$ *if* $y \in T$,

6. $d_G((m,b)_1) = q - u_0 + d_{G_1}(b)$ *if* $b \notin S$,

7. $d_G((m,b)_1) = q - u_0 - 1 + d_{G_1}(b)$ *if* $b \in S$,

Note that with the above mentioned labelling, the labels of G_0, G_1 are the elements of $GF(q)$ and the labels of H_0, H_1 are either the elements of $GF(q)$ itself or a proper subset according to S, T being empty or not.

Let $M_F := \{(u,v) : u, v \in GF(q) \text{ and } uv \in E(F)\}$, for $F \in \{H_0, H_1, G_0, G_1\}$. For each $(u,v) \in M_F$, we define $\omega((u,v)) = \pm(u-v) \in GF^*(q)$ to be its *weight* or *Cayley Colour*. We define $\Omega(F) := \{\omega((u,v)) : (u,v) \in M_F\}$ to be the *set of weights* or *set of Cayley Colours* of F, for $F \in \{H_0, H_1, G_0, G_1\}$.
Note that $\Omega(F_1) \cap \Omega(F_2) = \emptyset$ implies that $M_{F_1} \cap M_{F_2} = \emptyset$, for $F_1, F_2 \in \{H_0, H_1, G_0, G_1\}$ and $F_1 \neq F_2$, but the converse is false.

The following lemma generalizes theorems [1, Theorem 5] and [22, Theorem 2.8].

Theorem 4.9. *Let $T \subseteq S \subseteq GF(q)$ and let $0 \leqslant u_0 \leqslant u_1 < q-1$. Let H_0, H_1, G_0 and G_1 be defined as above and suppose that $M_{H_0} \cap M_{H_1} = \emptyset$, $M_{H_0} \cap M_{G_1} = \emptyset$, $M_{H_1} \cap M_{G_0} = \emptyset$ and $\Omega(G_0) \cap \Omega(G_1) = \emptyset$. Then the amalgam $B_q^*(S,T,u_0,u_1)$ has girth at least 5 and order $2q^2 - q(u_0+u_1) - |S| - |T|$.*

Proof. Suppose first that $u_0 = u_1 = 0$ and so $B_q^*(S,T) = B_q^*(S,T,u_0,u_1)$ (cf. Reduction 2).

Let C be a shortest cycle in $B_q^*(S,T)$ and suppose, by contradiction, that $|C| \leqslant 4$. Therefore, $C = (xyz)$ or $C = (wxyz)$. Since B_q has girth 6 and H_0, H_1, G_0, G_1 have girth at least 5, then C cannot be completely contained in B_q or in H_0, H_1, G_0 or G_1. Then, w.l.o.g. the path xyz in C is such that $x, y \in P_i$ and $z \in L_m$ for some $i, m \in GF(q)$. Since the edges between P_i and L_m form a matching, then $xz \notin E(B_q)$ and hence $xz \notin E(B_q^*(S,T))$. Thus $|C| > 3$ and we can assume $|C| = 4$ and $C = (wxyz)$,with xyz taken as before.

If $w \in P_i$, by the same argument, $wz \notin E(B_q^*(S,T))$ and we have a contradiction. There are no edges between P_i and P_j in $B_q^*(S,T)$, so $w \notin P_j$ for $j \in GF(q) - \{i\}$, which implies that $w \in L_n$ for some $n \in GF(q)$. If $n \neq m$, we have a contradiction since there are no edges between L_m and L_n in $B_q^*(S,T)$. Therefore $x, y \in P_i$ and $w, z \in L_m$. Let $x = (i, a_1)_0, y = (i, a_2)_0, w = (m, b_1)_1$ and $z = (m, b_2)_1$ as in the labelling chosen in Section 2. Then $wx, yz \in E(B_q^*(S,T))$ imply that $a_1 = m \cdot i + b_1$ and $a_2 = m \cdot i + b_2$, respectively.

If m or i are zero, i.e. if $xy \in H_0$ or $wz \in H_1$, then the above equations are satisfied if and only if $a_1 = b_1$ and $a_2 = b_2$, but this contradicts at least one of $M_{H_0} \cap M_{H_1} = \emptyset$, $M_{H_0} \cap M_{G_1} = \emptyset$ and $M_{H_1} \cap M_{G_0} = \emptyset$.

If m and i are both non-zero, i.e. if $xy \in G_0$ and $wz \in G_1$, then the above equations are satisfied if and only if $a_1 - a_2 = b_1 - b_2$, implying that $\pm(a_1 - a_2) \in \Omega(G_0)$ and $\pm(a_1 - a_2) = \pm(b_1 - b_2) \in \Omega(G_1)$ which contradicts $\Omega(G_0) \cap \Omega(G_1) = \emptyset$.

Hence $B_q^*(S,T)$ has girth at least five. Since $B_q^*(S,T,u_0,u_1)$ is a subgraph of $B_q^*(S,T)$, for $0 \leqslant u_0 \leqslant u_1 < q-1$, then also $B_q^*(S,T,u_0,u_1)$ ha girth five, completing the proof. $\square$

5 New $(\{r, 2r-5\}; 5)$-cages for $r \geqslant 8$.

In this section we will construct a new family of $(\{r, 2r-5\}; 5)$-cages, for $r = q+1$ and $q \geqslant 7$ a prime, applying Reduction 1, 2 and Amalgam (cf. Theorem 4.9 and Lemma 4.8) to the graph B_q as previously described.

Recall that every prime q is either congruent to 1 or 3 modulo 4, we will now treat these two cases separately, when $q = 4n+1$ or $q = 4n+3$. In each case we will specify the sets S and T to be deleted from P_0 and L_0, the integers u_0 and u_1 of number of blocks to be deleted, and the graphs H_0, H_1, G_0 and G_1 to be used for the amalgam into $B_q^*(S, T, u_0, u_1)$.

Since q is a prime we can consider that $GF(q)$ coincides with $\mathbb{Z}_q$, and the addition operations are modulo q. We present the case $q \equiv 3 \mod 4$ first, since the smallest case of the construction occurs for $q = 7 \equiv 3 \mod 4$. In what follows, recall that $D(F)$ denotes the degree set of a graph F.

5.1 Construction for primes $q = 4n+3$, $n \geqslant 1$.

Let $B_q^*(S, T, u_0, u_1)$ be the graph resulting from the following choice of its parameters:

$S = \{\frac{q+1}{4}, -\frac{q+1}{4}\} = \{\frac{q+1}{4}, \frac{3q-1}{4}\}$; $T = \emptyset$; $u_0 = 0$; $u_1 = 1$			
Graph	Vertices	Edges	Description
H_0	$\mathbb{Z}_q - \{\frac{q+1}{4}, \frac{3q-1}{4}\}$	$\{(j, j+\frac{q-1}{2}) \mid j \in \mathbb{Z}_q - \{\frac{q+1}{4}, \frac{3q-1}{4}, \frac{3q+3}{4}\}\} \cup \{(\frac{3q+3}{4}, \frac{q-3}{4})\}$ sums modulo q	$(q-2)$–cycle $\Omega(H_0) = \{\frac{q-1}{2}, \frac{q-3}{2}\}$
G_0	$\mathbb{Z}_q$	$\{(j, j+\frac{q-1}{2}) \mid j \in \mathbb{Z}_q\}$ sums modulo q	q–cycle $\Omega(G_0) = \{\frac{q-1}{2}\}$
$H_1 \cong G_1$	$\mathbb{Z}_q$	$\{(0, j) : j \in \mathbb{Z}_q^* - \{\frac{q-1}{2}, \frac{q+1}{2}\}\} \cup \{(\frac{q+1}{4}, \frac{q-1}{2}), (\frac{3q-1}{4}, \frac{q+1}{2})\}$	$\Omega(H_1) = \Omega(G_1) = \mathbb{Z}_q^* - \{\frac{q-1}{2}\}$

To illustrate the construction we present in Figure 1 the graph $B_q^*(S, T, u_0, u_1)$ without the edges from B_q, for $q = 7$. Each line style represents a different weight (or Cayley Colour). As we will proved in the Theorem 5.10, this is an $(\{8, 11\}; 5)$–cage.

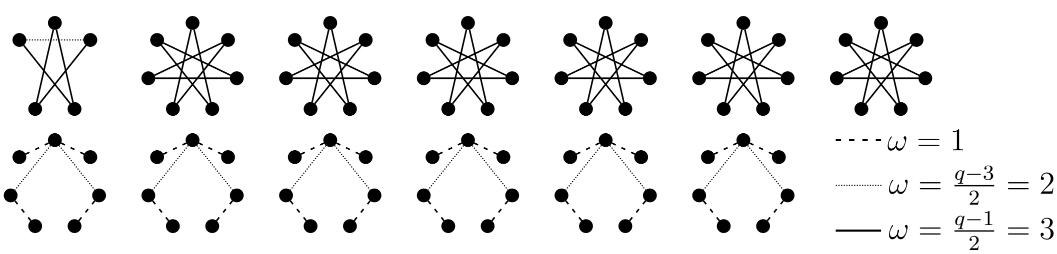

Figure 1: $B_7^*(S, T, 0, 1) - E(B_7)$ with $S = \{3, 5\}$ and $T = \emptyset$

Theorem 5.10. *Let $q = 4n+3$ be a prime, for $n \geqslant 1$. Let $S, T, u_0, u_1, H_0, H_1, G_0$ and G_1 be defined as above. Then the amalgam graph $B_q^*(S, T, u_0, u_1)$ is an $(\{r, 2r-5\}; 5)$–cage of order $r(2r-5)+1$, where $r = q+1$.*

Proof. From Lemma 4.8, the degree set $D(B_q^*(S,T,u_0,u_1)) = \{r, 2r-5\}$, where $r = q+1$. In fact, all vertices in $V_0 \cap B_q(S,T,u_0,u_1)$ have degree $q-1$. Moreover, $D(H_0) = D(G_0) = \{2\}$ since H_0 and G_0 are cycles. Hence, all the vertices of $V_0 \cap V(B_q^*(S,T,u_0,u_1))$ have degree $q+1 = r$. In V_1 we distinguish three subsets of vertices: $V_1' := \{(m,0)_1 : m \in \mathbb{Z}_q\}$, $V_1'' := \{(m,t)_1 : m \in \mathbb{Z}_q, t \in \frac{q+1}{4}, \frac{3q-1}{4}\}$ and $V_1''' := \{(m,t)_1 : t \in \mathbb{Z}_q^* - \{\frac{q+1}{4}, \frac{3q-1}{4}\}\}$. The vertices in V_1''' have degree q in $B_q(S,T,u_0,u_1)$, and their degree is 1 in H_1 and G_1. Thus these vertices have degree $q+1 = r$ in $B_q^*(S,T,u_0,u_1)$. The vertices of V_1'' have degree $q-1$ in $B_q(S,T,u_0,u_1)$, whereas their degree is 2 in H_1 and G_1. Hence, they have degree $q+1 = r$ in $B_q^*(S,T,u_0,u_1)$. Finally, the vertices in V_1' have degree q in $B_q(S,T,u_0,u_1)$, while they have degree $q-3$ in H_1 and G_1. Hence, they have degree $q+q-3 = 2q-3 = 2(r-1)-3 = 2r-5$ in $B_q^*(S,T,u_0,u_1)$.

The graph $B_q^*(S,T,u_0,u_1)$ has girth at least 5, since we are under the hypothesis of Theorem 4.9. In fact, $\Omega(H_0) \cap \Omega(H_1) = \Omega(H_0) \cap \Omega(G_1) = \{\frac{q+3}{2}\}$, but $M_{H_0} \cap M_{H_1} = M_{H_0} \cap M_{G_1} = \emptyset$, since the only edge of weight $\frac{q+3}{2}$ in H_0 is $(\frac{3q+3}{4}, \frac{q-3}{4})$, while the edges $(0, \frac{q-3}{2})$ $(0, \frac{q+3}{2})$ are those of weight $\pm\frac{q+3}{2}$ in H_1 and G_1. We also have that $\Omega(G_0) \cap \Omega(H_1) = \Omega(G_0) \cap \Omega(G_1) = \emptyset$, which implies that $M_{G_0} \cap M_{H_1} = M_{G_0} \cap M_{G_1} = \emptyset$. Moreover, the girth is exactly five, since the 5–cycle $((0,0)_0, (0, \frac{q+1}{4})_0, (0, \frac{q-1}{2})_0, (0, \frac{q-1}{2})_1, (0,0)_1)$ lies in $B_q^*(S,T,u_0,u_1)$.

Finally, the order of $B_q^*(S,T,u_0,u_1)$ is $2q^2 - q(u_0+u_1) - |S| - |T| = 2q^2 - q - 2 = r(2r-5)+1$, by Lemma 4.7, which satisfies exactly Down's bound (1)

$$n(\{r, 2r-5\}; 5) = 1 + \sum_{i=1}^{2} (2r-5)(r-1)^{i-1} = r(2r-5)+1.$$

Hence, the graph $B_q^*(S,T,0,1)$ is an $(\{r, 2r-5\}; 5)$-cage. □

5.2 Construction for primes $q = 4n+1$, $n \geqslant 3$.

Let $B_q^*(S,T,u_0,u_1)$ be the graph resulting from the following choice of its parameters:

$S = \{\frac{q-1}{4}, -\frac{q-1}{4}\} = \{\frac{q-1}{4}, \frac{3q+1}{4}\}$; $T = \emptyset$; $u_0 = 0$; $u_1 = 1$			
Graph	Vertices	Edges	Description
H_0	$\mathbb{Z}_q - \{\frac{q-1}{4}, \frac{3q+1}{4}\}$	$\{(j, j+\frac{q-1}{2}) \mid j \in \mathbb{Z}_q - \{\frac{q-1}{4}, \frac{3q+1}{4}, \frac{3q-3}{4}\}\} \cup \{(\frac{3q-3}{4}, \frac{q+3}{4})\}$ sums modulo q	$(q-2)$–cycle $\Omega(H_0) = \{\frac{q-1}{2}, \frac{q+3}{2}\}$
G_0	$\mathbb{Z}_q$	$\{(j, j+\frac{q-1}{2}) \mid j \in \mathbb{Z}_q\}$ sums modulo q	q–cycle $\Omega(G_0) = \{\frac{q-1}{2}\}$
$H_1 \cong G_1$	$\mathbb{Z}_q$	$\{(0,j) : j \in \mathbb{Z}_q^* - \{\frac{q-1}{2}, \frac{q+1}{2}\}\} \cup \{(\frac{q-1}{4}, \frac{q-1}{2}), (\frac{3q+1}{4}, \frac{q+1}{2})\}$	$\Omega(H_1) = \Omega(G_1) = \mathbb{Z}_q^* - \{\frac{q-1}{2}\}$

Theorem 5.11. *Let $q = 4n+1$ be a prime, for $n \geqslant 3$. Let $S, T, u_0, u_1, H_0, H_1, G_0$ and G_1 be defined as above. Then the amalgam graph $B_q^*(S,T,u_0,u_1)$ is an $(\{r, 2r-5\}; 5)$–cage of order $r(2r-5)+1$, where $r = q+1$.*

Proof. Analogous to the proof of Theorem 5.10. □

6 New semiregular $(\{r, r+1\}; 5)$-cages for $r = 5, 6$.

In this section we construct two new semiregular cages, namely a $(\{5, 6\}; 5)$-cage with 31 vertices and a $(\{6, 7\}; 5)$-cage with 43 vertices. For the first one, we choose $S, T, u_0, u_1, H_0, H_1, G_0$ and G_1 to construct the amalgam graph $B_q^*(S, T, u_0, u_1)$ and the result is obtained as a consequence of Theorem 4.9 and Lemma 4.8. The second one is a sporadic example in which we adapt and slightly generalize the techniques that we have used so far.

6.1 Construction of the $(\{5, 6\}; 5)$-cage.

Let $GF(4) = \{0, 1, \alpha, \alpha^2\}$ be the finite field of order 4 and let $B_q^*(S, T, u_0, u_1)$ be the graph resulting from the following choice of its parameters (c.f. Figure 2):

$S = \{0\}$; $T = \emptyset$; $u_0 = 0$; $u_1 = 0$			
Graph	Vertices	Edges	Description
H_0	$GF^*(4)$	$\{(1, \alpha^2), (\alpha^2, \alpha)\}$	2–path $\Omega(H_1) = \{1, \alpha\}$
G_0	$GF(4)$	$\{(0, \alpha), (1, \alpha^2)\}$	Two disjoint edges $\Omega(G_1) = \{\alpha\}$
$H_1 \cong G_1$	$GF(4)$	$\{(\alpha^2, 0), (0, 1), (1, \alpha)\}$	3–path $\Omega(H_2) = \Omega(G_2) = \{1, \alpha^2\}$

Theorem 6.12. *Let $q = 4$ and let $S, T, u_0, u_1, H_0, H_1, G_0$ and G_1 be defined as above. Then the amalgam graph $B_4^*(S, T, u_0, u_1)$ is a $(\{5, 6\}; 5)$–cage of order 31.*

Proof. From Lemma 4.8 we have that the degree set $D(B_4^*(S, T, u_0, u_1)) = \{5, 6\}$, and moreover, the set of vertices of degree 6 is $\{(i, 1)_0 : i \in GF(4)\} \cup \{(0, \alpha^2)_1\}$.

The graph $B_4^*(S, T, u_0, u_1)$ has girth at least 5, since we are under the hypothesis of Theorem 4.9. In fact, $\Omega(H_0) \cap \Omega(H_1) = \Omega(H_0) \cap \Omega(G_1) = \{1\}$, but $M_{H_0} \cap M_{H_1} = M_{H_0} \cap M_{G_1} = \emptyset$, since (α^2, α) is the only edge of weight 1 in H_0, while the only edges of such weight is $(0, 1)$ in H_1 and G_1. We also have that $\Omega(G_0) \cap \Omega(H_1) = \Omega(G_0) \cap \Omega(G_1) = \emptyset$, which implies that $M_{G_0} \cap M_{H_1} = M_{G_0} \cap M_{G_1} = \emptyset$. Moreover, the girth is exactly five, since the 5–cycle $((0, 0)_0, (0, 1)_0, (0, \alpha)_0, (1, \alpha)_1, (1, 0)_1)$ lies in $B_4^*(S, T, u_0, u_1)$.

Finally, by Lemma 4.7 we have that the order of the graph is $2q^2 - q(u_0 + u_1) - |S| - |T| = 2(16) - 1 = 31$. $\square$

6.2 Construction of the $(\{6, 7\}; 5)$-cage.

For this construction we need to modify Reduction 1 and apply the amalgam operation accordingly as follows:

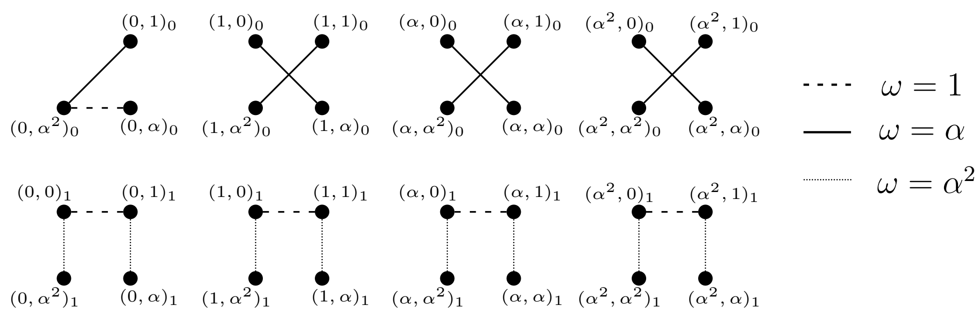

Figure 2: $B_4^*(S,T,0,0) - E(B_4)$ with $S = \{0\}$ and $T = \emptyset$

Let $q = 5$ and consider the graph B_5. We modify Reduction 1 removing vertices from P_0 and vertices of V_1 from each block L_j for $j \in GF(5)$ (whereas in Reduction 1 we delete vertices of V_1 only from the block L_0).

Let $T := \{3\} \subset GF(5)$, $S := \{0,3\} \subset GF(5)$, $S_0 = \{(0,y)_0 | y \in S\} \subseteq P_0$, $T_j = \{(j,b)_1 | b \in T\} \subseteq L_j$ for $j \in GF(5)$, and let $B_5(S,TT) := B_5 - S_0 - \bigcup_{j \in GF(5)} T_j$. Note that the graph $B_5(S,TT)$ has degree set $D(B_5(S,TT)) = \{q-1, q\} = \{4,5\}$ and order $2q^2 - |S| - q|T| = 50 - 2 - 5 = 43$. Moreover, each vertex in $V_0 - P_0$ has degree $q - |T| = 4$, each vertex of V_1 with second coordinate in $S - T$ has degree $q - |S - T| = 4$, and all other vertices have degree $q = 5$.

Now we will amalgam some graphs into $B_5(S,TT)$. Let $H_0 := \{(2,4),(4,1)\}$, $G_0 := \{(0,2),(2,4),(4,1),(1,3),(3,0)\}$ and $H_1 := \{(4,0),(0,1),(1,2)\}$. These graphs are a 2–path, a 5–cycle and a 3–path, respectively, with weights $\Omega(H_0) = \Omega(G_0) = \{2\}$ and $\Omega(H_1) = \{1\}$.

Let $B_5^*(S,TT)$ be the graph obtained from the amalgam of H_0 into $P_0' := P_0 - S_0$, G_0 into P_i, for all $i \in GF^*(5)$, and H_1 into $L_j' = L_j - T_j$, for all $j \in GF(5)$ (c.f. Figure 3 for an illustration).

Theorem 6.13. *Let $q = 5$ and let S, T, H_0, H_1 and G_0 be defined as above. Then the amalgam graph $B_5^*(S,TT)$ is a $(\{6,7\};5)$–cage of order* 43.

Proof. Using the same reasoning as in the proof of Theorem 4.9, the amalgam graph $B_5^*(S,TT)$ has girth at least 5, since $\Omega(H_0) \cap \Omega(H_1) = \Omega(G_0) \cap \Omega(H_1) = \emptyset$. The girth is exactly 5, since G_0 is a 5–cycle. Moreover, the degree set $D(B_5^*(S,TT)) = \{6,7\}$, since all vertices of $B_5(S,TT)$ of degree 4 obtain two new edges in $B_5^*(S,TT)$, and similarly the vertices of $B_5(S,TT)$ of degree 5 obtain one or two new edges in $B_5^*(S,TT)$. Hence, $B_5^*(S,TT)$ is a $(\{6,7\};5)$–cage as desired, since its order satisfies Down's bound. □

There is a further and simpler way of constructing this graph: let G be the Hoffman-Singleton Graph [26], consider a vertex $x \in V(G)$ and $N(x) = \{x_1, x_2, \ldots, x_7\}$ its set

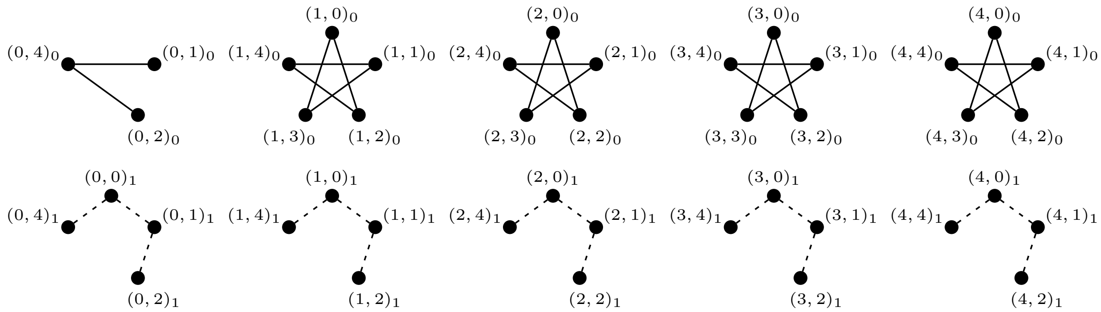

Figure 3: $B_5^*(S,TT) - E(B_5)$ with $S = \{0,3\}$ and $T = \{3\}$

of neighbors, then the graph H obtained from G by deleting the set of vertices $\{x\} \cup (N(x) - x_1)$ is clearly a graph of girth 5 and 43 vertices with 6 vertices of degree 7 and the rest of degree 6. The graph H is a $(\{6,7\};5)$-cage as desired, since its order satisfies Down's bound. It is not difficult to prove that this graph H is isomorphic to the previous $B_5^*(S,TT)$.

It is widely known that Moore graphs of odd girth are very rare: complete graphs, the Petersen graph, the Hoffman-Singleton Graph and maybe the $(57,5)$-cage. Therefore, the deletion technique carried out for the Hoffman-Singleton Graph can be only applied for this case and may be for $r = 57$.

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