

Enumeration of (16,4,16,4) Relative Difference Sets

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Abstract

A complete enumeration of relative difference sets (RDS) with parameters $(16, 4, 16, 4)$ in a group of order 64 with a normal subgroup N of order 4 is given. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 nonabelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs.

Keywords: Relative difference set; symmetric net.

1 Introduction

A relative difference set (RDS) with parameters (m, n, k, λ) in a finite group G of order mn relative to a normal subgroup N of order n is a k -subset R of G such that every element of $g \in G \setminus N$ appears exactly λ times in the multiset $S = \{ab^{-1} \mid a, b \in R, a \neq b\}$, and no element of N appears in S [1]. An RDS is called *abelian* if G is abelian, and *nonabelian* otherwise.

Relative difference sets are closely related to difference sets, group-divisible designs, generalized Hadamard matrices, symmetric nets, and finite geometry [1], [4], [6]. A comprehensive survey on RDS is the paper by Pott [5]. The existence problem of (p^a, p^b, p^a, p^{a-b}) RDSs is considered to be one of the most important questions concerning RDSs [5].

In [7], Schmidt studied the existence of abelian (p^a, p^b, p^a, p^{a-b}) RDS, and settled the existence problem of abelian $(16, 4, 16, 4)$ RDS completely.

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In this paper, a complete enumeration of $(16, 4, 16, 4)$ RDSs is given, for all groups, abelian and nonabelian, of order 64. In summary, RDSs exist in 6 of the 11 abelian groups of order 64, as well as in 195 of the 256 nonabelian groups of order 64. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. The computer algebra package Magma [2] was used in the computations.

2 RDS and symmetric nets

Our approach to the enumeration of $(16, 4, 16, 4)$ RDSs is based on their link to incidence structures known as symmetric $(4, 4)$ -nets.

A *symmetric $(4, 4)$ -net*¹ is an incidence structure $I = (X, \mathcal{B})$ consisting of a set X of 64 *points* and a collection $\mathcal{B}$ of 64 *blocks*, each block being a subset of 16 points of X , having the following properties:

- Each point belongs to 16 blocks.
- There exists a partition $\mathcal{P}$ of the point set X into 16 subsets of size 4, called *groups*, so that every two points belonging to different groups appear together in exactly 4 blocks, while every two points belonging to the same group do not appear together in any block.
- The 64 blocks are partitioned into 16 *parallel classes*, each class consisting of 4 pairwise disjoint blocks, so that every two blocks belonging to different parallel classes share exactly 4 points.

Other terms used for a structure with the above properties are group-divisible design, or a transversal design [1].

An *automorphism* of an incidence structure I is any permutation of the point set which preserves the collection of blocks. The set of all automorphisms of I form a group, called the *full automorphism group*, $\text{Aut}(I)$, of I . The subgroups of $\text{Aut}(I)$ are called *automorphism groups*.

A symmetric $(4, 4)$ -net is *class-regular* if it admits an automorphism group N of order 4 which acts transitively on each group of points and each parallel class of blocks. The group N is then called a group of *bitranslations*.

If R is a $(16, 4, 16, 4)$ RDS in a group G of order 64, relative to a normal subgroup $N \leq G$ of order 4, one can associate with R a class-regular $(4, 4)$ -net I with point set G and blocks being the subsets $B_g \subseteq G$ of the form

$$B_g = \{Rg \mid g \in G\}.$$

¹More generally, a net is defined as a resolvable 1-design, and a symmetric net is a net with equal number of points and blocks. For more definitions concerning designs see [1].

The partition $\mathcal{P}$ of the points into subsets of size 4 is defined as the partition of G into cosets of N . Consequently, G acts as an automorphism group of I , and the subgroup N acts transitively on each point group and each parallel class.

Thus, any $(16, 4, 16, 4)$ RDS corresponds to a class-regular symmetric $(4, 4)$ -net which admits a regular automorphism group.

All nonisomorphic class-regular symmetric $(4, 4)$ -nets were enumerated by Harada, Lam and Tonchev in [4], and, implicitly, by Gibbons and Mathon in [3] (two incidence structures are isomorphic if there is an incidence preserving bijection between their point sets). Up to isomorphism, there are exactly 226 nets with group of bitranslations $N = Z_2 \times Z_2$, and 13 nets with $N = Z_4$.

These results reduce the enumeration of $(16, 4, 16, 4)$ RDSs to finding sharply transitive regular subgroups G of the full automorphism groups of those class-regular symmetric $(4, 4)$ -nets which admit automorphism groups acting transitively on the points, such that N is a normal subgroup of G . We used Magma to find the conjugacy classes of sharply transitive regular subgroups.

There are 267 groups of order 64, of which 11 are abelian and 256 are nonabelian. Among the 226 nets with group of bitranslations $Z_2 \times Z_2$, only 200 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 6 were abelian and 194 were nonabelian. Among the 13 nets with group of bitranslations Z_4 , only 26 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 3 were abelian and 23 were nonabelian.

3 The results

Tables 1 and 2 list the nets with automorphism groups which admit regular subgroups with normal subgroup N . Each entry is as follows:

- *#*: The index of the net within the list of nets with a group of bitranslations $N = Z_2 \times Z_2$ available at <http://www.math.mtu.edu/~tonchev/Z2Z2nets> and at <http://www.math.mtu.edu/~tonchev/Z4nets> for the nets with $N = Z_4$. Missing indices indicate that the corresponding nets do not have transitive automorphism groups.
- *Order*: The order of the automorphism group.
- *2-Rank*: The 2-rank of the incidence matrix of the net.
- *Total*: In the format x/y , x indicates the total number of conjugacy classes of regular subgroups containing the group of bitranslations, found within the automorphism group of each net, while y is the number of nonisomorphic regular subgroups.
- *Abelian* and *Nonabelian*: In the format x/y , x indicates the number of conjugacy classes of each type of subgroup found within the net's automorphism group, while y is the number of nonisomorphic subgroups.

- *List of indices:* Below the previous data is a list of the indices of the regular subgroups of order 64 found in each automorphism group according to Magma's list of the groups of order 64. Entries of the form $x(y)$ indicate that groups isomorphic to group x appeared in y distinct conjugacy classes. Entries marked with an asterisk are abelian, and all others are nonabelian.

Tables 3 and 4 give details about the structure of the regular abelian subgroups of order 64 found. Only such subgroups containing the relevant group N of bitranslations were considered. Groups with the following indices had a single regular abelian subgroup isomorphic to $\mathbb{Z}_2^6$, and are not listed in Table 3: 6, 99, 100, 103, 104, 105, 107, 111, 113, 120, 121, 127, 128, 131, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 181, 182, 192, 193, 194, 195, 196, 197, 198, 212, 214, 221, 224, 225, 226.

Tables 5 and 6 summarize the structures of the regular abelian subgroups of order 64 found within the nets.

Table 7 gives the structures of all abelian groups of order 64 which do not appear as a regular subgroup in any net. Consequently, those are the groups which do not support any $(16, 4, 16, 4)$ RDS.

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
1	1105920 4(2), 9(2), 18(2), 20(2), 23(5), 25(2), 32(6), 33(2), 34, 35(3), 36, 37(3), 56(8), 60(2), 74, 77, 80, 88(4), 90(5), 92(2), 100(2), 102(2), 132, 165, 192(4)*, 193(4), 194, 198(2), 199(3), 200(2), 202(3), 206(3), 207(3), 210(4), 214(3), 215(3), 217(3), 219(4), 223(8), 224(5), 226(2), 227(4), 229(2), 230(2), 232(10), 236, 237(6), 239(4), 240, 241(2), 242(9), 243, 244(5), 245, 262(3), 264(2), 267(2)*	16	173/57	6/2	167/55
2	13824 60(2), 88, 132, 192*, 193, 262, 267*	18	8/7	2/2	6/5
4	4608 56(2), 88(2), 100, 132, 262(2)	18	8/5	0/0	8/5
5	18432 4(2), 9(2), 20, 23, 32(4), 35(2), 56(14), 60(2), 74(3), 88(3), 90, 92, 100, 132(2), 192(5)*, 193(5), 202, 207(2), 242(5), 262(3), 267*	16	61/21	6/2	55/19
6	4608 20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92, 100, 109, 193, 202(2), 204, 207, 247, 267*	20	28/19	1/1	27/18
8	384 88, 132(2), 262	20	4/3	0/0	4/3
9	4608 60(2), 88, 132, 192*, 193, 262, 267*	18	8/7	2/2	6/5

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
22	6144 4(2), 9(2), 20, 23, 32(2), 56(29), 60, 66, 67(2), 69(3), 71(2), 73(2), 75(3), 78(2), 88(3), 90, 92, 109, 131, 164, 192(5)*, 193(8), 194(8), 260*, 261	16	84/25	6/2	78/23
24	1536 18, 22, 25(2), 88(2), 93(2), 97, 100(5), 102, 111, 114, 116, 117, 120, 122, 132(5), 133, 143, 145, 149, 151, 156, 158, 163, 166(3), 168, 170, 200, 206, 217, 229, 230, 249, 252, 255, 262	20	48/35	0/0	48/35
25	1536 56(3), 88(2), 100, 132, 262(2)	18	9/5	0/0	9/5
30	1536 18, 22, 25(2), 33, 36, 91, 93(2), 97(2), 99, 100(2), 105, 111, 119, 120, 121, 122, 129, 130, 131, 132(2), 133, 142, 143, 144, 145(3), 146, 148, 149, 151, 156, 158, 166(2), 168, 170, 176, 178, 200, 249, 251, 252, 254, 255	20	50/42	0/0	50/42
31	1536 23, 24, 32(6), 35(4), 58, 59, 60, 61(2), 66, 67(2), 69, 72(2), 74, 75, 78, 85, 90(2), 101(2), 136(3), 138(2), 202(2), 207, 212, 242(2), 255(2), 260*	20	45/26	1/1	44/25
32	128 146, 148, 206, 246*	20	4/4	1/1	3/3
33	512 18, 25, 32(4), 33(4), 35(2), 86(2), 88, 91(2), 97(2), 99, 100, 102(2), 105, 108(3), 115, 116, 119, 122, 132, 133, 135, 138, 139(3), 145, 146(2), 148(2), 149, 151, 155, 158(2), 163, 166(3), 170, 179, 206(2), 217, 223(2), 227, 232, 237, 242, 246*, 251, 254	20	66/44	1/1	65/43
36	73728 4(2), 9(2), 18(6), 20(6), 23(13), 25(6), 32(14), 33(10), 34(5), 35(7), 36(5), 37(7), 56(13), 60(3), 62, 74(4), 77(2), 80, 88(6), 90(11), 92(3), 93, 99(2), 100(4), 102(6), 132(2), 192(4)*, 193(6), 194(2), 195(5), 196(5), 197(3), 198(13), 199(15), 200(5), 201(6), 202(7), 203(6), 204(6), 205(7), 206(29), 207(7), 209(8), 210(52), 211(3), 212(3), 213(12), 214(17), 215(16), 216(12), 217(19), 218(9), 219(56), 220(44), 221(18), 222(26), 223(68), 224(12), 225(13), 226(15), 227(58), 228(19), 229(22), 230(11), 231(5), 232(69), 233(46), 234(38), 235(23), 236(20), 237(33), 238(9), 239(5), 240(18), 241(27), 242(15), 243(24), 244(31), 260*, 261, 262(4), 263, 264(5), 265, 267*	16	1158/85	6/3	1152/82
78	6144 4, 5, 9(4), 57, 62, 67, 68, 69(2), 70, 73, 74, 77, 78, 79, 81, 82, 87, 112, 131, 132, 164, 165, 208, 263	20	28/24	0/0	28/24

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
99	1152 60(3), 267*	20	$4/2$	$1/1$	$3/1$
100	1152 60(3), 267*	22	$4/2$	$1/1$	$3/1$
103	384 60(3), 267*	20	$4/2$	$1/1$	$3/1$
104	384 60(3), 267*	22	$4/2$	$1/1$	$3/1$
105	1152 60(3), 267*	18	$4/2$	$1/1$	$3/1$
107	6912 60(3), 193, 267*	20	$5/3$	$1/1$	$4/2$
109	384 100, 109, 204, 247	22	$4/4$	$0/0$	$4/4$
111	512 20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92(2), 100, 109, 193(2), 202(6), 204(2), 207(2), 247, 267*	20	$52/19$	$1/1$	$51/18$
113	1536 20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92, 100, 109, 193(2), 202(3), 204(2), 207(2), 247, 267*	20	$48/19$	$1/1$	$47/18$
115	256 9(2), 132(2)	20	$4/2$	$0/0$	$4/2$
116	768 9(2), 132(2)	20	$4/2$	$0/0$	$4/2$
117	128	21	$0/0$	$0/0$	$0/0$
120	1536 20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92(2), 100, 109, 193, 202(4), 204, 207, 247, 267*	20	$31/19$	$1/1$	$30/18$
121	384 60(3), 267*	22	$4/2$	$1/1$	$3/1$
122	128 100, 109, 204, 247	22	$4/4$	$0/0$	$4/4$
123	192 20	22	$1/1$	$0/0$	$1/1$
124	64 23	22	$1/1$	$0/0$	$1/1$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
125	64 132	22	$1/1$	$0/0$	$1/1$
126	192 132	22	$1/1$	$0/0$	$1/1$
127	768 60(4), 193, 267*	20	$6/3$	$1/1$	$5/2$
128	256 60(4), 193(2), 267*	20	$7/3$	$1/1$	$6/2$
131	768 60(4), 193, 267*	20	$6/3$	$1/1$	$5/2$
132	512 23, 24, 32, 33(2), 58, 59(2), 61, 62, 63, 65(2), 66, 67(3), 68(3), 69(7), 70, 71, 72, 75, 77(3), 78(2), 80, 81(2), 85, 90(2), 98(2), 101(2), 139, 192*, 197, 204, 205, 212	20	$52/32$	$1/1$	$51/31$
134	256 58, 59(2), 61, 63, 65, 68(2), 69(10), 70(5), 71(4), 72, 73(2), 75(9), 77(4), 78(7), 79(3), 80(3), 81(2), 195(2), 196, 197, 204(2)	20	$64/21$	$0/0$	$64/21$
136	512 23, 24, 32, 35(2), 55*, 58(2), 59, 61, 63(3), 66(2), 67(4), 69(8), 71, 72(4), 74(3), 75, 76, 78(2), 81(2), 90(2), 98, 101(2), 104, 136, 197, 208, 255, 262	20	$52/28$	$1/1$	$51/27$
137	768 57, 59, 69(4), 70(3), 72, 77(3), 78(2), 79(2), 80(3), 81, 82, 197, 212	20	$24/13$	$0/0$	$24/13$
138	1536 23, 24, 32(2), 33(4), 34(4), 55*, 58, 61, 62(2), 65, 67, 69, 70, 71, 75, 77, 79, 90(2), 98, 101(2), 104, 134(2), 139(3), 194, 196, 203, 205(2), 209, 241(2), 254	20	$45/30$	$1/1$	$44/29$
139	512 18, 20, 25(2), 89, 91(2), 92, 95, 97, 98, 100(3), 102(2), 105, 112, 115, 117, 119, 120, 129, 131(2), 132, 133(2), 145(2), 148, 151, 159, 160, 163, 164, 165, 166, 169(2), 198, 200, 207(2), 217(2), 227(2), 229, 252, 255, 264	21	$52/40$	$0/0$	$52/40$
140	512 18, 25, 32(2), 33(6), 35(2), 86(2), 88, 91(2), 97, 100(3), 102(2), 105, 108(2), 109, 116, 117, 120, 122, 132(2), 137, 138, 139(3), 143, 148(2), 151(4), 156, 158, 160, 165, 166(3), 168, 182, 204, 206, 217, 222, 225, 230, 232, 233, 244, 247, 252, 255	21	$66/44$	$0/0$	$66/44$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
142	512 20(2), 23(2), 32(2), 60(10), 66(2), 67(2), 69(4), 71(3), 77(4), 78(4), 88, 90, 92, 109(2), 193(2), 195(2), 202(5), 205(2), 267*	19	$52/19$	$1/1$	$51/18$
143	6144 5, 9(2), 20, 23(2), 32, 57, 58, 60(5), 61, 62, 66(2), 67, 68, 69(2), 70, 71, 77(2), 78, 79, 81, 88, 90(4), 92, 109, 112, 164, 165, 193, 195, 202, 205, 208, 267*	20	$45/33$	$1/1$	$44/32$
144	128 60(3), 267*	20	$4/2$	$1/1$	$3/1$
145	256 60(21), 69(21), 77(7), 80(7), 209(7), 267*	20	$64/6$	$1/1$	$63/5$
146	64 267*	22	$1/1$	$1/1$	$0/0$
147	128 60(3), 267*	20	$4/2$	$1/1$	$3/1$
148	128 60(3), 267*	22	$4/2$	$1/1$	$3/1$
149	128 60(3), 267*	22	$4/2$	$1/1$	$3/1$
150	384 60(3), 267*	20	$4/2$	$1/1$	$3/1$
151	1152 60, 267*	21	$2/2$	$1/1$	$1/1$
152	512 18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255	21	$66/43$	$0/0$	$66/43$
156	256 35, 84, 91, 97, 170, 172(2), 214	25	$8/7$	$0/0$	$8/7$
157	256 86, 89, 95, 97, 98, 99, 105, 109(3), 116(2), 117(2), 118, 119, 122, 123, 128, 130, 133(2), 141, 144, 146(4), 147(2), 149(3), 150(2), 151, 155, 157(2), 159, 160(2), 162, 163(4), 164(2), 166, 170, 171, 179, 180, 216, 218, 219, 221(2), 223, 226, 227(2), 228, 232, 233, 248, 254	21	$68/48$	$0/0$	$68/48$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
158	256 85, 88, 96(2), 97, 100, 104, 106, 108, 109, 115(2), 116(2), 120(2), 121, 122, 129, 132(2), 133, 143, 145, 148(2), 149(3), 151(7), 158, 159, 160(4), 162, 163, 165(2), 166(4), 169, 172, 180, 182, 201, 204, 206, 210, 214, 220, 228, 229, 230, 232, 233, 235, 247, 252	21	$68/47$	$0/0$	$68/47$
166	1024 9(2), 20(2), 23(2), 33, 35, 58, 62(3), 63, 65, 66, 67(3), 68(2), 69(2), 72(3), 74, 75(2), 77, 85, 88(2), 90(2), 92(2), 93(4), 100, 104, 109, 114, 192*, 195(2), 204, 205, 212, 217, 262	20	$52/33$	$1/1$	$51/32$
167	512 18(3), 20(2), 22, 25(2), 33, 36, 83*, 85, 92, 100, 105(3), 111, 112, 114, 119, 120, 121, 122, 131, 132, 145(2), 146, 148, 149, 151, 155, 156(2), 157, 158(2), 166(2), 169(2), 180(2), 199, 200, 207, 210, 214, 223, 232, 237	21	$52/40$	$1/1$	$51/39$
168	512 20(2), 23(2), 32(6), 33(2), 35(2), 58, 59, 62(2), 66(5), 67(4), 68(3), 69(2), 70(2), 71, 72, 75(3), 76, 78(2), 79(2), 90, 91(2), 93, 94(2), 100, 109, 194, 195(2), 196, 204(2), 205(2), 212, 232(4), 247	19	$66/33$	$0/0$	$66/33$
169	512 5(2), 9(4), 59(5), 63, 68(2), 69(8), 70, 72(2), 74, 76, 77(2), 78(4), 79(4), 80, 81, 113(2), 132, 164(2), 165, 197(2), 212	20	$48/21$	$0/0$	$48/21$
175	512 4, 87	20	$2/2$	$0/0$	$2/2$
177	64 132	22	$1/1$	$0/0$	$1/1$
178	64 233	22	$1/1$	$0/0$	$1/1$
181	384 60(3), 267*	21	$4/2$	$1/1$	$3/1$
182	1536 20(2), 23(2), 32(4), 33(4), 34(2), 35(2), 60(3), 88, 90, 92, 99, 100, 193(2), 195, 196, 202(7), 203, 204, 205, 207, 209(2), 211, 216(4), 219(8), 227(8), 232(6), 241(6), 263, 267*	19	$76/29$	$1/1$	$75/28$
183	512 25(2), 86(2), 91(2), 99, 100(3), 102(2), 108(2), 119, 120, 121, 122, 129, 131, 132(2), 135, 137, 139(2), 142, 143, 145(2), 146, 148, 149, 151, 156, 158, 159, 160, 163, 166(3), 168, 169, 178, 182, 217(2), 251, 252, 254, 255	21	$52/39$	$0/0$	$52/39$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
184	256 96(2), 98, 99, 106, 109, 119(2), 120(2), 121(2), 122, 123, 129(3), 131(2), 132, 133(2), 142, 143, 144(2), 145(4), 146(2), 147(2), 151(4), 157, 158, 159(2), 160(4), 161(2), 162, 165(3), 166(2), 169(2), 170, 172, 176, 178, 180, 182, 252, 254	21	$64/38$	$0/0$	$64/38$
185	512 20(2), 23(2), 33, 35, 58, 59, 62(2), 66(3), 67(3), 68(3), 69(4), 70(2), 71(2), 72, 74, 75(5), 76, 77(2), 79(2), 80, 90, 91, 93, 94, 100, 109, 194, 195, 204, 205, 212, 247	19	$52/32$	$0/0$	$52/32$
186	512 18, 19, 83*, 85, 86(2), 88(2), 89(3), 91, 94, 97, 101, 102, 104(2), 105(2), 108, 110, 112(3), 114(3), 115(2), 116(4), 117(6), 126(2), 127(2), 135, 137, 138, 139, 206, 247, 248, 256	21	$52/31$	$1/1$	$51/30$
187	512 5(2), 9(4), 58, 59, 61, 62(2), 63, 66(4), 67(2), 69(2), 70(3), 72(2), 73, 75(5), 77, 78(3), 79(4), 113(2), 131, 164(3), 196, 204, 212	20	$48/23$	$0/0$	$48/23$
190	1536 4(2), 9(4), 62(3), 67(4), 73(2), 74(3), 82(2), 87(2), 131(2), 132(2), 263(2)	20	$28/11$	$0/0$	$28/11$
192	128 60(3), 267*	21	$4/2$	$1/1$	$3/1$
193	128 60(3), 267*	22	$4/2$	$1/1$	$3/1$
194	384 60(3), 267*	20	$4/2$	$1/1$	$3/1$
195	128 60(3), 267*	21	$4/2$	$1/1$	$3/1$
196	640 60(3), 267*	22	$4/2$	$1/1$	$3/1$
197	384 60, 267*	22	$2/2$	$1/1$	$1/1$
198	384 60(3), 267*	20	$4/2$	$1/1$	$3/1$
201	64 20	22	$1/1$	$0/0$	$1/1$
202	64 23	22	$1/1$	$0/0$	$1/1$
203	128 97, 108, 206, 247	22	$4/4$	$0/0$	$4/4$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
204	64 20	22	$1/1$	$0/0$	$1/1$
206	64 9	22	$1/1$	$0/0$	$1/1$
207	128 90(2)	22	$2/1$	$0/0$	$2/1$
208	64 109	22	$1/1$	$0/0$	$1/1$
209	64 233	22	$1/1$	$0/0$	$1/1$
211	30720 18(4), 20(4), 23(4), 25(4), 32(8), 33(8), 34(4), 35(4), 36(4), 37(4), 60, 62, 88(2), 90(2), 92, 93, 99(2), 100(2), 102(4), 192*, 193(2), 194, 195(9), 196(17), 197(5), 198(22), 199(17), 200(7), 201(22), 202(7), 203(17), 204(15), 205(17), 206(62), 207(7), 208(9), 209(10), 210(128), 211(2), 212(4), 213(32), 214(34), 215(29), 216(32), 217(36), 218(31), 219(132), 220(124), 221(66), 222(66), 223(116), 224(19), 225(31), 226(34), 227(122), 228(65), 229(46), 230(19), 231(19), 232(135), 233(122), 234(122), 235(65), 236(34), 237(65), 238(19), 239(11), 240(34), 241(47), 242(15), 243(63), 244(66), 245(5), 262(2), 263(4), 264(6), 265(2), 267*	18	$2316/78$	$2/2$	$2314/76$
212	2048 5, 9(2), 20, 23(4), 32, 58(3), 59, 60(10), 61(3), 62(3), 64, 66(6), 67(3), 68(5), 69(6), 70, 71(2), 74, 76, 77(4), 78(3), 79(2), 80, 81, 88, 90(8), 92, 109, 112, 132, 164, 193(2), 195(2), 197, 202(2), 205(3), 208, 267*	20	$92/38$	$1/1$	$91/37$
213	2048 4, 5, 9(4), 59, 62(2), 64, 67(3), 68(5), 69(6), 70, 73, 74(3), 76, 77, 78(3), 79(2), 80, 81, 82, 87, 112, 131, 132(2), 164, 197, 208, 263	20	$48/27$	$0/0$	$48/27$
214	1024 4, 9(2), 24(2), 55*, 57, 59, 66(2), 68, 69(2), 71(3), 72, 75(2), 77, 80, 87, 88, 89, 197, 207, 208(2), 212(2), 222, 247	20	$32/23$	$1/1$	$31/22$
221	768 60(7), 69(7), 77(3), 80(3), 209(3), 267*	19	$24/6$	$1/1$	$23/5$
222	256 35, 90, 114, 131, 145, 169, 178, 222	22	$8/8$	$0/0$	$8/8$

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
223	192 20	22	$1/1$	$0/0$	$1/1$
224	768 60(7), 69(7), 77(3), 80(3), 209(3), 267*	20	$24/6$	$1/1$	$23/5$
225	256 60(21), 69(21), 77(7), 80(7), 209(7), 267*	20	$64/6$	$1/1$	$63/5$
226	320 267*	21	$1/1$	$1/1$	$0/0$

 Table 2: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_4$

#	Order	2-Rank	Total	Abelian	Nonabelian
1	512 58, 59(2), 61(2), 72(4), 85, 175, 208, 212, 262	22	$14/9$	$0/0$	$14/9$
5	256 59, 61, 63, 65, 70(4), 72(2), 195, 197(2), 204(2), 212	22	$16/10$	$0/0$	$16/10$
6	512 58, 59, 61, 63, 65, 70(2), 72, 85, 168, 194, 197, 204, 212	22	$14/13$	$0/0$	$14/13$
8	512 55*, 58(2), 59(3), 61, 70, 104, 142, 192*, 195, 196(2)	22	$14/10$	$2/2$	$12/8$
9	7680 55*, 57, 58(2), 61, 104, 143, 197, 260*	22	$9/8$	$2/2$	$7/6$
10	128 59, 72, 197, 212	22	$4/4$	$0/0$	$4/4$
11	384 55*, 70	22	$2/2$	$1/1$	$1/1$
12	64 238	24	$1/1$	$0/0$	$1/1$
13	384 57, 212	22	$2/2$	$0/0$	$2/2$

Table 3: Structures of abelian regular subgroups in $\mathbb{Z}_2 \times \mathbb{Z}_2$ nets

#	Order	Abelian	Group structure
1	1105920	$6/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(4), \mathbb{Z}_2^6(2)$
2	13824	$2/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
5	18432	$6/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^6$
9	4608	$2/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
22	6144	$6/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^4 \times \mathbb{Z}_4$
31	1536	$1/1$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$
32	128	$1/1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
33	512	$1/1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
36	73728	$6/3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(4), \mathbb{Z}_2^4 \times \mathbb{Z}_4, \mathbb{Z}_2^6$
132	512	$1/1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
136	512	$1/1$	$\mathbb{Z}_4^3$
138	1536	$1/1$	$\mathbb{Z}_4^3$
166	1024	$1/1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
167	512	$1/1$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
186	512	$1/1$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
211	30720	$2/2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
214	1024	$1/1$	$\mathbb{Z}_4^3$

Table 4: Structures of abelian regular subgroups in $\mathbb{Z}_4$ nets.

#	Order	Abelian	Group structure
8	512	$2 / 2$	$\mathbb{Z}_4^3, \mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
9	7680	$2 / 2$	$\mathbb{Z}_4^3, \mathbb{Z}_2^4 \times \mathbb{Z}_4$
11	384	$1 / 1$	$\mathbb{Z}_4^3$

Table 5: All regular abelian subgroups of order 64 appearing in GDDs with bitranslation group $\mathbb{Z}_4$.

Group	Structure
55	$\mathbb{Z}_4^3$
192	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
260	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$

Table 6: All regular abelian subgroups of order 64 appearing in GDDs with bi-translation group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Group	Structure
55	$\mathbb{Z}_4^3$
83	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
192	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
246	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
260	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$
267	$\mathbb{Z}_2^6$

Table 7: Abelian groups which do not contain any $(16, 4, 16, 4)$ RDS

Group	Structure
1	$\mathbb{Z}_{64}$
2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
26	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
50	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$
183	$\mathbb{Z}_2^2 \times \mathbb{Z}_{16}$

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