Nonexistence of almost Moore digraphs of diameter four

J. Conde*, J. Gimbert*

Dept. de Matemàtica, Universitat de Lleida
Jaume II, 69, 25001 Lleida, Spain

{jconde, joangim}@matematica.udl.cat

J. González†

Dept. de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya
Víctor Balaguer s/n, 08800 Vilanova i Geltrú, Spain

josepg@ma4.upc.edu

J.M. Miret‡, R. Moreno‡

Dept. de Matemàtica, Universitat de Lleida
Jaume II, 69, 25001 Lleida, Spain

{miret, ramiro}@matematica.udl.cat

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Abstract

Regular digraphs of degree $d > 1$, diameter $k > 1$ and order $N(d,k) = d + \cdots + d^k$ will be called *almost Moore* (d,k)*-digraphs*. So far, the problem of their existence has only been solved when $d = 2, 3$ or $k = 2, 3$. In this paper we prove that almost Moore digraphs of diameter 4 do not exist for any degree d.

Keywords: Almost Moore digraph, characteristic polynomial, cyclotomic polynomial.

1 Introduction

The *degree/diameter problem* finds, given two natural numbers d and k, the largest possible number of vertices in a [directed] graph with maximum [out-]degree d and diameter k (for

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a survey of it see [12]). In the directed case, W.G. Bridges and S. Toueg in [4] proved that this number of vertices is less than the *Moore bound*, $M(d,k) = 1 + d + \cdots + d^k$, unless $d = 1$ or $k = 1$. Then, the question of finding for which values of $d > 1$ and $k > 1$ there exist digraphs of order

$$N(d,k) = M(d,k) - 1$$

becomes an interesting problem. In this case, any extremal digraph turns out to be d-regular (see [10]). From now on, regular digraphs of degree $d > 1$, diameter $k > 1$ and order $N(d,k)$ will be called *almost Moore* (d,k)*-digraphs* (or (d,k)*-digraphs* for short).

The problem of the existence of almost Moore (d,k)-digraphs has been solved when $d = 2,3$ or $k = 2,3$. M. Miller and I. Fris [11] proved that the $(2,k)$-digraphs do not exist for values of $k > 2$ and Baskoro et al. [3] established the nonexistence of $(3,k)$-digraphs unless $k = 2$. On the other hand, Fiol et al. [6] showed that the $(d,2)$-digraphs do exist for any degree. Their classification was completed by J. Gimbert in [8]. Moreover, J. Conde et al. [5] proved the nonexistence of $(d,3)$-digraphs.

In this paper we prove that almost Moore digraphs of diameter four do not exist for any degree. The paper is organized as follows: Section 2 is devoted to determine the characteristic polynomial of a $(d,4)$-digraph in terms of the polynomials $F_{n,4}(x) = \Phi_n(1+x+x^2+x^3+x^4)$, being $\Phi_n(x)$ the nth cyclotomic polynomial and $2 \leqslant n \leqslant N(d,4)$. In Section 3, assuming the cyclotomic conjecture (see [7]) for $k = 4$, which says that $F_{n,4}(x)$ is irreducible unless $n = 3,6$, we prove the nonexistence of $(d,4)$-digraphs for $d \geqslant 2$. Finally, in Section 4 we show the conjecture for $k = 4$.

2 On the characteristic polynomial of a $(d,4)$-digraph

Given a (d,k)-digraph G, its adjacency matrix A fulfills the equation

$$I + A + \cdots + A^k = J + P, \tag{1}$$

where J denotes the all-one matrix and $P = (p_{ij})$ is the $(0,1)$-matrix associated with a distinguished permutation r of the set of vertices $V(G) = \{1,\ldots,N\}$; that is to say, $p_{ij} = 1$ iff $r(i) = j$ (see [1]).

Notice that r has a *cycle structure* which corresponds to its unique decomposition in disjoint cycles. The number of permutation cycles of G of each length $n \leqslant N$ will be denoted by m_n and the vector $(m_1,\ldots,m_N)$ will be referred to as the *permutation cycle structure* of G.

The factorization of $\det(xI - (J+P))$ in $\mathbb{Q}[x]$ in terms of the cyclotomic polynomials $\Phi_i(x)$ is given by (see [2, 5])

$$\det(xI - (J+P)) = (x-(N+1))(x-1)^{m(1)-1}\prod_{n=2}^{N}\Phi_n(x)^{m(n)}, \tag{2}$$

where $m(n) = \sum_{n|i} m_i$ represents the total number of permutation cycles of order multiple of n.

From Equations (1) and (2), the problem of the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of G, $\phi(G,x) = \det(xI - A)$, was connected by J. Gimbert in [7] with the study of the irreducibility in $\mathbb{Q}[x]$ of the polynomials

$$F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k).$$

The idea is that, when such polynomials are irreducible, they appear as factors of the characteristic polynomial of G.

Proposition 1. *Let $(m_1, \dots, m_N)$ be the permutation cycle structure of a (d,k)-digraph G and $2 \leqslant n \leqslant N$. If $F_{n,k}(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$, then it is a factor of $\phi(G,x)$ and its multiplicity is $m(n)/k$.*

This result was proved in [7]. Moreover, it was proved that $F_{2,k}(x) = 2 + x + \cdots + x^k$ is irreducible in $\mathbb{Q}[x]$, for any positive integer k. On the other hand, it was shown that for each $n > 2$ there are infinitely many values of k for which $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$. More precisely,

Lemma 2. *Let $n > 2$ and $k > 1$ be integers. Then, the following statements hold.*

(i) If n is odd and $k \equiv -2 \pmod{2n}$, then $\Phi_{2n}(x)$ divides $F_{n,k}(x)$.

(ii) If $n \equiv 0 \pmod 4$ and $k \equiv -2 \pmod n$, then $\Phi_n(x)$ divides $F_{n,k}(x)$.

(iii) If $n \equiv 2 \pmod 4$ and $k \equiv -2 \pmod{\frac{n}{2}}$, then $\Phi_{\frac{n}{2}}(x)$ divides $F_{n,k}(x)$.

On the other hand, in [7] it was conjectured that $F_{n,k}(x)$ is irreducible in $\mathbb{Q}[x]$ if n and k do no satisfy any of the conditions of Lemma 2.

Conjecture 3. Let $n > 2$ and $k > 1$ be integers. One has that

(i) If k is even, then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $n \mid (k+2)$, in which case $F_{n,k}(x)$ has just two factors.

(ii) If k is odd, then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if n is even and $n \mid 2(k+2)$, in which case $F_{n,k}(x)$ has just two factors.

We will refer to this conjecture as the cyclotomic conjecture. The case $k = 2$ was proved by H.W. Lenstra Jr. and B. Poonen [9] and, recently, the authors proved the case $k = 3$ in [5].

The remainder of this section is devoted to finding the conditions in order to obtain a factorization of the characteristic polynomial of a $(d,4)$-digraph G in terms of $F_{n,4}(x)$. Thus, let G be a $(d,4)$-digraph of degree $d > 3$ and let $(m_1, \dots, m_N)$ be its permutation cycle structure, where $N = d + d^2 + d^3 + d^4$.

We will assume the cyclotomic conjecture is true for $k = 4$, that is $F_{n,4}(x)$ is irreducible in $\mathbb{Q}[x]$ except $n = 3, 6$, which will be proven in the last section. From now on, we will write $F_n(x)$ instead of $F_{n,4}(x)$.

Then, by applying Proposition 1 we have that

$$\prod_{\substack{2\leqslant n\leqslant N\\ n\neq 3,6}} (F_n(x))^{\frac{m(n)}{4}} \quad \text{is a factor of } \phi(G,x).$$

The remaining factors of $\phi(G,x)$ are derived as follows:

- Since G is d-regular and strongly connected, $\phi(G,x)$ has the linear factor $x-d$ with multiplicity 1;

- Taking into account that $x-1$ is a factor of $\det(xI-(J+P))$ with multiplicity $m(1)-1$ and since
$$F_1(x) = (x+1)(x^2+1)x,$$
we have that $x+1$, x^2+1 and x are factors of $\phi(G,x)$ with multiplicities a_1, a_2 and a_3, respectively, where $a_1+2a_2+a_3 = m(1)-1$;

- Since $\Phi_3(x) = x^2+x+1$ is a factor of $\det(xI-(J+P))$ with multiplicity $m(3)$ and taking into account the factorization of $F_3(x)$ in $\mathbb{Q}[x]$,
$$F_3(x) = (x^2-x+1)(x^6+3x^5+5x^4+6x^3+7x^2+6x+3),$$
we have that $\Phi_6(x) = x^2-x+1$ and $F_3(x)/\Phi_6(x)$ are factors of $\phi(G,x)$ with multiplicities b_1 and b_2, respectively, where $2b_1+6b_2 = 2m(6)$; that is, $b_1 = m(3)-3b_2$. Analogously, since the factorization of $F_6(x)$ in $\mathbb{Q}[x]$ is
$$F_6(x) = (x^2+x+1)(x^6+x^5+x^4+2x^3+x^2+1),$$
we have that $\Phi_3(x)$ and $F_6(x)/\Phi_3(x)$ are factors of $\phi(G,x)$ with multiplicities c_1 and c_2, respectively, where $c_1 = m(6)-3c_2$.

As a result, the characteristic polynomial of G is

$$\phi(G,x) = (x-d)(x+1)^{a_1}(x^2+1)^{a_2}x^{a_3}\Phi_6(x)^{b_1}(F_3(x)/\Phi_6(x))^{b_2} \tag{3}$$
$$\times\, \Phi_3(x)^{c_1}(F_6(x)/\Phi_3(x))^{c_2} \prod_{\substack{2\leqslant n\leqslant N\\ n\neq 3,6}} (F_n(x))^{\frac{m(n)}{4}}. \tag{4}$$

3 On the nonexistence of $(d,4)$-digraphs

In this section, we will derive the nonexistence of a $(d,4)$-digraph from the irreducibility of the polynomials $F_n(x)$ which appear in the factorization of its characteristic polynomial and from the behaviour of the first three powers of its adjacency matrix.

Theorem 4. *Assuming that the cyclotomic conjecture is true for $k=4$, there is no almost Moore digraph of diameter four.*

Proof. Let G be a $(d,4)$-digraph with adjacency matrix A. We compute the graph spectral invariants $\operatorname{Tr} A^\ell$ ($\ell = 1,2,3$) in terms of the sum of the ℓth powers of the roots of each factor of $\phi(G,x)$.

Given a monic polynomial of degree $n \geqslant 1$, $a(x) = x^n + \sum_{i=1}^n a_{n-i}x^{n-i}$, and given an integer $\ell \geqslant 1$, we define $S_\ell(a(x))$ to be the sum of the ℓth powers of all the roots of $a(x)$. Using Newton's formulas [14], which express $S_\ell(a(x))$ in terms of the coefficients of $a(x)$, we have

$$\begin{aligned} S_1(a(x)) &= -a_{n-1}, \\ S_2(a(x)) &= a_{n-1}^2 - 2a_{n-2}, \\ S_3(a(x)) &= -a_{n-1}^3 + 3a_{n-1}a_{n-2} - 3a_{n-3}. \end{aligned}$$

Since $S_\ell(a(x)b(x)) = S_\ell(a(x))S_\ell(b(x))$, for all pairs of polynomials, and taking into account that

$$F_n(x) = \Phi_n(1+x+x^2+x^3+x^4) = (1+x+x^2+x^3+x^4)^{\varphi(n)} + O(x^{4\varphi(n)-4}),$$

where $\varphi(n)$ stands for Euler's function, we obtain

$$S_\ell(F_n(x)) = \varphi(n)S_\ell(x^4+x^3+x^2+x+1) = -\varphi(n), \quad \ell = 1,2,3.$$

Besides, it can be easily checked that

	S_1	S_2	S_3
$x+1$	-1	1	-1
x^2+1	0	-2	0
$\Phi_6(x)$	1	-1	-2
$\Phi_3(x)$	-1	-1	2

Now, for each $\ell = 1,2,3$ we can express the trace of the ℓth power of the adjacency matrix A of G in terms of the sums S_ℓ of all factors of $\phi(G,x)$. Thus,

$$\begin{aligned} \operatorname{Tr} A &= d - a_1 + b_1 - 3b_2 - c_1 - c_2 - \frac{1}{4}T, \\ \operatorname{Tr} A^2 &= d^2 + a_1 - 2a_2 - b_1 - b_2 - c_1 - c_2 - \frac{1}{4}T, \\ \operatorname{Tr} A^3 &= d^3 - a_1 - 2b_1 + 2c_1 - 4c_2 - \frac{1}{4}T, \end{aligned}$$

where $T = \sum_{\substack{2\leqslant n\leqslant N \\ n\neq 3,6}} m(n)\varphi(n)$. From the identity $\sum_{n=1}^{N} m(n)\varphi(n) = N$ (see [7]),

$$T = N - m(1) - 2m(3) - 2m(6).$$

So, taking into account that $b_1 = m(3) - 3b_2$ and $c_1 = m(6) - 3c_2$,

$$\begin{aligned} \operatorname{Tr} A &= d - \frac{1}{4}N + \frac{1}{4}m(1) + \frac{3}{2}m(3) - \frac{1}{2}m(6) - a_1 - 6b_2 + 2c_2, \\ \operatorname{Tr} A^2 &= d^2 - \frac{1}{4}N + \frac{1}{4}m(1) - \frac{1}{2}m(3) - \frac{1}{2}m(6) + a_1 - 2a_2 + 2b_2 + 2c_2, \\ \operatorname{Tr} A^3 &= d^3 - \frac{1}{4}N + \frac{1}{4}m(1) - \frac{3}{2}m(3) + \frac{5}{2}m(6) - a_1 + 6b_2 - 10c_2. \end{aligned}$$

Since G has no cycles of length $\leqslant 3$, we know that $\operatorname{Tr} A^\ell = 0$ ($\ell = 1, 2, 3$). As a consequence,

$$\begin{array}{rcrcrcrcl} 4a_1 & & & + & 24b_2 & - & 8c_2 & = & 4d - N + m(1) + 6m(3) - 2m(6), \\ -4a_1 & + & 8a_2 & - & 8b_2 & - & 8c_2 & = & 4d^2 - N + m(1) - 2m(3) - 2m(6), \\ 4a_1 & & & - & 24b_2 & + & 40c_2 & = & 4d^3 - N + m(1) - 6m(3) + 10m(6). \end{array}$$

Applying Gauss reduction method to the previous linear system, it follows that

$$8a_2 + 16b_2 - 16c_2 = 4d^2 + 4d - 2N + 2m(1) + 4m(3) - 4m(6), \tag{5}$$
$$-48b_2 + 48c_2 = 4d^3 - 4d - 12m(3) + 12m(6). \tag{6}$$

Taking into account that $N = d^4 + d^3 + d^2 + d$, from (5) and (6) we derive that

$$24a_2 = 4d^3 + 12d^2 + 8d + 6m(1) - 6N.$$

Notice that $m(1) = \sum_{n=1}^{N} m_n$ takes its maximum value when all permutation cycles are short as possible. Moreover, the number of selfrepeats m_1 of a (d,k)-digraph is either 0 or k, if $k \geqslant 3$ (see [1]). So, $m(1) \leqslant 4 + \frac{N-4}{2}$ and, consequently,

$$24a_2 \leqslant 4d^3 + 12d^2 + 8d + 12 - 3N = -3d^4 + d^3 + 9d^2 + 5d + 12.$$

Hence, if $d > 3$ then $a_2 < 0$, which is impossible since a_2 is a nonnegative integer. $\square$

4 The cyclotomic conjecture for $k = 4$

This section is devoted to proving the cyclotomic conjecture in the case $k = 4$, that is, we show that the polynomial $F_n(x) = \Phi_n(1 + x + x^2 + x^3 + x^4)$ is irreducible in $\mathbb{Q}[x]$, when $n > 1$ and $n \neq 3, 6$.

As a first step, we show that the condition of being $F_n(x)$ reducible in $\mathbb{Q}[x]$ implies a divisibility relation by a cyclotomic polynomial. In order to prove this, let us suppose that $F_n(x)$ is reducible in $\mathbb{Q}[x]$ and let us consider a root ε of $F_n(x)$. Denoting

$$p_1(x, z) = 1 - z + x + x^2 + x^3 + x^4, \tag{7}$$

and taking a suitable primitive nth root of unity ζ_n, we get

$$p_1(\varepsilon, \zeta_n) = 0.$$

Using properties about the degrees of the algebraic extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(\varepsilon),$$

we derive that $F_n(x)$ has an irreducible factor in $\mathbb{Q}[x]$ of degree $\varphi(n)$ or $2\varphi(n)$. We can assume that ε is a root of such a factor. In particular, ε is an algebraic integer and $[\mathbb{Q}(\varepsilon) : \mathbb{Q}(\zeta_n)]$ is either 1 or 2.

If $[\mathbb{Q}(\varepsilon) : \mathbb{Q}(\zeta_n)] = 1$, we consider the element $\overline{\varepsilon}/\varepsilon \in \mathbb{Q}(\varepsilon, \overline{\varepsilon})$, where $^-$ denotes the complex conjugation. By using arguments given in [5] we obtain that $\overline{\varepsilon}/\varepsilon$ is a root of unity and hence the same procedure given for diameter 3 to state the irreducibility of $F_n(x)$ follows.

Now, assume that $[\mathbb{Q}(\varepsilon) : \mathbb{Q}(\zeta_n)] = 2$ for all ε such that $p_1(\varepsilon, \zeta_n) = 0$. We denote by ε' the conjugate root of ε over $\mathbb{Q}(\zeta_n)$, that is to say, the polynomial $p_1(x, \zeta_n)/((x-\varepsilon)(x-\varepsilon'))$ is irreducible in $\mathbb{Q}(\zeta_n)[x]$. Changing the root of $p_1(x, \zeta_n)$ if necessary, we can assume that $\varepsilon\varepsilon'$ is not real. Since ε is an algebraic integer and $1 - \zeta_n$ is a unity or a prime element of $\mathbb{Z}[\zeta_n]$, $\varepsilon\varepsilon'$ is also a unity or a prime element of $\mathbb{Z}[\zeta_n]$. Therefore,

$$\alpha = \frac{\overline{\varepsilon\varepsilon'}}{\varepsilon\varepsilon'} \in \mathbb{Z}[\zeta_n]$$

is a unity of $\mathbb{Z}[\zeta_n]$ whose conjugates have absolute value 1. Hence, $\alpha \neq 1$ is a root of unity of order $2n$ [15, Lemma 1.6]. Notice that if n is even, α is a root of unity of order n.

Now, we search for a polynomial relation between ζ_n and $\alpha = \beta\beta'$, where $\beta = \overline{\varepsilon}/\varepsilon$ and $\beta' = \overline{\varepsilon'}/\varepsilon'$. In order to find such an expression we give first a relation between ζ_n and β. We use the following identities:

$$\begin{aligned} 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^4 &= \zeta_n, \\ \overline{\varepsilon} &= \beta\varepsilon. \end{aligned}$$

From them, and taking into account that $\overline{\zeta_n} = 1/\zeta_n$, it can be seen that $p_2(\varepsilon, \beta, \zeta_n) = 0$ where

$$p_2(x, y, z) = 1 - z - xyz - x^2y^2z - x^3y^3z - x^4y^4z. \tag{8}$$

Similarly, $p_2(\varepsilon', \beta', \zeta_n) = 0$. Notice as well that $p_3(\alpha, \beta, \beta') = 0$ where

$$p_3(y, y', w) = w - yy'.$$

Therefore, the relation between ζ_n and α we are looking for is $R(\zeta_n, \alpha) = 0$, where

$$R_1(y, z) = \mathrm{Res}(p_1(x, z), p_2(x, y, z), x), \tag{9}$$
$$R_2(y', z, w) = \mathrm{Res}(R_1(y, z), p_3(y, y', w), y), \tag{10}$$
$$R(z, w) = \mathrm{Res}(R_1(y', z), R_2(y', z, w), y'). \tag{11}$$

This polynomial factorizes as follows

$$R(z, w) = (z - 1)^{50} q_1(z, w) q_2^2(z, w) q_3^2(z, w) q_4^4(z, w), \tag{12}$$

where $q_1(z, w)$ has degree 14 in z and 16 in w, $q_2(z, w)$ and $q_3(z, w)$ have degree 21 in z and 24 in w, and $q_4(z, w)$ has degree 27 in z and 36 in w.

Proposition 5. *Let $n > 2$ be an integer and $F_n(x) = \Phi_n(1 + x + x^2 + x^3 + x^4)$. If $F_n(x)$ is reducible in $\mathbb{Q}[x]$ then:*

– *If n is even, then there exists an integer k, $1 \leqslant k < n$, such that $\Phi_n(x)$ divides one of the polynomials $q_i(x, x^k)$, $i \in \{1, 2, 3, 4\}$, given in* (12).

– *If n is odd, then there exists an integer k, $1 \leqslant k < n$, such that $\Phi_n(x)$ divides one of the polynomials $q_i(x, x^k)$ or $q_i(x, -x^k)$, $i \in \{1, 2, 3, 4\}$, given in* (12).

Proof. Since the cyclotomic polynomial $\Phi_n(x)$ is irreducible in $\mathbb{Q}[x]$ and it does not divide $x - 1$, then when n is even it must divide at least one of the polynomials $q_i(x, x^k)$, $i \in \{1, 2, 3, 4\}$, $1 \leqslant k < n$. When n is odd, α or $-\alpha$ is a root of unity of order n. Hence, $\Phi_n(x)$ must divide $q_i(x, x^k)$ or $q_i(x, -x^k)$, $i \in \{1, 2, 3, 4\}$. □

Our main goal is to show that $F_n(x)$ is irreducible in $\mathbb{Q}[x]$, for $n > 1$ and $n \neq 3, 6$. It is enough to prove that $\Phi_n(x)$ does not divide, for $i \in \{1, 2, 3, 4\}$, any of the polynomials $q_i(x, x^k)$, $1 \leqslant k < n$, when n is even and it does not divide any of the polynomials $q_i(x, x^k)$ or $q_i(x, -x^k)$, $1 \leqslant k < n$, when n is odd. This is equivalent to proving that $\Phi_{2n}(x)$ does not divide any of the polynomials $q_i(x^2, x^\ell)$, $1 \leqslant \ell < 2n$.

Theorem 6. *The polynomial $F_n(x)$ is irreducible in $\mathbb{Q}[x]$ for $n > 1$, unless $n = 3, 6$.*

Proof. If $F_n(x)$ is reducible, then taking into account Proposition 5 there exist polynomials $q_i(x^2, x^\ell)$, $i \in \{1, 2, 3, 4\}$, given by (12) such that the cyclotomic polynomial $\Phi_{2n}(x)$ divides one of them. Now, we show that $\Phi_{2n}(x)$ does not divide $q_1(x^2, x^\ell)$. To see this, from part (i) of Lemma 3 in [5] (see also [13]), we know that

$$\Phi_{2n}(x) \equiv \Phi_r(x)^{\varphi(p^e)} \pmod{p\mathbb{Z}[x]},$$

where p is a prime number dividing $2n$ with $2n = p^e r$ and $(p, r) = 1$. Consequently

$$\Phi_r(x)^{\varphi(p^e)-1} \mid \gcd\left(q_1(x^2, x^\ell),\, xq_1'(x^2, x^\ell)\right) \pmod{p\mathbb{Z}[x]}.$$

Now, we consider the polynomial

$$A_1(z, w) = 2z\frac{\partial}{\partial z}q_1(z, w) + \ell w\frac{\partial}{\partial w}q_1(z, w) \in \mathbb{Z}[z, w],$$

that is $A_1(x^2, x^\ell) = xq_1'(x^2, x^\ell)$. Therefore

$$\Phi_r(x)^{\varphi(p^e)-1} \mid P_1(x) \pmod{p\mathbb{Z}[x]}, \tag{13}$$

where $P_1(x)$ is the following resultant

$$P_1(x) = \operatorname{Res}\left(q_1(x^2, w), A_1(x^2, w), w\right).$$

It can be checked that

$$P_1(x) = 5^4 x^{264}\Phi_1^{82}(x)\Phi_2^{82}(x)\Phi_4^{12}(x)\Phi_3^{6}(x)\Phi_6^{6}(x)\Phi_{12}^{6}(x)P_{1,0}^2(x)P_{1,\ell}(x), \tag{14}$$

with $P_{1,0}(x)$ a polynomial of degree 36 and $P_{1,\ell}(x)$ a polynomial of degree at most 60.

Notice that for those integers n which have a prime factor p such that $P_1(x) \neq 0 \pmod{p\mathbb{Z}[x]}$ for all $\ell \pmod{p}$, the degree of $P_1(x) \pmod{p\mathbb{Z}[x]}$ provides us an upper bound K for $\varphi(n)$. Hence, for those values of n such that $\varphi(n) > K$, $F_n(x)$ is irreducible in $\mathbb{Q}[x]$, and for those n with $\varphi(n) \leqslant K$, we can computationally check the irreducibility of $F_n(x)$ unless $n = 3, 6$.

The coefficients of $P_{1,0}(x)$ do not depend on ℓ and its gcd is one. Hence, this polynomial does not vanish for any prime p. The polynomial $P_{1,\ell}(x)$ is given by

$$P_{1,\ell}(x) = \sum_{i=0}^{30} a_i(\ell)x^{2i},$$

where the coefficients $a_i(\ell)$ are polynomials on $\mathbb{Q}[\ell]$ of degree 16 given by the expressions

$$\begin{aligned}
a_0(\ell) &= \quad 2^{32}5^{12}(\ell+1)^{16},\\
a_1(\ell) &= -2^{26}5^{11}(\ell+1)^{12}(9353\ell^4 + 37412\ell^3 + 57248\ell^2 + 39552\ell + 10368),\\
a_2(\ell) &= \quad 2^{17}5^{10}(\ell+1)^{8}(338813683\ell^8 + 2710509464\ell^7 + 9562778864\ell^6\\
&\qquad + 19424004608\ell^5 + 24833262080\ell^4 + 20453500928\ell^3 + 10593286144\ell^2\\
&\qquad + 3152707584\ell + 412581888),\\
&\ \ \vdots\\
a_{29}(\ell) &= -2^{26}5^{11}(\ell+1)^{12}(9353\ell^4 + 37412\ell^3 + 57248\ell^2 + 39552\ell + 10368),\\
a_{30}(\ell) &= \quad 2^{32}5^{12}(\ell+1)^{16}.
\end{aligned}$$

From the first coefficient it turns out that the factors which can vanish $P_{1,\ell}(x)$ are 2, 5 and those that divide $\ell + 1$. The polynomials $a_j(\ell)$, $j = 4, \ldots, 26$, are not divisible by $\ell + 1$. The greatest common divisor of the remaining divisions of these polynomials by $\ell + 1$ in $\mathbb{Z}[x]$ is 1. Thus, there are no primes dividing $\ell + 1$ that vanish $P_{1,\ell}(x)$. For the prime $p = 2$, the polynomial $P_{1,\ell}(x)$ only vanishes when ℓ is even. Concerning the prime $p = 5$, the polynomial $P_{1,\ell}(x)$ only vanishes when $\ell \equiv 4 \pmod{5}$.

Now, if the factorization of n has a prime factor p different from 2 and 5, by using (13) and taking into account the factorization of $P_1(x) \pmod{p\mathbb{Z}[x]}$ given in (14), the degree of the maximum power $\Phi_r(x)$ that could divide $P_1(x) \pmod{p\mathbb{Z}[x]}$ is bounded by $\deg P_1(x) - \deg x^{264} = 368$. This is a bound for $(\varphi(p^e) - 1)\varphi(r)$. Hence,

$$\varphi(n) \leqslant \varphi(2n) = \varphi(p^e)\varphi(r) \leqslant 368 + \varphi(r) \leqslant 736.$$

For these integers n which have a prime factor different from 2 and 5 and such that $\varphi(n) > 736$, $F_n(x)$ is irreducible in $\mathbb{Q}[x]$. For those integers n such that $\varphi(n) \leqslant 736$, it has been computationally checked that $F_n(x)$ is reducible in $\mathbb{Q}[x]$ only when $n = 3$ and $n = 6$. Therefore, the remaining cases to consider are $n = 2^e5^d$, with $e \geqslant 1$ or $d \geqslant 1$.

The previous method works as well taking $p = 2$ in (13) when ℓ is odd. On the other hand, if $5 \mid n$ and $p = 5$, then $P_1(x) = 0 \pmod{p\mathbb{Z}[x]}$ but the following relation holds

$$\Phi_r(x)^{\varphi(p^e)-2} \mid Q_1(x) \pmod{p\mathbb{Z}[x]}, \tag{15}$$

where $Q_1(x)$ is the resultant

$$Q_1(x) = \mathrm{Res}\left(q_1(x^2, w), B_1(x^2, w), w\right), \tag{16}$$

being

$$B_1(z, w) = 2z\frac{\partial}{\partial z}A_1(z, w) + kw\frac{\partial}{\partial w}A_1(z, w).$$

Since we must consider the cases $n = 2^e5^d$, we can apply (15) with $p = 5$ and we proceed in the same way as in (13). Nevertheless, the polynomial $Q_1(x) \pmod{5\mathbb{Z}[x]}$ is identically zero only for $\ell \equiv 4 \pmod 5$. Thus, taking into account these remarks, the cases we must study have been reduced to the following:

i) $n = 2^e5^d$, with $e \geqslant 0$, $d > 0$, ℓ even and $\ell \equiv 4 \pmod 5$,

ii) $n = 2^e$, with $e \geqslant 1$, and ℓ even.

i) We shall prove that $\Phi_{2n}(x) \pmod{5\mathbb{Z}[x]}$ does not divide $q_1(x^2, x^\ell) \pmod{5\mathbb{Z}[x]}$, for ℓ even and $\ell \equiv 4 \pmod 5$. It is known that $\Phi_{2n}(x) = \Phi_{2^{e+1}}(x)^{4\cdot 5^{d-1}} \pmod{5\mathbb{Z}[x]}$, where

$$\Phi_{2^m}(x) \pmod{5\mathbb{Z}[x]} = \begin{cases} x+4 & \text{if } m = 0, \\ x+1 & \text{if } m = 1, \\ (x^{2^{m-2}}+2)(x^{2^{m-2}}+3) & \text{if } m \geqslant 2. \end{cases}$$

We have that

$$q_1(z, w) = q_{1,1}(z, w)^2 q_{1,2}(z, w) q_{1,3}(z, w) q_{1,4}(z, w) \pmod{5\mathbb{Z}[z, w]},$$

where

$$\begin{aligned}
q_{1,1}(z, w) = {} & w^2z - 1, \\
q_{1,2}(z, w) = {} & w^4z^4 - 2w^4z^3 + w^4z^2 + w^3z^2 - 2w^2z^3 + w^2z^2 - 2w^2z + wz^2 + z^2 - 2z + 1, \\
q_{1,3}(z, w) = {} & w^4z^4 - 2w^4z^3 + w^4z^2 - 2w^3z^3 - 2w^3z^2 - 2w^2z^3 + 2w^2z^2 - 2w^2z - 2wz^2 \\
& -2wz + z^2 - 2z + 1, \\
q_{1,4}(z, w) = {} & w^4z^4 - 2w^4z^3 + w^4z^2 - w^3z^3 - 2w^2z^3 + w^2z^2 - 2w^2z - wz + z^2 - 2z + 1.
\end{aligned}$$

So, we will prove that $\Phi_{2^{e+1}5^d}(x) \pmod{5\mathbb{Z}[x]}$ does not divide $q_{1,i}(x^2, x^\ell) \pmod{5\mathbb{Z}[x]}$, for any $i \in \{1, 2, 3, 4\}$, when $e > 0$ and $e = 0$.

- *Case* $e > 0$. First, we claim that

$$\gcd\left(\Phi_{2^{e+1}}(x) \pmod{5\mathbb{Z}[x]},\ q_{1,1}(x^2, x^\ell) \pmod{5\mathbb{Z}[x]}\right) = 1.$$

Indeed, let γ be a root of $\Phi_{2^{e+1}}(x) \pmod{5\mathbb{Z}[x]}$, that is $\gamma^{2^{e-1}}$ is equal to 2 or 3. Then, $\gamma^{2^{e+1}}$ is the smallest power of γ equal to 1. Therefore, if γ is a root of $q_{1,1}(x^2, x^\ell) = x^{2(\ell+1)} - 1$ then $2^{e+1} \mid 2(\ell+1)$, which contradicts that ℓ is even.

Assume $\Phi_{2^{e+1}}(x) \pmod{5\mathbb{Z}[x]}$ divides $q_{1,2}(x^2, x^\ell)q_{1,3}(x^2, x^\ell)q_{1,4}(x^2, x^\ell)$. Then each irreducible divisor of $\Phi_{2^{e+1}}(x) \pmod{5\mathbb{Z}[x]}$ is a divisor of some of the polynomials $q_{1,i}(x^2, x^\ell)$

(mod $5\mathbb{Z}[x]$), $i \in \{2,3,4\}$, with multiplicity greater than 1. Then, for $i \in \{2,3,4\}$ we consider the resultant

$$T_{1,i}(x) = \operatorname{Res}(q_{1,i}(x^2,w), S_{1,i}(x^2,w), w),$$

where

$$S_{1,i}(z,w) = 2z\frac{\partial}{\partial z}q_{1,i}(z,w) + \ell w\frac{\partial}{\partial w}q_{1,i}(z,w).$$

When $\ell = 4 \pmod 5$, the polynomials $T_{1,i}(x) \pmod{5\mathbb{Z}[x]}$ are as follows:

$$\begin{aligned}
T_{1,2}(x) = &\ x^{20}(1+x)^6(2+x)^2(3+x)^2(4+x)^6(1+x+x^2)^2(1+4x+x^2)^2,\\
T_{1,3}(x) = &\ x^{20}(1+x)^4(4+x)^4(4+2x+x^2)^4(4+3x+x^2)^4,\\
T_{1,4}(x) = &\ x^{20}(1+x)^6(2+x)^2(3+x)^2(4+x)^6(1+x+x^2)^2(1+4x+x^2)^2.
\end{aligned}$$

Therefore, e must be 1 and $\Phi_4(x)^7 \pmod{5\mathbb{Z}[x]}$ is the greatest power of $\Phi_4(x) \pmod{5\mathbb{Z}[x]}$ which could divide $q_{1,2}(x^2,x^\ell)q_{1,3}(x^2,x^\ell)q_{1,4}(x^2,x^\ell)$. Since

$$\Phi_{2^{e+1}5^d}(x) = \Phi_{2^{e+1}}(x)^{4\cdot 5^{d-1}} \pmod{5\mathbb{Z}[x]},$$

for $d > 1$ the polynomial $\Phi_{2^{e+1}5^d}(x) \pmod{5\mathbb{Z}[x]}$ does not divide $q_1(x^2,x^\ell) \pmod{5\mathbb{Z}[x]}$. For $n = 2\cdot 5$ we can check that $F_n(x)$ is irreducible in $\mathbb{Q}[x]$.

• *Case* $e = 0$. In this case $\Phi_{2\cdot 5^d}(x) = (x+1)^{4\cdot 5^{d-1}} \pmod{5\mathbb{Z}[x]}$. Set $\ell + 1 = 5^k m$ with m odd and $\gcd(5,m) = 1$. Since $\ell + 1 = 0 \pmod 5$ and $\ell + 1 \leqslant 2\cdot 5^d$, it is clear that $1 \leqslant k \leqslant d$. The polynomial $(x+1)^{2\cdot 5^k}$ is the greatest power of $x+1$ which divides $q_{1,1}(x^2,x^\ell)^2 = (x^{2(\ell+1)}-1)^2 = (x^m-1)^{2\cdot 5^k}(x^m+1)^{2\cdot 5^k} \pmod{5\mathbb{Z}[x]}$. From the following equalities

$$\operatorname{Res}(q_{1,1}(x^2,w), q_{1,i}(x^2,w), w) = 4x^{10}(x+1)^2(4+x)^2, \quad 2 \leqslant i \leqslant 4,$$

we get that $(x+1)^{2\cdot 5^k+6}$ is the greatest power of $x+1$ dividing $q_1(x^2,x^\ell)$. Hence, $4\cdot 5^{d-1} \leqslant 2\cdot 5^k + 6$ and, thus, $k = d$. So, $\ell + 1$ must be either 5^d or $2\cdot 5^d$. Since ℓ is even, $\ell = 5^d - 1$. Therefore, only for this value of ℓ the polynomial $\Phi_{2\cdot 5^d}(x)$ can divide $q_1(x^2,x^\ell)$. Nevertheless, since the roots of $\Phi_{2\cdot 5^d}(x)$ satisfy that $x^{5^d} = -1$, the polynomial $\Phi_{2\cdot 5^d}(x)$ should divide

$$q_1(x^2,-1/x) = 25(-1+x)^4(1+x)^6(1-x+x^2),$$

which leads to a contradiction.

ii) In this case $n = 2^e$, with $e \geqslant 1$ and $\ell = 2k$. We shall prove that $\Phi_{2^e}(x) \pmod{5\mathbb{Z}[x]}$ does not divide $q_1(x,x^k) \pmod{5\mathbb{Z}[x]}$. With the same arguments used in the above case, we obtain that

$$\gcd\big(\Phi_{2^e}(x) \pmod{5\mathbb{Z}[x]},\ q_{1,1}(x,x^k) \pmod{5\mathbb{Z}[x]}\big) = 1.$$

Let $\gamma \in \mathbb{F}_{5^{2^{e-2}}}$ such that $\Phi_{2^e}(\gamma) = 0$, where $\mathbb{F}_{5^{2^{e-2}}}$ is the finite field with 5^{e-2} elements. Since $\Phi_{2^e}(x) = (x^{2^{e-2}} + 2)(x^{2^{e-2}} + 3)$ is the decomposition in irreducible factors in $\mathbb{F}_5$, we know that $\mathbb{F}_{5^{2^{e-2}}} = \mathbb{F}_5(\gamma)$ and $\gamma^{2^{e-2}} = a$, where a is either 2 or 3. Moreover,

$$\mathrm{Tr}(\gamma^m) = \begin{cases} \varphi(2^e)/2 & \text{if } \gcd(m, 2^e) = 2^e, \\ -\varphi(2^e)/2 & \text{if } \gcd(m, 2^e) = 2^{e-1}, \\ \pm a\varphi(2^e)/2 & \text{if } \gcd(m, 2^e) = 2^{e-2}, \\ 0 & \text{otherwise}, \end{cases}$$

where Tr denotes the trace $\mathrm{Tr}_{\mathbb{F}_{5^{2^{e-2}}}/\mathbb{F}_5}$ and the sign of a depends on the class $\dfrac{m}{2^{e-2}} \pmod 4$. We can assume that $e > 5$ and, thus, when $\gcd(m, 2^e) \mid 8$ we have $\mathrm{Tr}(\gamma^m) = 0$. If

$$q_{1,4}(\gamma, \gamma^\ell) = 1 + \sum_{i>0} a_i \gamma^i = 0,$$

taking traces we obtain $\mathrm{Tr}(1) = \varphi(2^e)/2 = 0 \pmod 5$ which is impossible.

If $q_{1,2}(\gamma, \gamma^\ell) = 0$, taking traces we obtain

$$\mathrm{Tr}(1) + \mathrm{Tr}(\gamma^{2+\ell}) + \mathrm{Tr}(\gamma^{2+3\ell}) = 0 \pmod 5. \tag{17}$$

Notice that $\mathrm{Tr}(\gamma^{2+\ell})\mathrm{Tr}(\gamma^{2+3\ell}) = 0 \pmod 5$. From (17), we get that either $\gcd(2^e, 2+\ell) = 2^{e-1}$ or $\gcd(2^e, 2+3\ell) = 2^{e-1}$. In the first case, $2 + \ell = 2^{e-1}$ and $\gamma^{2-\ell} = -1$. In the second case, $2 + 3\ell = 2^{e-1}$ and $\gamma^{2+3\ell} = -1$. Since $\Phi_1(x)$ is the unique cyclotomic polynomial dividing

$$\mathrm{Res}(q_1(x, w), x^2 w + 1, w) \cdot \mathrm{Res}(q_1(x, w), x^2 w^3 + 1, w),$$

it follows that $q_{1,2}(\gamma, \gamma^\ell) \neq 0$.

If $q_{1,3}(\gamma, \gamma^\ell) = 0$, taking traces we obtain

$$\mathrm{Tr}(1) - 2\mathrm{Tr}(\gamma^{2+\ell}) - 2\mathrm{Tr}(\gamma^{2+3\ell}) = 0 \pmod 5. \tag{18}$$

As above $\mathrm{Tr}(\gamma^{2+\ell})\mathrm{Tr}(\gamma^{2+3\ell}) = 0 \pmod 5$. Since $\mathrm{Tr}(\gamma^{2+h\ell}) = \mathrm{Tr}(1)/2 \pmod 5$, $h \in \{1, 2\}$, is not possible, neither is the equality (18).

Consequently, $\Phi_n(x)$ does not divide $q_{1,i}(x, x^\ell) \pmod{5\mathbb{Z}[x]}$, $i \in \{1, 2, 3, 4\}$, and thus $\Phi_n(x)$ does not divide $q_1(x, x^\ell) \pmod{5\mathbb{Z}[x]}$.

The non divisibility with respect to the other factors $q_i(x^2, x^\ell)$, $i \in \{2, 3, 4\}$, can be proved in a similar way. Indeed, for $2 \leqslant i \leqslant 4$, let $P_i(x)$ be the polynomials in $\mathbb{Z}[x]$ obtained as in (14) but from the polynomial $q_i(z, w)$ instead of $q_1(z, w)$. Let us consider

$$U_i(x) = \mathrm{Res}\left(q_i(x, w),\ x\frac{\partial}{\partial x}q_i(x, w) + kw\frac{\partial}{\partial w}q_i(x, w),\ w\right).$$

Concerning $q_2(x^2, x^\ell)$ and $q_3(x^2, x^\ell)$, the polynomials $P_i(x)$ are non identically zero modulo $p\mathbb{Z}[x]$, except for $p = 2$ with ℓ even. Therefore, if n has a factor $p \neq 2$, using

(13), it turns out that $\Phi_{2n}(x) \nmid q_2(x^2, x^\ell)$ and $\Phi_{2n}(x) \nmid q_3(x^2, x^\ell)$ When $n = 2^e$, since the polynomials $U_2(x)$ and $U_3(x)$ satisfy $U_i(x) \not\equiv 0 \pmod{2\mathbb{Z}[x]}$, it turns out that $\Phi_n(x) \nmid q_2(x, x^k)$ and $\Phi_n(x) \nmid q_3(x, x^k)$.

Regarding $q_4(x^2, x^\ell)$, the polynomial $P_4(x)$ is non identically zero modulo $p\mathbb{Z}[x]$, except for $p = 2$ and ℓ even or $p = 5$. Moreover, $U_4(x) \not\equiv 0 \pmod{2\mathbb{Z}[x]}$. So, we have only to consider the case $n = 5^d$, $d \geqslant 1$. In such a case, we can derive that the corresponding polynomial $Q_4(x)$ obtained as in (16) from $q_4(z, w)$ is not identically zero $\pmod{5\mathbb{Z}[x]}$, unless $\ell \equiv 4 \pmod 5$. On the other hand, we have that

$$q_4(z, w) = \prod_{i=1}^{10} q_{4,i}(z, w) \pmod{5\mathbb{Z}[z, w]},$$

where

$$\begin{aligned}
q_{4,1}(z, w) &= w^2 z - 1,\\
q_{4,2}(z, w) &= (w^2 z + 1)^2,\\
q_{4,3}(z, w) &= w^2 z^2 - w^2 z - wz - z + 1,\\
q_{4,4}(z, w) &= w^4 z^3 - w^4 z^2 + w^3 z^2 + 2w^2 z^2 + 2w^2 z - 2wz + z - 1,\\
q_{4,5}(z, w) &= w^4 z^3 - w^4 z^2 + 2w^3 z^2 + 2w^2 z^2 + 2w^2 z - wz + z - 1,\\
q_{4,6}(z, w) &= w^4 z^3 - w^4 z^2 - w^3 z^2 + 2w^2 z^2 - 2w^2 z + wz + z - 1,\\
q_{4,7}(z, w) &= w^4 z^3 - w^4 z^2 + w^3 z^2 + 2w^2 z^2 - 2w^2 z - wz + z - 1\\
q_{4,8}(z, w) &= w^4 z^3 - w^4 z^2 + w^3 z^2 - 2w^2 z^2 - 2w^2 z - 2wz + z - 1,\\
q_{4,9}(z, w) &= w^4 z^3 - w^4 z^2 + 2w^3 z^2 - 2w^2 z^2 - 2w^2 z - wz + z - 1,\\
q_{4,10}(z, w) &= w^4 z^4 - 2w^4 z^3 + w^4 z^2 + w^3 z^3 - w^3 z^2 + 2w^2 z^3 + 2w^2 z - wz^2 + wz + z^2 - 2z + 1.
\end{aligned}$$

Now, by using a similar argument as the one given for $q_1(z, w)$ and $n = 5^d$ we obtain that $\ell + 1 = 5^d$, which leads us to a contradiction, since the polynomial

$$q_4(x^2, -1/x) = 5(-1 + x)^5 x^2 (1 + x)^5 (9 + 46x^2 + 9x^4)$$

is never a multiple of $\Phi_{5^d}(x)$. □

As we have shown in Theorem 6 the cyclotomic conjecture for $k = 4$, we can apply Theorem 4 to prove the nonexistence of almost Moore digraph of diameter $k = 4$.

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