Products and sums divisible by central binomial coefficients

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Abstract

In this paper we study products and sums divisible by central binomial coefficients. We show that

$$2(2n+1)\binom{2n}{n} \,\Big|\, \binom{6n}{3n}\binom{3n}{n} \quad \text{for all } n=1,2,3,\ldots.$$

Also, for any nonnegative integers k and n we have

$$\binom{2k}{k} \,\Big|\, \binom{4n+2k+2}{2n+k+1}\binom{2n+k+1}{2k}\binom{2n-k+1}{n}$$

and

$$\binom{2k}{k} \,\Big|\, (2n+1)\binom{2n}{n}C_{n+k}\binom{n+k+1}{2k},$$

where C_m denotes the Catalan number $\frac{1}{m+1}\binom{2m}{m}=\binom{2m}{m}-\binom{2m}{m+1}$. On the basis of these results, we obtain certain sums divisible by central binomial coefficients.

Keywords: central binomial coefficients; divisibility; congruences

1 Introduction

Central binomial coefficients are given by $\binom{2n}{n}$ with $n \in \mathbb{N} = \{0,1,2,\ldots\}$. The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \quad (n=0,1,2,\ldots)$$

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play important roles in combinatorics (cf. R. P. Stanley [13, pp. 219-229]). There are many sophisticated congruences involving central binomial coefficients and Catalan numbers (see, e.g., [15, 18, 19]).

In 1998 N. J. Calkin [5] proved that $\binom{2n}{n} \mid \sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^m$ for any $m, n \in \mathbb{Z}^+$. See also V.J.W. Guo, F. Jouhet and J. Zeng [9], and H. Q. Cao and H. Pan [6] for further extensions of Calkin's result.

In this paper we investigate a new kind of divisibility problems involving central binomial coefficients.

Our first theorem is as follows.

Theorem 1. (i) *For any positive integer n we have*

$$2(2n+1)\binom{2n}{n} \,\Big|\, \binom{6n}{3n}\binom{3n}{n}. \tag{1}$$

(ii) *Let k and n be nonnegative integers. Then*

$$\binom{2k}{k} \,\Big|\, \binom{4n+2k+2}{2n+k+1}\binom{2n+k+1}{2k}\binom{2n-k+1}{n} \tag{2}$$

and

$$\binom{2k}{k} \,\Big|\, (2n+1)\binom{2n}{n}C_{n+k}\binom{n+k+1}{2k}. \tag{3}$$

In view of (1) it is worth introducing the sequence

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}} \quad (n = 1, 2, 3, \ldots).$$

Here we list the values of $S_1, \ldots, S_8$:

$$5,\ 231,\ 14568,\ 1062347,\ 84021990,$$
$$7012604550,\ 607892634420,\ 54200780036595.$$

The author generated this sequence as A176898 at N.J.A Sloane's OEIS (cf. [16]). By Stirling's formula, $S_n \sim 108^n/(8n\sqrt{n\pi})$ as $n \to +\infty$. Set $S_0 = 1/2$. Using `Mathematica` we find that

$$\sum_{k=0}^{\infty} S_k x^k = \frac{\sin(\frac{2}{3}\arcsin(6\sqrt{3x}))}{8\sqrt{3x}} \quad \left(0 < x < \frac{1}{108}\right)$$

and in particular

$$\sum_{k=0}^{\infty} \frac{S_k}{108^k} = \frac{3\sqrt{3}}{8}.$$

`Mathematica` also yields that

$$\sum_{k=0}^{\infty} \frac{S_k}{(2k+3)108^k} = \frac{27\sqrt{3}}{256}.$$

It would be interesting to find a combinatorial interpretation or recursion for the sequence $\{S_n\}_{n\geqslant 1}$.

One can easily show that $S_p \equiv 15 - 30p + 60p^2 \pmod{p^3}$ for any odd prime p. Below we present a conjecture concerning congruence properties of the sequence $\{S_n\}_{n\geqslant 1}$.

Conjecture 2. (i) Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then S_n is odd if and only if n is a power of two. Also, $3S_n \equiv 0 \pmod{2n+3}$.

(ii) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{S_k}{108^k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -1 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

Remark 3. Part (i) of Conjecture 2 might be shown by our method for proving Theorem 1(i).

Our following conjecture is concerned with a companion sequence of $\{S_n\}_{n\geqslant 0}$.

Conjecture 4. There are positive integers $T_1, T_2, T_3, \ldots$ such that

$$\sum_{k=0}^{\infty} S_k x^{2k+1} + \frac{1}{24} - \sum_{k=1}^{\infty} T_k x^{2k} = \frac{\cos(\frac{2}{3}\arccos(6\sqrt{3}x))}{12}$$

for all real x with $|x| \leqslant 1/(6\sqrt{3})$. Also, $T_p \equiv -2 \pmod{p}$ for any prime p.

Here we list the values of $T_1, \ldots, T_8$:

$$1,\ 32,\ 1792,\ 122880,\ 9371648,$$
$$763363328,\ 65028489216,\ 5722507051008.$$

In 1914 Ramanujan [12] obtained that

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}$$

and

$$\sum_{k=0}^{\infty} (20k+3) \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10})^k} = \frac{8}{\pi}.$$

(See also [2, 3, 4] for such series.) Actually the first identity was originally proved by G. Bauer in 1859. Both identities can be proved via the WZ (Wilf-Zeilberger) method (cf. M. Petkovšek, H. S. Wilf and D. Zeilberger [11], and Zeilberger [21] for this method). For WZ proofs of the two identities, see S. B. Ekhad and D. Zeilberger [7] and Guillera [8]. van Hamme [20] conjectured that the first identity has a p-adic analogue. This conjecture was first proved by E. Mortenson [10], and recently re-proved in [22] via the WZ method.

On the basis of Theorem 1, we deduce the following new result.

Theorem 5. *For any positive integer n we have*

$$4(2n+1)\binom{2n}{n} \ \Big| \ \sum_{k=0}^{n}(4k+1)\binom{2k}{k}^3(-64)^{n-k} \tag{4}$$

and

$$4(2n+1)\binom{2n}{n} \ \Big| \ \sum_{k=0}^{n}(20k+3)\binom{2k}{k}^2\binom{4k}{2k}(-2^{10})^{n-k}. \tag{5}$$

Now we pose two more conjectures.

Conjecture 6. (i) For any $n \in \mathbb{Z}^+$ we have

$$a_n := \frac{1}{8n^2\binom{2n}{n}^2}\sum_{k=0}^{n-1}(205k^2+160k+32)(-1)^{n-1-k}\binom{2k}{k}^5 \in \mathbb{Z}^+.$$

(ii) Let p be an odd prime. If $p \neq 3$ then

$$\sum_{k=0}^{(p-1)/2}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2 + \frac{896}{3}p^5 B_{p-3} \pmod{p^6},$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. If $p \neq 5$ then

$$\sum_{k=0}^{p-1}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5 \equiv 32p^2 + 64p^3 H_{p-1} \pmod{p^7},$$

where $H_{p-1} = \sum_{k=1}^{p-1} 1/k$.

Remark 7. . Note that $a_1 = 1$ and

$$4(2n+1)^2 a_{n+1} + n^2 a_n = (205n^2+160n+32)\binom{2n-1}{n}^3 \quad \text{for } n = 1, 2, \ldots.$$

The author generated the sequence $\{a_n\}_{n>0}$ at OEIS as A176285 (cf. [16]). In 1997 T. Amdeberhan and D. Zeilberger [1] used the WZ method to obtain

$$\sum_{k=1}^{\infty}\frac{(-1)^k(205k^2-160k+32)}{k^5\binom{2k}{k}^5} = -2\zeta(3).$$

Conjecture 8. (i) For any odd prime p, we have

$$\sum_{k=0}^{p-1}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k} \equiv 3p^2 - \frac{7}{2}p^5 B_{p-3} \pmod{p^6},$$

and
$$\sum_{k=0}^{(p-1)/2}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2+6\left(\frac{-1}{p}\right)p^4E_{p-3}\ (\mathrm{mod}\ p^5),$$
where $E_0,E_1,E_2,\ldots$ are Euler numbers.

(ii) For any integer $n>1$, we have
$$\sum_{k=0}^{n-1}(28k^2+18k+3)\binom{2k}{k}^4\binom{3k}{k}(-64)^{n-1-k}\equiv 0\ \left(\mathrm{mod}\ (2n+1)n^2\binom{2n}{n}^2\right).$$
Also,
$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Remark 9. The conjectural series for $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ was first announced by the author in a message to Number Theory Mailing List (cf. [17]) on April 4, 2010.

For more conjectures similar to Conjectures 6 and 8 the reader may consult [14] and [16].

In the next section we will establish three auxiliary inequalities involving the floor function. Sections 3 and 4 are devoted to the proofs of Theorem 1 and Theorem 5 respectively.

2 Three auxiliary inequalities

In this section, for a rational number x we let $\{x\}=x-\lfloor x\rfloor$ be the fractional part of x, and set $\{x\}_m=m\{x/m\}$ for any $m\in\mathbb{Z}^+$.

Theorem 10. *Let $m>1$ be an integer. Then for any $n\in\mathbb{Z}$ we have*
$$\left\lfloor\frac{n}{m}\right\rfloor+\left\lfloor\frac{6n}{m}\right\rfloor\geqslant\left\lfloor\frac{2n}{m}\right\rfloor+\left\lfloor\frac{2n+1}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor. \tag{6}$$

Proof. Let $A_m(n)$ denote the left-hand side of (6) minus the right-hand side. Then
$$A_m(n)=\left\{\frac{2n}{m}\right\}+\left\{\frac{2n+1}{m}\right\}+\left\{\frac{3n}{m}\right\}-\frac{1}{m}-\left\{\frac{n}{m}\right\}-\left\{\frac{6n}{m}\right\},$$
which only depends on n modulo m. So, without any loss of generality we may simply assume that $n\in\{0,\ldots,m-1\}$. Hence $A_m(n)\geqslant 0$ if and only if
$$\left\{\frac{2n}{m}\right\}+\left\{\frac{2n+1}{m}\right\}+\left\{\frac{3n}{m}\right\}\geqslant\frac{n+1}{m}. \tag{7}$$
(Note that $2n+(2n+1)+3n-(n+1)=6n$.)

(6) is obvious when $n = 0$. If $1 \leqslant n < m/2$, then $\{2n/m\} = 2n/m \geqslant (n+1)/m$ and hence (7) holds. In the case $n \geqslant m/2$, (7) can be simplified as

$$\frac{3n}{m} + \left\{\frac{3n}{m}\right\} \geqslant 2,$$

which holds since $3n \geqslant m + m/2$.

By the above we have proved (6). $\square$

Theorem 11. *Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then we have*

$$\left\lfloor\frac{4n+2k+2}{m}\right\rfloor - \left\lfloor\frac{2n+k+1}{m}\right\rfloor + 2\left\lfloor\frac{k}{m}\right\rfloor - 2\left\lfloor\frac{2k}{m}\right\rfloor \geqslant \left\lfloor\frac{n}{m}\right\rfloor + \left\lfloor\frac{n-k+1}{m}\right\rfloor, \qquad (8)$$

unless $2 \mid m$ and $k \equiv n+1 \equiv m/2 \pmod{m}$ in which case the right-hand side of the inequality equals the left-hand side plus one.

Proof. Since

$$(4n+2k+2) - (2n+k+1) + 2k - 2(2k) = n + (n-k+1),$$

(8) has the following equivalent form:

$$\left\{\frac{4n+2k+2}{m}\right\} - \left\{\frac{2n+k+1}{m}\right\} + 2\left\{\frac{k}{m}\right\} - 2\left\{\frac{2k}{m}\right\} \leqslant \left\{\frac{n}{m}\right\} + \left\{\frac{n-k+1}{m}\right\}. \qquad (9)$$

Note that this only depends on k and n modulo m. So, without any loss of generality, we may simply assume that $k, n \in \{0, \ldots, m-1\}$.

Case 1. $k < m/2$ and $\{2n+k+1\}_m < m/2$.

In this case, (9) can be simplified as

$$\frac{n+2k}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant \left\{\frac{2n+k+1}{m}\right\},$$

which is true since the left-hand side is nonnegative and $(n+2k) + (n-k+1) \equiv 2n + k + 1 \pmod{m}$.

Case 2. $k < m/2$ and $\{2n+k+1\}_m \geqslant m/2$.

In this case, (9) can be simplified as

$$\frac{n+2k}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant \left\{\frac{2n+k+1}{m}\right\} - 1,$$

which holds trivially since the right-hand side is negative.

Case 3. $k \geqslant m/2$ and $\{2n+k+1\}_m < m/2$.

In this case, (9) can be simplified as

$$\frac{n+2k}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant 2 + \left\{\frac{2n+k+1}{m}\right\}.$$

Since $(n+2k)+(n-k+1)=2n+k+1$, this is equivalent to

$$n+2k+\{n-k+1\}_m \geqslant 2m.$$

If $k>n+1$, then

$$n+2k+\{n-k+1\}_m = n+2k+(n-k+1+m) = 2n+k+1+m \geqslant 2m$$

since $2n+k+1>k\geqslant m/2$ and $\{2n+k+1\}_m<m/2$.

Now assume that $k\leqslant n+1$. Clearly

$$n+2k+\{n-k+1\}_m = n+2k+(n-k+1) = 2n+k+1 \geqslant 3k-1.$$

If $k>m/2$ then $3k-1\geqslant 3(m+1)/2-1>3m/2$. If $k\leqslant n$ then $2n+k+1>3k\geqslant 3m/2$. So, except the case $k=n+1=m/2$ we have

$$n+2k+\{n-k+1\}_m = 2n+k+1 \geqslant 3m/2$$

and hence $n+2k+\{n-k+1\}_m = 2n+k+1\geqslant 2m$ since $\{2n+k+1\}_m<m/2$.

When $k=n+1=m/2$, the left-hand side of (9) minus the right-hand side equals

$$\frac{m-2}{m}-\frac{m/2-1}{m}+2\frac{m/2}{m}-\frac{m/2-1}{m}=1.$$

Case 4. $k\geqslant m/2$ and $\{2n+k+1\}_m\geqslant m/2$.

In this case, clearly $m\neq 1$, and (9) can be simplified as

$$\frac{n+2k}{m}+\left\{\frac{n-k+1}{m}\right\}\geqslant 1+\left\{\frac{2n+k+1}{m}\right\}$$

which is equivalent to

$$n+2k+\{n-k+1\}_m \geqslant m.$$

If $k\leqslant n+1$, then

$$n+2k+\{n-k+1\}_m = n+2k+(n+1-k) = 2n+k+1 \geqslant 3k-1 \geqslant \frac{3m}{2}-1\geqslant m.$$

If $k>n+1$, then

$$n+2k+\{n-k+1\}_m = n+2k+(n+1-k)+m = 2n+k+1+m > m.$$

In view of the above, we have completed the proof of Theorem 11. □

Theorem 12. *Let $m\in\mathbb{Z}^+$ and $k,n\in\mathbb{Z}$. Then we have*

$$\begin{aligned}\left\lfloor\frac{2n+2k}{m}\right\rfloor-\left\lfloor\frac{n+k}{m}\right\rfloor+2\left\lfloor\frac{k}{m}\right\rfloor-2\left\lfloor\frac{2k}{m}\right\rfloor\\ \geqslant 2\left\lfloor\frac{n}{m}\right\rfloor-\left\lfloor\frac{2n+1}{m}\right\rfloor+\left\lfloor\frac{n-k+1}{m}\right\rfloor,\end{aligned}\tag{10}$$

unless $2\mid m$ and $k\equiv n+1\equiv m/2 \pmod m$ in which case the right-hand side of the inequality equals the left-hand side plus one.

Proof. Since

$$2n + 2k - (n + k) + 2k - 2(2k) = 2n - (2n + 1) + (n - k + 1),$$

(10) is equivalent to the following inequality:

$$\begin{aligned} &\left\{\frac{2n+2k}{m}\right\} - \left\{\frac{n+k}{m}\right\} + 2\left\{\frac{k}{m}\right\} - 2\left\{\frac{2k}{m}\right\} \\ &\leqslant 2\left\{\frac{n}{m}\right\} - \left\{\frac{2n+1}{m}\right\} + \left\{\frac{n-k+1}{m}\right\}. \end{aligned} \tag{11}$$

As (11) only depends on k and n modulo m, without loss of generality we simply assume that $k, n \in \{0, \ldots, m - 1\}$.

Case 1. $k < m/2$ and $\{n + k\}_m < m/2$.

In this case, (11) can be simplified as

$$\frac{2n+2k}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant \left\{\frac{2n+1}{m}\right\} + \left\{\frac{n+k}{m}\right\}$$

which holds since

$$\frac{2n+2k}{m} - \left\{\frac{n+k}{m}\right\} + \left\{\frac{n-k+1}{m}\right\} \geqslant 0$$

and $2n + 2k - (n + k) + (n - k + 1) = 2n + 1$.

Case 2. $k < m/2$ and $\{n + k\}_m \geqslant m/2$.

In this case, (11) can be simplified as

$$\frac{2n+2k}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant \left\{\frac{2n+1}{m}\right\} + \left\{\frac{n+k}{m}\right\} - 1$$

which holds since

$$\frac{2n+2k}{m} \geqslant \frac{n+k}{m} \geqslant \left\{\frac{n+k}{m}\right\} \text{ and } \left\{\frac{n-k+1}{m}\right\} \geqslant 0 > \left\{\frac{2n+1}{m}\right\} - 1.$$

Case 3. $k \geqslant m/2$ and $\{n + k\}_m < m/2$.

In this case, we must have $n + k \geqslant m$ and hence $\{n + k\}_m = n + k - m$. Thus (11) can be simplified as

$$\frac{n+k-m}{m} + \left\{\frac{n-k+1}{m}\right\} \geqslant \left\{\frac{2n+1}{m}\right\}$$

which holds trivially since $n + k - m + (n - k + 1) \equiv 2n + 1 \pmod{m}$.

Case 4. $k \geqslant m/2$ and $\{n + k\}_m \geqslant m/2$.

In this case, (11) can be simplified as

$$\frac{2n+2k}{m} - \left\{\frac{n+k}{m}\right\} + \left\{\frac{n-k+1}{m}\right\} \geqslant 1 + \left\{\frac{2n+1}{m}\right\}$$

which is equivalent to

$$\frac{2(n+k)}{m} - \left\{\frac{n+k}{m}\right\} + \left\{\frac{n-k+1}{m}\right\} \geqslant 1 \tag{12}$$

since $2n+2k-(n+k)+(n-k+1) = 2n+1$.

Clearly (12) holds if $n+k \geqslant m$. If $n+k < m$ and $k > n+1$, then the left-hand side of the inequality (12) is

$$\frac{n+k}{m} + \frac{n+1-k}{m} + 1 = \frac{2n+1}{m} + 1 > 1.$$

Now assume that $n+k < m$ and $k \leqslant n+1$. Then (12) is equivalent to $2n+1 \geqslant m$. If $k \leqslant n$ then $2n+1 > 2k \geqslant m$. If $k = n+1 \neq m/2$, then $k = n+1 \geqslant (m+1)/2$ and hence $2n+1 = 2(n+1)-1 \geqslant m$.

When $k = n+1 = m/2$, the left-hand side of (11) minus the right-hand side equals

$$\frac{m-2}{m} - \frac{m-1}{m} + 2\frac{m/2}{m} - 2\frac{m/2-1}{m} + \frac{m-1}{m} = 1.$$

Combining the discussion of the four cases we obtain the desired result. □

3 Proof of Theorem 1

For a prime p, the p-adic evaluation of an integer m is given by

$$\nu_p(m) = \sup\{a \in \mathbb{N} :\ p^a \mid m\}.$$

For a rational number $x = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we set $\nu_p(x) = \nu_p(m) - \nu_p(n)$ for any prime p. Note that a rational number x is an integer if and only if $\nu_p(x) \geqslant 0$ for all primes p.

Proof of Theorem 1. (i) Fix $n \in \mathbb{Z}^+$, and define $A_m(n)$ for $m > 1$ as in the proof of Theorem 10. Observe that

$$Q := \frac{\binom{6n}{3n}\binom{3n}{n}}{(2n+1)\binom{2n}{n}} = \frac{n!(6n)!}{(2n)!(2n+1)!(3n)!}.$$

So, for any prime p we have

$$\nu_p(Q) = \sum_{i=1}^{\infty} A_{p^i}(n) \geqslant 0$$

by Theorem 10. Therefore Q is an integer.

Choose $j \in \mathbb{Z}^+$ such that $2^{j-1} \leqslant n < 2^j$. As $2n+1 \leqslant 2(2^j-1)+1 < 2^{j+1}$, we have

$$\begin{aligned}
&\left\lfloor \frac{n}{2^{j+1}} \right\rfloor + \left\lfloor \frac{6n}{2^{j+1}} \right\rfloor - \left\lfloor \frac{2n}{2^{j+1}} \right\rfloor - \left\lfloor \frac{2n+1}{2^{j+1}} \right\rfloor - \left\lfloor \frac{3n}{2^{j+1}} \right\rfloor \\
&= \left\lfloor \frac{3n}{2^j} \right\rfloor - \left\lfloor \frac{3n}{2^{j+1}} \right\rfloor = \left\lfloor \frac{3n+2^j}{2^{j+1}} \right\rfloor \geqslant \left\lfloor \frac{2n+2^j}{2^{j+1}} \right\rfloor \geqslant 1.
\end{aligned}$$

Therefore

$$\nu_2(Q) = \sum_{i=1}^{\infty} A_{2^i}(n) \geqslant A_{2^{j+1}}(n) \geqslant 1.$$

and hence Q is even. This proves (1).

(ii) (2) and (3) are obvious in the case $k = 0$. If $k > n+1$, then

$$\binom{2n+k+1}{2k} = \binom{n+k+1}{2k} = 0$$

and hence (2) and (3) hold trivially. Below we assume that $1 \leqslant k \leqslant n+1$.

Recall that for any nonnegative integer m and prime p we have

$$\nu_p(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor .$$

Since

$$\frac{\binom{4n+2k+2}{2n+k+1}\binom{2n+k+1}{2k}\binom{2n+k+1}{n}}{\binom{2k}{k}} = \frac{(4n+2k+2)!(k!)^2}{(2n+k+1)!((2k)!)^2 n!(n-k+1)!}$$

and

$$\frac{(2n+1)\binom{2n}{n}C_{n+k}\binom{n+k+1}{2k}}{\binom{2k}{k}} = \frac{(2n+1)!(2n+2k)!(k!)^2}{(n!)^2(n+k)!((2k)!)^2(n-k+1)!},$$

it suffices to show that for any prime p we have

$$\sum_{i=1}^{\infty} C_{p^i}(n,k) \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{\infty} D_{p^i}(n,k) \geqslant 0,$$

where

$$\begin{aligned} C_m(n,k) = & \left\lfloor \frac{4n+2k+2}{m} \right\rfloor - \left\lfloor \frac{2n+k+1}{m} \right\rfloor + 2\left\lfloor \frac{k}{m} \right\rfloor - 2\left\lfloor \frac{2k}{m} \right\rfloor \\ & - \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{n-k+1}{m} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} D_m(n,k) = & \left\lfloor \frac{2n+2k}{m} \right\rfloor - \left\lfloor \frac{n+k}{m} \right\rfloor + 2\left\lfloor \frac{k}{m} \right\rfloor - 2\left\lfloor \frac{2k}{m} \right\rfloor \\ & - 2\left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{2n+1}{m} \right\rfloor - \left\lfloor \frac{n-k+1}{m} \right\rfloor . \end{aligned}$$

(a) By Theorem 11, $C_{p^i}(n,k) \geqslant 0$ unless $p = 2$ and $k \equiv n+1 \equiv 2^{i-1} \pmod{2^i}$ in which case $C_{2^i}(n,k) = -1$. Suppose that $k \equiv n+1 \equiv 2^{i-1} \pmod{2^i}$, $k = 2^{i-1}k_0$ and $n+1 = 2^{i-1}n_0$, where $1 \leqslant k_0 \leqslant n_0$ and k_0 and n_0 are odd. If $i \geqslant 2$, then

$$C_{2^{i-1}}(n,k) = 4n_0 + 2k_0 - 1 - (2n_0 + k_0 - 1) + 2k_0 - 4k_0 - (n_0 - 1) - (n_0 - k_0) = 1$$

and hence $C_{2^{i-1}}(n,k) + C_{2^i}(n,k) = 1 + (-1) = 0$. So it remains to consider the case $k \equiv n+1 \equiv 1 \pmod 2$.

Assume that k is odd and n is even. Write $k+1 = 2^j k_1$ and $n = 2n_1$ with $k_1, n_1 \in \mathbb{Z}^+$ and $2 \nmid k_1$. Then it is easy to see that

$$\begin{aligned}
C_{2^{j+1}}(n,k) = & \left\lfloor \frac{4n_1}{2^j} \right\rfloor + k_1 - \left\lfloor \frac{2n_1 - 2^{j-1} + 2^{j-1}(k_1-1)}{2^j} \right\rfloor \\
& + 2\left\lfloor \frac{k_1}{2} \right\rfloor - 2\left\lfloor \frac{2^j k_1 - 1}{2^j} \right\rfloor - \left\lfloor \frac{n_1}{2^j} \right\rfloor - \left\lfloor \frac{n_1 + 1 - 2^{j-1}k_1}{2^j} \right\rfloor \\
= & \left\lfloor \frac{4n_1}{2^j} \right\rfloor + k_1 - \left\lfloor \frac{2n_1 - 2^{j-1}}{2^j} \right\rfloor - \frac{k_1+1}{2} + k_1 - 1 - 2(k_1-1) \\
& - \left\lfloor \frac{n_1}{2^j} \right\rfloor - \left\lfloor \frac{n_1 + 1 + 2^{j-1}}{2^j} \right\rfloor + \frac{k_1+1}{2} \\
= & 1 + \left\lfloor \frac{n_1 + (n_1 + 1 + 2^{j-1}) + (2n_1 - 2^{j-1})}{2^j} \right\rfloor \\
& - \left\lfloor \frac{n_1}{2^j} \right\rfloor - \left\lfloor \frac{n_1 + 1 + 2^{j-1}}{2^j} \right\rfloor - \left\lfloor \frac{2n_1 - 2^{j-1}}{2^j} \right\rfloor \\
\geqslant & 1
\end{aligned}$$

and hence $C_2(n,k) + C_{2^{j+1}}(n,k) \geqslant 0$.

By the above, we do have $\sum_{i=1}^{\infty} C_{p^i}(n,k) \geqslant 0$ for any prime p. So (2) holds.

(b) By Theorem 11, $D_{p^i}(n,k) \geqslant 0$ unless $p = 2$ and $k \equiv n+1 \equiv 2^{i-1} \pmod{2^i}$ in which case $D_{2^i}(n,k) = -1$. So, to prove (2) it suffices to find a positive integer j such that $D_{2^j}(n,k) \geqslant 1$.

Clearly there is a unique positive integer j such that $2^{j-1} \leqslant n+k < 2^j$. Note that $k \leqslant (n+k)/2 < 2^{j-1}$ and

$$D_{2^j}(n,k) = 1 + \left\lfloor \frac{2n+1}{2^j} \right\rfloor \geqslant 1.$$

This concludes the proof of (3).

The proof of Theorem 1 is now complete. □

4 Proof of Theorem 5

Proof of Theorem 5. (i) We first prove (4). For $k, n \in \mathbb{N}$ define

$$F(n,k) = \frac{(-1)^{n+k}(4n+1)}{4^{3n-k}} \binom{2n}{n}^2 \frac{\binom{2n+2k}{n+k}\binom{n+k}{2k}}{\binom{2k}{k}}$$

and

$$G(n,k) = \frac{(-1)^{n+k}(2n-1)^2\binom{2n-2}{n-1}^2}{2(n-k)4^{3(n-1)-k}} \binom{2(n-1+k)}{n-1+k} \frac{\binom{n-1+k}{2k}}{\binom{2k}{k}}.$$

Clearly $F(n,k) = G(n,k) = 0$ if $n < k$. By [7],

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k)$$

for all $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$.

Fix a positive integer N. Then

$$\begin{aligned}\sum_{n=0}^{N} F(n,0) - F(N,N) &= \sum_{n=0}^{N} F(n,0) - \sum_{n=0}^{N} F(n,N)\\ &= \sum_{k=1}^{N}\left(\sum_{n=0}^{N} F(n,k-1) - \sum_{n=0}^{N} F(n,k)\right)\\ &= \sum_{k=1}^{N}\sum_{n=0}^{N}(G(n+1,k) - G(n,k)) = \sum_{k=1}^{N} G(N+1,k).\end{aligned}$$

Note that

$$\sum_{n=0}^{N} F(n,0) = \sum_{n=0}^{N} \frac{4n+1}{(-64)^n}\binom{2n}{n}^3$$

and

$$F(N,N) = \frac{4N+1}{4^{2N}}\binom{2N}{N}\binom{4N}{2N} = \frac{(4N+1)(2N+1)}{4^{2N}}\binom{2N}{N}C_{2N}.$$

Also,

$$\begin{aligned}\sum_{k=1}^{N} G(N+1,k) &= \frac{(2N+1)^2}{2}\sum_{k=1}^{N}\frac{(-1)^{N+k+1}}{4^{3N-k}}\binom{2N}{N}^2 C_{N+k}\frac{\binom{N+k+1}{2k}}{\binom{2k}{k}}\\ &= \frac{2(2N+1)\binom{2N}{N}}{(-64)^N}\sum_{k=1}^{N}(-4)^{k-1}\frac{(2N+1)\binom{2N}{N}C_{N+k}\binom{N+k+1}{2k}}{\binom{2k}{k}}.\end{aligned}$$

and

$$\begin{aligned}\frac{\binom{2N}{N}C_{N+1}\binom{N+2}{2}}{\binom{2}{1}} &= \binom{2N-1}{N-1}\binom{2N+2}{N+1}\frac{N+1}{2}\\ &= \binom{2N-1}{N-1}\binom{2N+1}{N+1}(N+1)\\ &= \binom{2N-1}{N-1}(2N+1)\binom{2N}{N}\\ &= 2(2N+1)\binom{2N-1}{N-1}^2 \equiv 0 \pmod 2.\end{aligned}$$

So, with the help of (3) we see that $\sum_{n=0}^{N}(4n+1)\binom{2n}{n}^3(-64)^{N-n}$ is divisible by $4(2N+1)\binom{2N}{N}$.

(ii) Now we turn to the proof of (5).

For $n, k \in \mathbb{N}$, define

$$F(n,k) := \frac{(-1)^{n+k}(20n-2k+3)}{4^{5n-k}} \cdot \frac{\binom{2n}{n}\binom{4n+2k}{2n+k}\binom{2n+k}{2k}\binom{2n-k}{n}}{\binom{2k}{k}}.$$

and

$$G(n,k) := \frac{(-1)^{n+k}}{4^{5n-4-k}} \cdot \frac{n\binom{2n}{n}\binom{4n+2k-2}{2n+k-1}\binom{2n+k-1}{2k}\binom{2n-k-1}{n-1}}{\binom{2k}{k}}.$$

Clearly $F(n,k) = G(n,k) = 0$ if $n < k$. By [22],

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k)$$

for all $k \in \mathbb{Z}^+$ and $n \in \mathbb{N}$.

Fix a positive integer N. As in part (i) we have

$$\sum_{n=0}^{N} F(n,0) - F(N,N) = \sum_{k=1}^{N} G(N+1,k).$$

Observe that

$$\sum_{n=0}^{N} F(n,0) = \sum_{n=0}^{N} \frac{20n+3}{(-2^{10})^n}\binom{2n}{n}^2\binom{4n}{2n}$$

and

$$F(N,N) = \frac{18N+3}{2^{8N}}\binom{6N}{3N}\binom{3N}{N}.$$

Also,

$$\sum_{k=1}^{N} G(N+1,k) = \frac{2(2N+1)\binom{2N}{N}}{(-2^{10})^N}\sum_{k=1}^{N}(-4)^{k-1}\frac{\binom{4N+2k+2}{2N+k+1}\binom{2N+k+1}{2k}\binom{2N-k+1}{N}}{\binom{2k}{k}}.$$

Note that

$$\frac{\binom{4N+4}{2n+2}\binom{2N+2}{2}\binom{2N}{N}}{\binom{2}{1}} = 2\binom{4N+3}{2N+1}\binom{2N+2}{2}\binom{2N-1}{N-1} \equiv 0 \pmod 2.$$

Applying (2) we see that $(-2^{10})^N \sum_{k=1}^{N} G(N+1,k)$ is a multiple of $4(2N+1)\binom{2N}{N}$. By (1),

$$(-2^{10})^N \frac{18N+3}{2^{8N}}\binom{6N}{3N}\binom{3N}{N}$$

is divisible by $8(2N+1)\binom{2N}{N}$. Therefore

$$\sum_{n=0}^{N}(20n+3)\binom{2n}{n}^2\binom{4n}{2n}(-2^{10})^{N-n}$$

is a multiple of $4(2N+1)\binom{2N}{N}$.

Combining the above, we have completed the proof of Theorem 5. $\square$

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