A combinatorial proof of an identity for the divisor generating function

Masanori Ando
Department of Information and Media
Wakhok University
Hokkaido, Japan
m-ando@wakhok.ac.jp

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Abstract

In this paper, we give combinatorial proofs and new generalizations of q-series identities of Dilcher and Uchimura related to divisor function. Some interesting combinatorial results related to partition and arm length are also presented.

Keywords: integer partition; q-series; Uchimura's identity; Dilcher's identity

1 Introduction

In [6], Uchimura proved the following q-series identity

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k(1-q^k)}=\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}, \tag{1}$$

where $(a;b)_k=(1-a)(1-ab)(1-ab^2)\cdots(1-ab^{k-1})$. In fact the identity (1) has been known 100 years before ([3]). Moreover, it is obtained also as specialization of basic hypergeometric series (see [2] section 12 on page 34). And, this identity is an infinite version of the following q-series identity called *Problem 6407* in American Mathematical Monthly [7].

$$\sum_{k=1}^{m}(-1)^{k-1}\begin{bmatrix} m \\ k \end{bmatrix}\frac{q^{\frac{k(k+1)}{2}}}{1-q^k}=\sum_{k=1}^{m}\frac{q^k}{1-q^k}.$$

Here $\begin{bmatrix} m \\ k \end{bmatrix}$ is a q-binomial coefficient. Many authors have generalized these identities (see e.g. [8]). In this paper, we translate these identities and Dilcher's generalization [1] into combinatorics of partitions, and give a combinatorial proof of them. For example, we transform (1) as

$$\sum_{\lambda \in \mathcal{SP}} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)} q^{|\lambda|} = \sum_{n=1}^{\infty} \sigma_0(n) q^n.$$

It is a q-series identity about *strict partitions* and a *divisor function*.

The generalizations of (1) we give in this paper are the following.

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{b^k q^{\frac{k(k-1)}{2}+mk}}{(bq;q)_k (1-q^k)^m} = \sum_{j_1=1}^{\infty} \frac{b^{j_1} q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}},$$

$$\sum_{k=1}^{t} (-1)^{k-1} \frac{b^k q^{\frac{k(k-1)}{2}+mk}}{(1-q^k)^m} \left[\begin{array}{c} t \\ k \end{array} \right]_{q,b} = \sum_{j_1=1}^{t} \frac{b^{j_1} q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}}.$$

As a by-product of their proofs, we obtain some combinatorial results.

2 Young diagrams

Definition 1. Let n be a positive integer. A partition λ of n is an integer sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We call $\ell(\lambda) := \ell$ the length of λ, and each λ_i a part of λ. We let $\mathcal{P}$ and $\mathcal{P}(n)$ denote the set of partitions and the set of partitions of n.

Definition 2. A partition λ is said to be strict if $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0$. We let $\mathcal{SP}$ and $\mathcal{SP}(n)$ denote the set of partitions and the set of partitions of n.

Definition 3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. The Young diagram of λ is defined by

$$Y(\lambda) := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant \lambda_i\}.$$

We call $(i,j) \in Y(\lambda)$ the (i,j)-cell of λ. And the set of the corners of λ is defined by

$$C(\lambda) := \{(i,j) \in Y(\lambda) \mid (i+1,j), (i,j+1) \notin Y(\lambda)\}.$$

We put $c(\lambda) := \sharp C(\lambda)$, the number of the corners of λ.

Definition 4. Let $(i,j) \in Y(\lambda)$, The (i,j)-hook length of λ is defined by

$$h_{ij}(\lambda) := \sharp\{(i',j') \in Y(\lambda) \mid i'=i, j' \geqslant j \ \text{ or } \ j'=j, i' \geqslant i\ \}$$

And we put $a_{ij}(\lambda) := \lambda_i - j + 1$, the (i,j)-arm length of λ. We remark that our definition of arm length a_{ij} is different by 1 from the usual definition [4].

3 q-series identity for the divisor function

Theorem 5 (Uchimura's identity)**.**

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k(1-q^k)}=\sum_{k=1}^{\infty}\frac{q^k}{1-q^k},$$

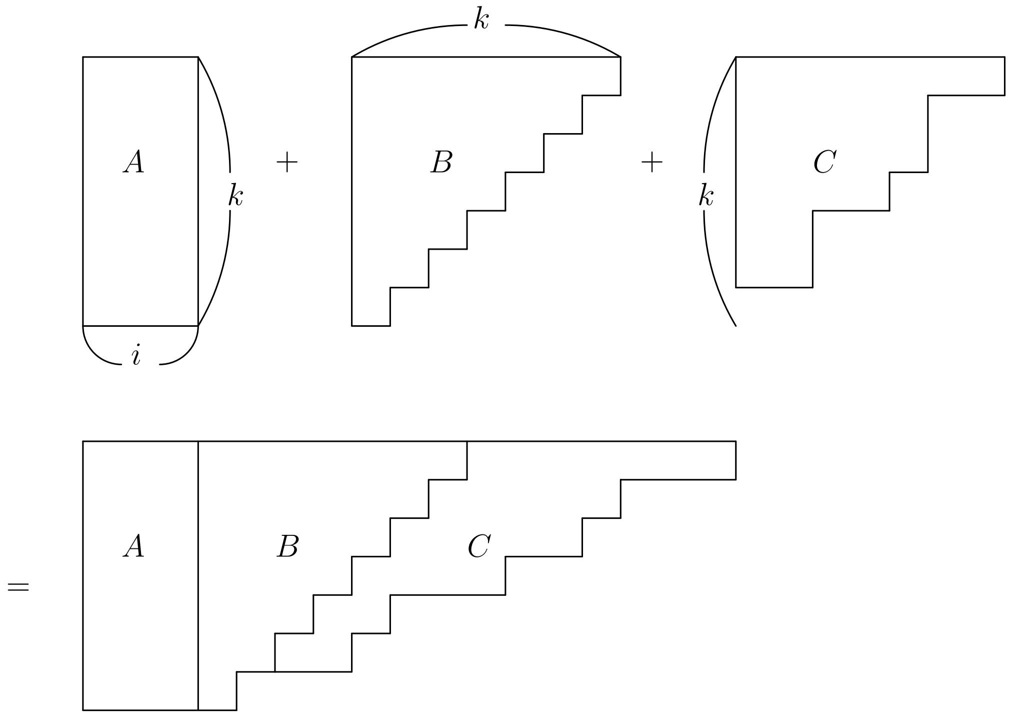
where $(a;b)_k=(1-a)(1-ab)(1-ab^2)\cdots(1-ab^{k-1})$.

Remark that the right-hand side is computed as

$$\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}=\sum_{k=1}^{\infty}(q^k+q^{2k}+q^{3k}+\cdots)=\sum_{n=1}^{\infty}\sigma_0(n)q^n,$$

where $\sigma_0(n)$ is the number of positive divisors of n. We now translate this identity into a language of Young diagrams. Then we are able to prove this identity combinatorially.

Figure 1.



Looking at each term of the left-hand side, $q^{\frac{k(k+1)}{2}}$ is translated into the stairs B in Figure 1. Since $\frac{1}{(q;q)_k}$ is the generating function of partitions whose lengths are at most k, this term corresponds to C in Figure 1. The leftover $\frac{1}{1-q^k}$ is the generating function of rectangular Young diagrams whose vertical lengths are equal to k. This part corresponds to A in Figure 1. Therefore the left-hand side of the identity is an alternating sum over $k \geqslant 1$ of $A+B+C$. As is noted in Figure 1, the "sum" $A+B+C$ is a strict partition. For a strict partition λ, we count the number of tuples (A, B, C) such that $A+B+C = \lambda$. Let λ be a fixed strict partition of length k. One can embed the stairs B into λ in λ_k ways. For each embedding the rectangle A and the partition C are uniquely determined, respectively. Therefore there are λ_k tuples (A, B, C), such that $A+B+C = \lambda$. Summing up over k, Theorem 5 reads

$$\sum_{\lambda \in \mathcal{SP}} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)} q^{|\lambda|} = \sum_{n=1}^{\infty} \sigma_0(n) q^n. \tag{2}$$

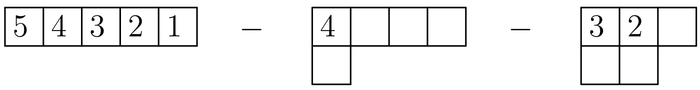
The proof of this identity will follow from the next combinatorial theorem.

Theorem 6. *For any positive integers n and k,*

$$\begin{aligned} & \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda); \textit{odd}\} \\ - \; & \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda); \textit{even}\} \\ = \; & \begin{cases} 1 & (k \mid n) \\ 0 & (k \nmid n). \end{cases} \end{aligned}$$

Example. Let $n = 5$. We draw Young diagrams $Y(\lambda)$ of all strict partitions of 5, and write arm length $a_{1j}(\lambda)$ in $(1, j)$-cell for $1 \leqslant j \leqslant \lambda_{\ell(\lambda)}$.

Figure 2.



$$= \quad \{5, 1\}.$$

Here the numbers are regarded as variables. Numbers 5 and 1 are the positive divisors of 5. We remark that it is check about Theorem 6 in all positive integer k at the same time.

Proof of Theorem 6.
We consider the set of strict partitions of n such that $a_{1,j} = k$ for some $1 \leqslant j \leqslant \lambda_{\ell(\lambda)}$:

$$\mathfrak{D}(n, k) := \{\lambda \in \mathcal{SP}(n) \mid \lambda_1 - \lambda_{\ell(\lambda)} < k \leqslant \lambda_1\}.$$

We divide these strict partitions into two classes A and B:

$$A = \{\lambda \in \mathfrak{D}(n, k) \mid k \nmid \lambda_i \text{for any } i\}, B = \{\lambda \in \mathfrak{D}(n, k) \mid k \mid \lambda_i \text{for some } i\}.$$

We consider a map between them that changes the length by 1.

$$\alpha_k : \; A \to B, \qquad \alpha_k(\lambda) = \lambda',$$

where $\lambda' \in B$ is defined in the following steps:

Step 1. Append 0 in the tail of λ to get $\{\lambda_1, \ldots, \lambda_{\ell+1}\}$.

Step 2. Subtract k from $\max\{\lambda_1, \ldots, \lambda_{\ell+1}\}$, and add k to $\lambda_{\ell+1}$.

Step 3. Repeat Step 2 till $\max\{\lambda_1, \ldots, \lambda_{\ell+1}\} - \min\{\lambda_1, \ldots, \lambda_{\ell+1}\}$ gets less than k.

Step 4. From the resulting composition we have the partition $\lambda' = (\lambda_1, \ldots, \lambda_{\ell+1})$ by arranging parts.

Example. Let $n = 13, k = 4$.

$$\mathfrak{D}(13, 4) = \{(13), (8, 5), (7, 6), (6, 4, 3)\}.$$

And we divide these partitions

$$A = \{(13), (7, 6)\}, B = \{(8, 5), (6, 4, 3)\}.$$

Then the map α_4 looks

$$\begin{aligned} \lambda &= (13) \to \{13, 0\} \to \{9, 4\} \to \{5, 8\} \to (8, 5) = \alpha_4(\lambda), \\ \mu &= (7, 6) \to \{7, 6, 0\} \to \{3, 6, 4\} \to (6, 4, 3) = \alpha_4(\mu). \end{aligned}$$

When $k = 4$, the pair of $\lambda = (8, 4, 1)$ seems not to exist. However, the partition λ is not an element of $\mathfrak{D}(13, 4)$ primarily.

By the above construction, we have $\ell(\lambda') = \ell(\lambda) + 1$, and

$$\sharp\{i \mid \lambda_i \equiv j (\mathrm{mod}\ k), 1 \leqslant i \leqslant \ell\} = \sharp\{i \mid \lambda'_i \equiv j (\mathrm{mod}\ k), 1 \leqslant i \leqslant \ell\}$$

for $1 \leqslant j \leqslant k - 1$. The partition that α_k can not pair up is $(\lambda_1, \ldots, \lambda_\ell) \equiv (0) \pmod k$. Therefore $\lambda = (n)$ is left when n is a multiple of k. $\square$

We add in the proof that the identity and proof equivalent to Theorem 6 appears in [5]. Moreover, the identity which generalized σ_0 to σ_n in Theorem 5 appears there.

Proof of theorem 5.

Sum of the left-hand side of Theorem 6 over k is

$$\begin{aligned} & \sum_{k=1}^{\infty} \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda); \text{odd}\} \\ - & \sum_{k=1}^{\infty} \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda); \text{even}\} \\ = & \sum_{\lambda \in \mathcal{SP}(n)} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)}. \end{aligned}$$

And sum of the right-hand side is

$$\sum_{k|n} 1 = \sigma_0(n).$$

They are the coefficients of q^n in (2). $\square$

Theorem 5 is the generating function for the total sum of Theorem 6. Taking the sum over k from 1 to m for Theorem 6, we have the following identity of the generating function.

Theorem 7 (Problem 6407)**.**

$$\sum_{k=1}^{m} (-1)^{k-1} \begin{bmatrix} m \\ k \end{bmatrix} \frac{q^{\frac{k(k+1)}{2}}}{1-q^k} = \sum_{k=1}^{m} \frac{q^k}{1-q^k},$$

where the q-binomial coefficient is defined as

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leqslant m-k, \ell(\lambda) \leqslant k}} q^{|\lambda|}.$$

We draw the same figure as Figure 1. The Young diagram C in Figure is restricted that the first row is at most $m-k$. Then, there are new restriction $a_{1j} \leqslant m$ on the left-hand side of identity. Here Theorem 7 reads

$$\sum_{\lambda \in \mathcal{SP}} \sum_{j \leqslant \lambda_\ell, a_{1,j} \leqslant m} (-1)^{\ell(\lambda)-1} q^{|\lambda|} = \sum_{n=1}^{\infty} \sum_{k|n, k \leqslant m} q^n. \tag{3}$$

Proof. Sum of the left-hand side of Theorem 6 over $1 \leqslant k \leqslant m$ is

$$\begin{aligned} & \sum_{k=1}^{m} \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{odd}\} \\ - & \sum_{k=1}^{m} \sharp\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geqslant k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{even}\} \\ = & \sum_{\lambda \in \mathcal{SP}(n)} \sum_{j \leqslant \lambda_\ell, a_{1,j} \leqslant m} (-1)^{\ell(\lambda)-1}. \end{aligned}$$

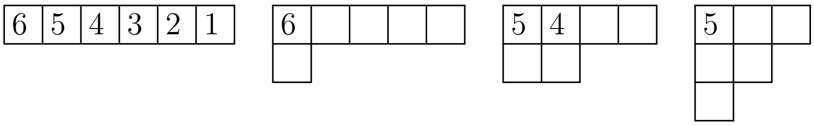
$\square$

Corollary 8. *For $n = 2(2m+1)$,*

$$\begin{aligned} & \sum_{\lambda \in \mathcal{SP}(n)} \sharp\{h_{1,j}(\lambda) \mid 1 \leqslant j \leqslant \lambda_{\ell(\lambda)}, h_{1,j}(\lambda) \textit{is odd}\} \\ = & \sum_{\lambda \in \mathcal{SP}(n)} \sharp\{h_{1,j}(\lambda) \mid 1 \leqslant j \leqslant \lambda_{\ell(\lambda)}, h_{1,j}(\lambda) \textit{is even}\}. \end{aligned}$$

Example. Let $n = 6$. We draw Young diagrams $Y(\lambda)$ of all strict partitions of 6, and write hook length $h_{1,j}(\lambda)$ in $(1, j)$-cell for $1 \leqslant j \leqslant \lambda_{\ell(\lambda)}$.

Figure 3.



The number of odd numbers equals the number of even numbers.

Proof. In Theorem 6, the strict partitions they have same arm length and different parity length are pair. Recall that $h_{1j}(\lambda) = a_{1j}(\lambda) + \ell(\lambda) - 1$. Therefore the parity of their hook length are different. And the leftovers are divisors of n. When n equals $2(2m + 1)$, the number of odd divisors of n equals the number of even divisors of n. $\square$

4 Generalizations

Theorem 9. *For any positive integers k, m and n, we have*

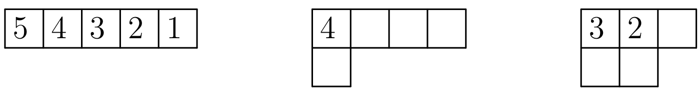
$$\begin{aligned}
& \sharp\left\{ (\lambda; i_1, \ldots, i_m) \;\middle|\; \begin{array}{c} \lambda \in \mathcal{SP}(n), 1 \leqslant i_1 < \ldots < i_m \leqslant \lambda_{\ell(\lambda)} \\ a_{1,i_m}(\lambda) = k, \ell(\lambda) : odd \end{array} \right\} \\
- \; & \sharp\left\{ (\lambda; i_1, \ldots, i_m) \;\middle|\; \begin{array}{c} \lambda \in \mathcal{SP}(n), 1 \leqslant i_1 < \ldots < i_m \leqslant \lambda_{\ell(\lambda)} \\ a_{1,i_m}(\lambda) = k, \ell(\lambda) : even \end{array} \right\} \\
= \; & \sharp\left\{ (\lambda; t_1, \ldots, t_{m-c(\lambda)}) \;\middle|\; \begin{array}{c} \lambda \in \mathcal{P}(n), c(\lambda) \leqslant m, \lambda_1 = k, \lambda_{t_i} = \lambda_{t_i+1} \\ 1 \leqslant t_1 < \ldots < t_{m-c(\lambda)} < \ell(\lambda) \end{array} \right\}.
\end{aligned}$$

When $m = 1$, this identity is specialized as

$$\begin{aligned}
& \sharp\{(\lambda; i) \mid \lambda \in \mathcal{SP}(n), 1 \leqslant i \leqslant \lambda_{\ell(\lambda)}, a_{1,i}(\lambda) = k, \ell(\lambda) : \text{odd}\} \\
- \; & \sharp\{(\lambda; i) \mid \lambda \in \mathcal{SP}(n), 1 \leqslant i \leqslant \lambda_{\ell(\lambda)}, a_{1,i}(\lambda) = k, \ell(\lambda) : \text{even}\} \\
= \; & \sharp\{\lambda \mid \lambda \in \mathcal{P}(n), c(\lambda) = 1, \lambda_1 = k\}.
\end{aligned}$$

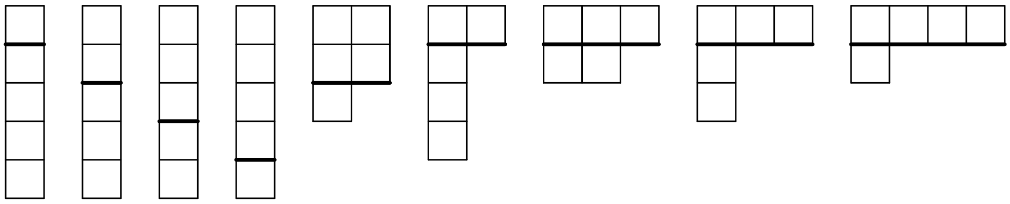
When $c(\lambda) = 1$, the shape of $Y(\lambda)$ is rectangle. Therefore this identity is equivalent to Theorem 6.
Example. For $n = 5, m = 2$, we draw the same figure as Figure 2.

Figure 4.



We count the pairs of arm lengths in a partition λ with sign $(-1)^{\ell(\lambda)}$. The pairs with positive sign are $(5,1)$, $(4,1)$, $(3,1)$, $(2,1)$, $(5,2)$, $(4,2)$, $(3,2)$, $(5,3)$, $(4,3)$, $(5,4)$. The pair with negative sign is $(3,2)$. On the other hand, all Young diagrams of partition of 5 that made by concatenating 2 rectangles are followings.

Figure 5.



There are 9 such Young diagrams. Here we count the ways of concatenating rectangles. Hence, for example, the first 4 diagrams must be thought of as the different ones. The number 9 equals the number of the pairs with positive sign minus the number of the pairs with negative sign. And more fix smaller number b, the number of pair (a,b) with positive sign minus the number of pair (a,b) with negative sign also equals the number of Young diagrams λ made by concatenating 2 rectangles that λ_1 equals b. For example, the number of the pairs that smaller number is 1 and the number of Young diagrams that size of first line is 1 are both 4.

Proof of Theorem 9. When $\lambda \in \mathcal{SP}(n)$, $i_1 < \ldots < i_m \leqslant \lambda_\ell$ are given, we consider the strict partitions $\lambda^{(1)}, \ldots, \lambda^{(m)}$ in the following procedure. First, we put $\lambda^{(1)} := \lambda$. When $\lambda^{(h)}$ is determined, we put $\mu := \lambda^{(h)}$ and $j_h := a_{1,i_{m+1-h}}(\lambda^{(h)})$. And we make a new strict partition $[\mu]$ by replacing μ_1 by $\mu_1 - j_h$ in μ. This operation keeps strictness, since μ is strict to modulus k. We repeat this operation until we get $a_{1,i_{m-h}}(\mu) \leqslant j_h$. We put $\lambda^{(h+1)}$ with μ obtained in this way. And let t_h be the number of times of the operations. The length of $\lambda^{(i)}$ is same as the length of λ. We pair up $(\lambda; i_1, i_2, \ldots i_m)$ and $(\lambda'; i'_1, i'_2, \ldots i'_m)$ when $\lambda^{(m)}$and $\lambda'^{(m)}$ correspond by α_{j_m} and $j_h = j'_h$ for all h. Here α_{j_m} is the map defined in the proof of Theorem 6. Such λ' is calculable by performing inverse operations to $\lambda'^{(m)}$. The inverse operation is adding j_h to the minimum part of $\lambda'^{(m)}$. And i'_h is determined from condition $j_h = j'_h$. Then there is only 1 difference between the lengths of λ and λ'. Leftovers are $\lambda^{(m)} = (\lambda_1^{(m)})$ that $\lambda_1^{(m)}$ is multiple of j_m. Since $j_1 \geqslant \ldots \geqslant j_m$, it corresponds with Young diagram that is made by concatenating rectangles $j_h \times t_h$, where $t_m = \dfrac{\lambda_1^{(m)}}{j_m}$. □

Example. Let $k = 13$, $m = 3$, $n = 70$, $\lambda = (23, 19, 16, 12)$, $i_1 = 4$, $i_2 = 5$, $i_3 = 11$.
First, we put $\lambda^{(1)} = (23, 19, 16, 12), j_1 = a_{1,11}(\lambda^{(1)}) = 13$.

$$\lambda^{(1)} = (23, 19, 16, 12) \overset{-13}{\to} (19, 16, 12, 10) \overset{-13}{\to} (16, 12, 10, 6).$$

Because $a_{1,5}(16, 12, 10, 6) = 12 \leqslant j_1$, operation stops. And we put $\lambda^{(2)} = (16, 12, 10, 6)$, $j_2 = a_{1,5}(\lambda^{(2)}) = 12$.

$$\lambda^{(2)} = (16, 12, 10, 6) \overset{-12}{\to} (12, 10, 6, 4).$$

We put $\lambda^{(3)} = (12, 10, 6, 4), j_3 = a_{1,4}(\lambda^{(3)}) = 9$. Then,

$$\begin{aligned}\lambda^{(3)} = (12, 10, 6, 4) &\to \{12, 10, 6, 4, 0\} \to \{3, 10, 6, 4, 9\} \\ &\to \alpha_9(\lambda^{(3)}) = (10, 9, 6, 4, 3).\end{aligned}$$

And we perform inverse operation,

$$\begin{aligned}a_9(\lambda^{(3)}) = (10, 9, 6, 4, 3) &\overset{12}{\to} (15, 10, 9, 6, 4) \\ &\overset{13}{\to} (17, 15, 10, 9, 6) \overset{13}{\to} (19, 17, 15, 10, 9).\end{aligned}$$

Here, $a_{1,2}(10, 9, 6, 4, 3)$, $a_{1,4}(15, 10, 9, 6, 4)$ and $a_{1,7}(19, 17, 15, 10, 9)$ are equal to j_3, j_2 and j_1 respectively. Then, we checked $((23, 19, 16, 12); 4, 5, 11)$ and $((19, 17, 15, 10, 9); 2, 4, 7)$ are pairs.
As another example, let $\lambda = (70), i_1 = 13, i_2 = 19, i_3 = 58$.
We put $\lambda^{(1)} = (70), j_1 = a_{1,58}(\lambda^{(1)}) = 13$.

$$\lambda^{(1)} = (70) \overset{-13}{\to} (57) \overset{-13}{\to} (44) \overset{-13}{\to} (31).$$

We put $\lambda^{(2)} = (31), j_2 = a_{1,19}(\lambda^{(2)}) = 13$.

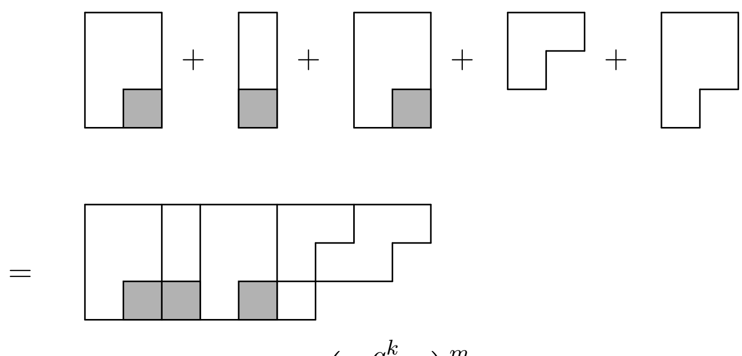
$$\lambda^{(2)} = (31) \overset{-13}{\to} (18).$$

We put $\lambda^{(3)} = (18), j_3 = a_{1,13}(\lambda^{(3)}) = 6$. Then, $\lambda^{(3)}_1 = 18$ is multiple of $j_3 = 6$. Therefore $((70); 13, 19, 58)$ corresponds with Young diagram concatenating with rectangles 13×3, 13×1 and 6×3.

Theorem 10 (Dilcher's identity 1)**.** *For any positive integer m,*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{\frac{k(k-1)}{2}+mk}}{(q;q)_k (1-q^k)^m} = \sum_{j_1=1}^{\infty} \frac{q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}}.$$

Figure 6.



In the left-hand side of Dilcher's identity, $\left(\dfrac{q^k}{1-q^k}\right)^m$ is the generating function of m-tuples of rectangular Young diagrams whose vertical lengths are equal to k. Note the lower right angle of each rectangle. For a strict partition λ, there are $\binom{\lambda_{\ell(\lambda)}}{m}$ decompositions to the partitions of figure type. Therefore, in a language of Young diagrams, this identity is equivalent to the following.

$$\sum_{\lambda\in\mathcal{SP}}(-1)^{\ell(\lambda)-1}\binom{\lambda_{\ell(\lambda)}}{m}q^{|\lambda|}=\sum_{\lambda\in\mathcal{P}}\binom{\ell(\lambda)-c(\lambda)}{m-c(\lambda)}q^{|\lambda|}. \tag{4}$$

Proof of Theorem 10. Total sum of the left-hand side of Theorem 9 is

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sharp\left\{(\lambda;i_1,\dots,i_m)\;\middle|\;\begin{array}{c}\lambda\in\mathcal{SP}(n),1\leqslant i_1<\ldots<i_m\leqslant\lambda_{\ell(\lambda)}\\ a_{1,i_m}(\lambda)=k,\ell(\lambda):\text{odd}\end{array}\right\}\\
-&\sum_{k=1}^{\infty}\sharp\left\{(\lambda;i_1,\dots,i_m)\;\middle|\;\begin{array}{c}\lambda\in\mathcal{SP}(n),1\leqslant i_1<\ldots<i_m\leqslant\lambda_{\ell(\lambda)}\\ a_{1,i_m}(\lambda)=k,\ell(\lambda):\text{even}\end{array}\right\}\\
=&\sum_{\lambda\in\mathcal{SP}(n)}(-1)^{\ell(\lambda)-1}\binom{\lambda_{\ell(\lambda)}}{m}.
\end{aligned}$$

And sum of right-hand side is

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sharp\left\{(\lambda;t_1,\dots,t_{m-c(\lambda)})\;\middle|\;\begin{array}{c}\lambda\in\mathcal{P}(n),c(\lambda)\leqslant m,\lambda_1=k,\lambda_{t_i}=\lambda_{t_i+1}\\ 1\leqslant t_1<\ldots<t_{m-c(\lambda)}<\ell(\lambda)\end{array}\right\}\\
=&\sum_{\lambda\in\mathcal{P}(n)}\binom{\ell(\lambda)-c(\lambda)}{m-c(\lambda)}.
\end{aligned}$$

They are the coefficients of q^n in (4) □

Analogously with Theorem 7, we have an identity by taking the sum over k from 1 to t for Theorem 9.

Theorem 11 (Dilcher's identity 2)**.** *For any positive integers m and t,*

$$\sum_{k=1}^{t}(-1)^{k-1}\frac{q^{\frac{k(k-1)}{2}+mk}}{(1-q^k)^m}\begin{bmatrix}t\\k\end{bmatrix}=\sum_{j_1=1}^{t}\frac{q^{j_1}}{1-q^{j_1}}\sum_{j_2=1}^{j_1}\frac{q^{j_2}}{1-q^{j_2}}\cdots\sum_{j_m=1}^{j_{m-1}}\frac{q^{j_m}}{1-q^{j_m}}.$$

Theorem 9 is a generalization of Dilcher's identities. The both sides are the coefficients of b^kq^n in the following generating functions

Theorem 12. *For any positive integer m,*

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{b^kq^{\frac{k(k-1)}{2}+mk}}{(bq;q)_k(1-q^k)^m}=\sum_{j_1=1}^{\infty}\frac{b^{j_1}q^{j_1}}{1-q^{j_1}}\sum_{j_2=1}^{j_1}\frac{q^{j_2}}{1-q^{j_2}}\cdots\sum_{j_m=1}^{j_{m-1}}\frac{q^{j_m}}{1-q^{j_m}}.$$

Theorem 13. *For any positive integers m and t,*

$$\sum_{k=1}^{t}(-1)^{k-1}\frac{b^k q^{\frac{k(k-1)}{2}+mk}}{(1-q^k)^m}\left[\begin{array}{c} t \\ k \end{array}\right]_{q,b} = \sum_{j_1=1}^{t}\frac{b^{j_1}q^{j_1}}{1-q^{j_1}}\sum_{j_2=1}^{j_1}\frac{q^{j_2}}{1-q^{j_2}}\cdots\sum_{j_m=1}^{j_{m-1}}\frac{q^{j_m}}{1-q^{j_m}},$$

where $\left[\begin{array}{c} t \\ k \end{array}\right]_{q,b}$ *is defined by*

$$\left[\begin{array}{c} t \\ k \end{array}\right]_{q,b} = \sum_{\substack{\lambda\in\mathcal{P} \\ \lambda_1\leqslant t-k,\ell(\lambda)\leqslant k}} b^{\lambda_1}q^{|\lambda|}.$$

When $b=1$, they are Dilcher's identities.

Proof of Theorem 13. By analogous transform in proof of Theorem 10, this identity is equivalent to the following.

$$\sum_{\lambda\in\mathcal{SP}}\sum_{i=1}^{\lambda_{\ell(\lambda)}}(-1)^{\ell(\lambda)-1}\binom{i-1}{m-1}b^{a_{1,i}(\lambda)}q^{|\lambda|} = \sum_{\lambda\in\mathcal{P}}\binom{\ell(\lambda)-c(\lambda)}{m-c(\lambda)}b^{\lambda_1}q^{\lambda}.$$

On the left-hand side of this identity, i is the column number which has the right most gray box in Figure 6. And, $\binom{i-1}{m-1}$ is the number of arrangement of other gray box. Then, we count the left-hand side of Theorem 9, taking care about i_m.

$$\begin{aligned}
&\sharp\left\{(\lambda;i_1,\ldots,i_m)\;\middle|\;\begin{array}{c}\lambda\in\mathcal{SP}(n),1\leqslant i_1<\ldots<i_m\leqslant\lambda_{\ell(\lambda)} \\ a_{1,i_m}(\lambda)=k,\ell(\lambda):\text{odd}\end{array}\right\} \\
-\;&\sharp\left\{(\lambda;i_1,\ldots,i_m)\;\middle|\;\begin{array}{c}\lambda\in\mathcal{SP}(n),1\leqslant i_1<\ldots<i_m\leqslant\lambda_{\ell(\lambda)} \\ a_{1,i_m}(\lambda)=k,\ell(\lambda):\text{even}\end{array}\right\} \\
=\;&\sum_{\lambda\in\mathcal{SP}(n)}\sum_{i\leqslant\lambda_\ell(\lambda),a_{1,i}(\lambda)=k}(-1)^{\ell(\lambda)-1}\binom{i-1}{m-1}.
\end{aligned}$$

And right-hand side is

$$\begin{aligned}
&\sharp\left\{(\lambda;t_1,\ldots,t_{m-c(\lambda)})\;\middle|\;\begin{array}{c}\lambda\in\mathcal{P}(n),c(\lambda)\leqslant m,\lambda_1=k,\lambda_{t_i}=\lambda_{t_i+1} \\ 1\leqslant t_1<\ldots<t_{m-c(\lambda)}<\ell(\lambda)\end{array}\right\} \\
=\;&\sum_{\lambda\in\mathcal{P}(n),\lambda_1=k}\binom{\ell(\lambda)-c(\lambda)}{m-c(\lambda)}.
\end{aligned}$$

Therefore the coefficients of b^kq^n of Theorem 12 are equal to the both sides of Theorem 9. □

Theorem 13 is the finite version of Theorem 12. The author does not have the good display of this analogue of q-binomial coefficient. We say that $\left[\begin{array}{c} t \\ k \end{array}\right]_{q,b}$ satisfy following recurrence formula.

$$(bq)^{t-k} \begin{bmatrix} t \\ k \end{bmatrix} + \begin{bmatrix} t \\ k+1 \end{bmatrix}_{q,b} = \begin{bmatrix} t+1 \\ k+1 \end{bmatrix}_{q,b}.$$

We remark that the first term is ordinary q-binomial coefficient and we will finish.

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