# Non-classical hyperplanes of $DW(5,q)$

Bart De Bruyn

Department of Mathematics
Ghent University
Belgium

bdb@cage.ugent.be

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**Abstract**

The hyperplanes of the symplectic dual polar space $DW(5,q)$ arising from embedding, the so-called classical hyperplanes of $DW(5,q)$, have been determined earlier in the literature. In the present paper, we classify non-classical hyperplanes of $DW(5,q)$. If $q$ is even, then we prove that every such hyperplane is the extension of a non-classical ovoid of a quad of $DW(5,q)$. If $q$ is odd, then we prove that every non-classical ovoid of $DW(5,q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5,q)$. If $DW(5,q)$, $q$ odd, has a semi-singular hyperplane, then $q$ is not a prime number.

**Keywords:** symplectic dual polar space, hyperplane, projective embedding

## 1 Introduction

The hyperplanes of the finite symplectic dual polar space $DW(5,q)$ that arise from some projective embedding, the so-called classical hyperplanes of $DW(5,q)$, have explicitly been determined earlier in the literature, see Cooperstein & De Bruyn [5], De Bruyn [7] and Pralle [21]. In the present paper, we give a rather complete classification for the non-classical hyperplanes of $DW(5,q)$. There are two standard constructions for such hyperplanes.

(1) Suppose $x$ is a point of $DW(5,q)$ and $O$ is a set of points of $DW(5,q)$ at distance 3 from $x$ such that every line at distance 2 from $x$ has a unique point in common with $O$. Then $x^{\perp} \cup O$ is a non-classical hyperplane of $DW(5,q)$, the so-called semi-singular hyperplane with deepest point $x$.

(2) Suppose $Q$ is a quad of $DW(5,q)$. Then the points and lines contained in $Q$ define a generalized quadrangle $\widetilde{Q}$ isomorphic to $Q(4,q)$. If $O$ is a non-classical ovoid of $\widetilde{Q}$, then

the set of points of $DW(5,q)$ at distance at most 1 from $O$ is a non-classical hyperplane of $DW(5,q)$, the so-called *extension of* $O$. Several classes of non-classical ovoids of $Q(4,q)$ are known, see Section 2.2 for a discussion.

The following is our main result.

**Theorem 1.** (1) *If $q$ is even, then every non-classical hyperplane of $DW(5,q)$ is the extension of a non-classical ovoid of a quad of $DW(5,q)$.*

(2) *If $q$ is odd, then every non-classical hyperplane of $DW(5,q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5,q)$.*

Up to present, no semi-singular hyperplane of $DW(5,q)$ is known to exist. If a semi-singular hyperplane of $DW(5,q)$ exists, then $q$ must be odd (Theorem 19) and not a prime (Corollary 18).

The lines and quads through a given point $x$ of $DW(5,q)$ define a projective plane isomorphic to $PG(2,q)$ which we denote by $Res(x)$. If $H$ is a hyperplane of $DW(5,q)$ and $x$ is a point of $H$, then $\Lambda_H(x)$ denotes the set of lines through $x$ contained in $H$. We regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If $\Lambda_H(x)$ is the whole set of points of $Res(x)$, then $x$ is called *deep with respect to* $H$.

The dual polar space $DW(5,q)$ has a nice full projective embedding $e$ in the projective space $PG(13,q)$, which is called the *Grassmann embedding* of $DW(5,q)$, see e.g. Cooperstein [4, Proposition 5.1]. A hyperplane of $DW(5,q)$ whose image under $e$ is contained in a hyperplane of of $PG(13,q)$ is said to arise from $e$. For a proof of the following proposition, we refer to Pasini [16, Theorem 9.3] or Cardinali & De Bruyn [3, Corollary 1.5].

**Proposition 2.** *If $H$ is a hyperplane of $DW(5,q)$ arising from the Grassmann embedding of $DW(5,q)$, then for every point $x$ of $H$, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$:* (1) *a point;* (2) *a line;* (3) *the union of two distinct lines;* (4) *a nonsingular conic;* (5) *the whole set of points of $Res(x)$.*

If $q \neq 2$, then the Grassmann embedding of $DW(5,q)$ is the so-called absolutely universal embedding of $DW(5,q)$ (Cooperstein [4, Theorem B], Kasikova & Shult [12, Section 4.6], Ronan [22]), implying that the classical hyperplanes of $DW(5,q)$ are precisely those hyperplanes arising from the Grassmann embedding. Combining Theorem 1 with Proposition 2, we easily find:

**Corollary 3.** *If $H$ is a hyperplane of $DW(5,q)$, $q \neq 2$, then for every point $x$ of $H$, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$:* (1) *the empty set;* (2) *a point;* (3) *a line;* (4) *the union of two distinct lines;* (5) *a nonsingular conic;* (6) *the whole set of points of $Res(x)$. If $\Lambda_H(x)$ is the empty set, then $H$ is a semi-singular hyperplane whose deepest point lies at distance* 3 *from $x$. If $H$ is not a semi-singular hyperplane, then case* (1) *cannot occur.*

The conclusion of Corollary 3 is false for the dual polar space $DW(5,2)$. If $x$ is a point of $DW(5,2)$, then for every set $Y$ of points of $Res(x) \cong \mathrm{PG}(2,2)$, there exists a hyperplane $H$ through $x$ such that $\Lambda_H(x) = Y$, see Pralle [21, Table 1].

If $n \geqslant 4$, then the symplectic dual polar space $DW(2n-1,q)$ has many full subgeometries isomorphic to $DW(5,q)$. So, Corollary 3 reveals information on the local structure of any hyperplane of any symplectic dual polar space $DW(2n-1,q)$, where $q \neq 2$ and $n \geqslant 4$.

Theorem 1 will be proved in Section 3. In Section 2, we give the basic definitions (including some of the notions already mentioned above) and basic properties which will play a role in the proof of Theorem 1.

# 2 Preliminaries

## 2.1 The dual polar space $DW(5,q)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with nonempty point-set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. A set $H \subsetneq \mathcal{P}$ is called a *hyperplane* of $\mathcal{S}$ if every line of $\mathcal{S}$ has either one or all of its points in $H$. A *full projective embedding* of $\mathcal{S}$ is an injective mapping $e$ from $\mathcal{P}$ to the point-set of a projective space $\Sigma$ satisfying (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $\{e(x) \,|\, (x,L) \in \mathrm{I}\}$ is a line of $\Sigma$ for every line $L$ of $\mathcal{S}$. If $e : \mathcal{S} \to \Sigma$ is a projective embedding of $\mathcal{S}$ and $\Pi$ is a hyperplane of $\Sigma$, then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. A hyperplane of $\mathcal{S}$ is said to be *classical* if it is of the form $e^{-1}(e(\mathcal{P}) \cap \Pi)$, where $e$ is some full projective embedding of $\mathcal{S}$ into a projective space $\Sigma$ and $\Pi$ is some hyperplane of $\Sigma$.

Distances $\mathrm{d}(\cdot,\cdot)$ in $\mathcal{S}$ will be measured in its collinearity graph. If $x$ is a point of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points of $\mathcal{S}$ at distance $i$ from $x$. Similarly, if $X$ is a nonempty set of points and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points at distance $i$ from $X$, i.e. the set of all points $y$ for which $\min\{\mathrm{d}(y,x) \,|\, x \in X\} = i$.

Let $W(5,q)$ be the polar space whose subspaces are the subspaces of $\mathrm{PG}(5,q)$ that are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(5,q)$, and let $DW(5,q)$ denote the associated dual polar space. The points and lines of $DW(5,q)$ are the totally isotropic planes and lines of $\mathrm{PG}(5,q)$, with incidence being reverse containment. The dual polar space $DW(5,q)$ belongs to the class of *near polygons* introduced by Shult and Yanushka in [23]. This means that for every point $x$ and every line $L$, there exists a unique point on $L$ nearest to $x$. The maximal distance between two points of $DW(5,q)$ is equal to 3. The dual polar space $DW(5,q)$ has $(q+1)(q^2+1)(q^3+1)$ points, $q+1$ points on each line and $q^2+q+1$ lines through each point.

If $x$ and $y$ are two points of $DW(5,q)$ at distance 2 from each other, then the smallest convex subspace $\langle x,y \rangle$ of $DW(5,q)$ containing $x$ and $y$ is called a *quad*. A quad $Q$ of $DW(5,q)$ consists of all totally isotropic planes of $W(5,q)$ that contain a given point $x_Q$ of $W(5,q)$. Any two lines $L$ and $M$ of $DW(5,q)$ that meet in a unique point are contained in a unique quad. We denote this quad by $\langle L,M \rangle$. Obviously, we have $\langle L,M \rangle = \langle x,y \rangle$ where $x$ and $y$ are arbitrary points of $L \setminus M$ and $M \setminus L$, respectively. The points and lines of $DW(5,q)$ that are contained in a given quad $Q$ define a point-line geometry $\widetilde{Q}$ isomorphic to the generalized quadrangle $Q(4,q)$ of the points and lines of a nonsingular parabolic quadric of $\mathrm{PG}(4,q)$. If $Q$ is a quad of $DW(5,q)$ and $x$ is a point not contained in $Q$, then $Q$ contains a unique point $\pi_Q(x)$ collinear with $x$ and $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_Q(x), y)$ for every point $y$ of $Q$. If $Q_1$ and $Q_2$ are two distinct quads of $DW(5,q)$, then $Q_1 \cap Q_2$ is either empty or a line of $DW(5,q)$. If $Q_1 \cap Q_2 = \emptyset$, then the map $Q_1 \to Q_2; x \mapsto \pi_{Q_2}(x)$ is an isomorphism between $\widetilde{Q_1}$ and $\widetilde{Q_2}$.

## 2.2 Hyperplanes of $Q(4,q)$

By Payne and Thas [18, 2.3.1], every hyperplane of the generalized quadrangle $Q(4,q)$ is either the perp $x^\perp$ of a point $x$, a $(q+1) \times (q+1)$-subgrid or an ovoid. An ovoid of $Q(4,q)$ is *classical* if it is an elliptic quadric $Q^-(3,q) \subseteq Q(4,q)$. For many values of $q$, non-classical ovoids of $Q(4,q)$ do exist: (i) $q = p^h$ with $p$ an odd prime and $h \geqslant 2$ [11]; (ii) $q = 2^{2h+1}$ with $h \geqslant 1$ [26]; (iii) $q = 3^{2h+1}$ with $h \geqslant 1$ [11]; (iv) $q = 3^h$ with $h \geqslant 3$ [24]; (v) $q = 3^5$ [19]. For several prime powers $q$, it is known that all ovoids of $Q(4,q)$ are classical:

**Proposition 4.** • **([2, 15])** *Every ovoid of* $Q(4,4)$ *is classical.*
- **([13, 14])** *Every ovoid of* $Q(4,16)$ *is classical.*
- **([1])** *Every ovoid of* $Q(4,q)$*,* $q$ *prime, is classical.*

A set $\mathcal{G}$ of hyperplanes of $Q(4,q)$ is called a *pencil of hyperplanes* if every point of $Q(4,q)$ is contained in either 1 or all elements of $\mathcal{G}$. The following lemma is precisely Lemma 3.2 and Corollary 3.3 of De Bruyn [8].

**Lemma 5.** *If* $G_1$ *and* $G_2$ *are two distinct classical hyperplanes of* $Q(4,q)$*, then through every point* $x$ *of* $Q(4,q)$ *not contained in* $G_1 \cup G_2$*, there exists a unique classical hyperplane* $G_x$ *satisfying* $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$*. As a consequence, any two distinct classical hyperplanes of* $Q(4,q)$ *are contained in a unique pencil of classical hyperplanes of* $Q(4,q)$*.*

## 2.3 Hyperplanes of $DW(5,q)$

Since $DW(5,q)$ is a near polygon, the set of points of $DW(5,q)$ at distance at most 2 from a given point $x$ is a hyperplane of $DW(5,q)$, the so-called *singular hyperplane with deepest point* $x$. If $O$ is a set of points of $DW(5,q)$ at distance 3 from a given point $x$ such that every line at distance 2 from $x$ has a unique point in common with $O$, then $x^\perp \cup O$ is a hyperplane of $DW(5,q)$, a so-called *semi-singular hyperplane of* $DW(5,q)$ *with deepest point* $x$. If $Q$ is a quad of $DW(5,q)$ and $G$ is a hyperplane of $\widetilde{Q} \cong Q(4,q)$, then $Q \cup \{x \in \Gamma_1(Q) \,|\, \pi_Q(x) \in G\}$ is a hyperplane of $DW(5,q)$, the so-called *extension* of $G$.

If $H$ is a hyperplane of $DW(5,q)$ and $Q$ is a quad, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of $Q \cong Q(4,q)$. If $Q \subseteq H$, then $Q$ is called a *deep quad.* If $Q \cap H = x^\perp \cap Q$ for some point $x \in Q$, then $Q$ is called *singular* with respect to $H$ and $x$ is called the *deep* *point* of $Q$. The quad $Q$ is called *ovoidal* (respectively, *subquadrangular*) with respect to $H$ if and only if $Q \cap H$ is an ovoid (respectively, a $(q+1) \times (q+1)$-subgrid) of $Q$. A hyperplane $H$ of $DW(5,q)$ is called *locally singular* (*locally subquadrangular*, respectively *locally ovoidal*) if every non-deep quad of $DW(5,q)$ is singular (subquadrangular, respectively ovoidal) with respect to $H$. A hyperplane that is locally singular, locally ovoidal or locally subquadrangular is also called a *uniform hyperplane*. In the following proposition, we collect a number of known results which we will need to invoke later in the proof of the Main Theorem.

**Proposition 6.** (1) *The dual polar space $DW(5,q)$, $q \neq 2$, has no locally subquadrangular hyperplanes.*

(2) *The dual polar space $DW(5,q)$ has no locally ovoidal hyperplanes.*

(3) *Every nonuniform hyperplane of $DW(5,q)$ admits a singular quad.*

Proposition 6(1) is due to Pasini & Shpectorov [17]. Locally ovoidal hyperplanes of $DW(5,q)$ are just ovoids and cannot exist by Thomas [25, Theorem 3.2], see also Cooperstein and Pasini [6]. Proposition 6(3) is due to Pralle [20].

The classical hyperplanes of the dual polar space $DW(5,q)$ have already been classified in the literature. The dual polar space $DW(5,q)$, $q \neq 2$, has six isomorphism classes of classical hyperplanes by Cooperstein & De Bruyn [5] and De Bruyn [7]. This fact is not true if $q = 2$. The dual polar space $DW(5,2)$ has twelve isomorphism classes of hyperplanes by Pralle [21], see also De Bruyn [7, Section 9]. Observe that all these hyperplanes are classical by Ronan [22, Corollary 2]. By De Bruyn [8], the classical hyperplanes of $DW(5,q)$ can be characterized as follows.

**Proposition 7.** *The classical hyperplanes of $DW(5,q)$ are precisely those hyperplanes $H$ of $DW(5,q)$ that satisfy the following property: if $Q$ is an ovoidal quad, then $Q \cap H$ is a classical ovoid of $Q$.*

## 2.4 Hyperbolic sets of quads of $DW(5,q)$

As in Section 2.1, let $W(5,q)$ be the polar space associated with a symplectic polarity of $\mathrm{PG}(5,q)$. If $L$ is a hyperbolic line of $\mathrm{PG}(5,q)$ (i.e. a line of $\mathrm{PG}(5,q)$ that is not a line of $W(5,q)$), then the set of the $q+1$ (mutually disjoint) quads of $DW(5,q)$ corresponding to the points of $L$ satisfy the property that every line that meets at least two of its members meets each of its members in a unique point. Any set of $q+1$ quads that is obtained in this way will be called a *hyperbolic set of quads* of $DW(5,q)$. Every two disjoint quads $Q_1$ and $Q_2$ of $DW(5,q)$ are contained in a unique hyperbolic set of quads of $DW(5,q)$. We will denote this hyperbolic set of quads by $\mathcal{H}(Q_1,Q_2)$. Considering all the lines meeting $Q_1$ and $Q_2$, we easily see that the following holds.

**Lemma 8.** *Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ and let $H$ be a hyperplane of $DW(5,q)$ such that $H \cap Q_1$ and $\pi_{Q_1}(H \cap Q_2)$ are distinct hyperplanes of $\widetilde{Q_1}$. Then $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leqslant i \leqslant q+1\}$ is a pencil of hyperplanes of $\widetilde{Q_1}$.*

# 3 Proof of Theorem 1

Throughout this section, we suppose that $H$ is an arbitrary hyperplane of $DW(5,q)$. In De Bruyn [9], we classified for every field $\mathbb{K}$ of size at least three the hyperplanes of $DW(5,\mathbb{K})$ containing a quad. The main theorem of [9] implies the following:

**Proposition 9.** *Every non-classical hyperplane of $DW(5,q)$, $q \neq 2$, containing a quad is the extension of a non-classical ovoid of a quad.*

We have already mentioned above that every hyperplane of $DW(5,2)$ is classical by Ronan [22, Corollary 2]. Since we are interested in the classification of all non-classical hyperplanes of $DW(5,q)$, we may by the above assume that the following holds:

**Assumption:** We have $q \geqslant 3$ and the hyperplane $H$ does not contain quads.

We denote by $v$ the total number of points of $H$ and by $l$ the total number of lines of $DW(5,q)$ contained in $H$. In Section 3.1, we prove that there are only three possible values for $v$, namely $q^5+q^3+q^2+q+1$, $q^5+q^4+q^3+q^2+2q+1$ or $q^5+q^4+q^3+q^2+q+1$. In Section 3.2, we prove that if $v=q^5+q^3+q^2+q+1$, then $H$ is a semi-singular hyperplane. We also prove there that semi-singular hyperplanes cannot exist if $q$ is even. In [10] (see also Corollary 18), the nonexistence of semi-singular hyperplanes was already shown for prime values of $q$. In Section 3.3, we prove that the case $v=q^5+q^4+q^3+q^2+2q+1$ cannot occur and in Section 3.4, we prove that $H$ must be classical if $v=q^5+q^4+q^3+q^2+q+1$. All these results together imply that Theorem 1 must hold.

## 3.1 The possible values of $v$

The following lemma is an immediate consequence of Proposition 6.

**Lemma 10.** *The hyperplane admits singular quads.*

**Lemma 11.** *We have $l = \frac{v\cdot(q^2+q+1)-(q^2+1)(q^3+1)(q^2+q+1)}{q}$.*

**Proof.** We count the number of lines not contained in $H$. There are $(q+1)(q^2+1)(q^3+1)-v$ points outside $H$ and each of these points is contained in $q^2+q+1$ lines which contain a unique point of $H$. Hence, the total number of lines not contained in $H$ is equal to $\frac{((q+1)(q^2+1)(q^3+1)-v)(q^2+q+1)}{q}$. Since the total number of lines of $DW(5,q)$ equals $(q^2+1)(q^3+1)(q^2+q+1)$, we have $l=(q^2+1)(q^3+1)(q^2+q+1)-\frac{((q+1)(q^2+1)(q^3+1)-v)(q^2+q+1)}{q} = \frac{v\cdot(q^2+q+1)-(q^2+1)(q^3+1)(q^2+q+1)}{q}$. □

**Lemma 12.** *If $Q$ is a singular quad with deep point $x$, then one of the following cases occurs:*

(1) $x^\perp \cap H = x^\perp \cap Q$;

(2) *there exists a line $L$ through $x$ not contained in $Q$ such that* $x^\perp \cap H = (x^\perp \cap Q) \cup L$;

(3) *there exists a quad $R$ through $x$ distinct from $Q$ such that* $x^\perp \cap H = (x^\perp \cap Q) \cup (x^\perp \cap R)$;

(4) $x^\perp \subseteq H$.

**Proof.** Since $x^\perp \cap Q \subseteq x^\perp \cap H$, $|\Lambda_H(x)| \geqslant q+1$. If $|\Lambda_H(x)| \in \{q+1, q+2\}$, then either case (1) or (2) of the lemma occurs. Suppose therefore that $|\Lambda_H(x)| \geqslant q+3$ and let $L_1$ and $L_2$ be two distinct lines through $x$ that are contained in $H$, but not in $Q$. Put $R := \langle L_1, L_2 \rangle$. Since $L_1 \subseteq R \cap H$, $L_2 \subseteq R \cap H$ and $R \cap Q \subseteq R \cap H$, $R$ is singular with deep point $x$ and hence every line of $R$ through $x$ is contained in $H$. So, $|\Lambda_H(x)| \geqslant 2q+1$.

If $|\Lambda_H(x)| = 2q+1$, then obviously case (3) of the lemma occurs. Suppose therefore that $|\Lambda_H(x)| \geqslant 2q+2$. Then there exists a line $L_3 \subseteq H$ through $x$ not contained in $Q \cup R$. If $Q'$ is a quad through $L_3$ distinct from $\langle L_3, Q \cap R \rangle$, then since $Q' \cap Q \subseteq H$, $Q' \cap R \subseteq H$ and $L_3 \subseteq H$, $Q'$ is singular with deep point $x$ and hence every line of $Q'$ through $x$ is contained in $H$. It follows that all lines of $DW(5,q)$ through $x$ are contained in $H$, except maybe for the $q-1$ lines through $x$ contained in $\langle L_3, Q \cap R \rangle$ and distinct from $L_3$ and $Q \cap R$. Let $L'$ be one of these $q-1$ lines and let $Q''$ be a quad through $L'$ distinct from $\langle L_3, Q \cap R \rangle$. Since $q \geqslant 3$ lines of $Q''$ through $x$ are contained in $H$, $Q''$ is singular with deep point $x$ and hence also $L'$ is contained in $H$. So, $x^\perp \subseteq H$ and case (4) of the lemma occurs. □

**Lemma 13.** *If $Q$ is a singular quad with deep point $x$, then $|\Gamma_3(x) \cap H| = q^5$.*

**Proof.** Every point of $\Gamma_3(x) \cap H$ is collinear with a unique point of $\Gamma_2(x) \cap Q$. Conversely, every point $u$ of $\Gamma_2(x) \cap Q$ is collinear with precisely $q^2$ points of $\Gamma_3(x) \cap H$. (One on each line through $u$ not contained in $Q$.) Hence, $|\Gamma_3(x) \cap H| = |\Gamma_2(x) \cap Q| \cdot q^2 = q^5$. □

**Lemma 14.** *Suppose $Q$ is a singular quad with deep point $x$.*

- *If case (1) of Lemma 12 occurs, then $v = q^5+q^4+q^3+q^2+q+1$ and $l = q^5+q^4+q^3+q^2+q+1$.*
- *If case (2) of Lemma 12 occurs, then $v = q^5+q^4+q^3+q^2+2q+1$ and $l = (q^2+q+1)(q^3+2)$.*
- *If case (3) of Lemma 12 occurs, then $v = q^5+q^4+q^3+q^2+q+1$ and $l = q^5+q^4+q^3+q^2+q+1$.*
- *If case (4) of Lemma 12 occurs, then $v = q^5+q^3+q^2+q+1$ and $l = q^2+q+1$.*

**Proof.** Suppose case (1) of Lemma 12 occurs. Then $x$ is contained in 1 singular quad that has $x$ as deep point (namely $Q$) and $q^2+q$ singular quads that do not have $x$ as deep point. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = q^2+q$, $|\Gamma_2(x) \cap H| = 1 \cdot 0 + (q^2+q) \cdot q^2$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (q^2+q) + (q^2+q) \cdot q^2 + q^5 = q^5+q^4+q^3+q^2+q+1$.

Suppose case (2) of Lemma 12 occurs. Then $x$ is contained in 1 singular quad with deep point equal to $x$, $q+1$ subquadrangular quads and $q^2-1$ singular quads with deep point different from $x$. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = (q+2)q = q^2+2q$, $|\Gamma_2(x) \cap H| = 1 \cdot 0 + (q+1) \cdot q^2 + (q^2-1) \cdot q^2 = q^4+q^3$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (q^2+2q) + (q^4+q^3) + q^5 = q^5+q^4+q^3+q^2+2q+1$.

Suppose case (3) of Lemma 12 occurs. Then $x$ is contained in 2 singular quads with deep point $x$, $q-1$ singular quads with deep point different from $x$ and $q^2$ subquadrangular quads. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = (2q+1)q = 2q^2 + q$, $|\Gamma_2(x) \cap H| = 2 \cdot 0 + (q-1) \cdot q^2 + q^2 \cdot q^2 = q^4 + q^3 - q^2$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (2q^2 + q) + (q^4 + q^3 - q^2) + q^5 = q^5 + q^4 + q^3 + q^2 + q + 1$.

Suppose case (4) of Lemma 12 occurs. Then $x$ is contained in $q^2+q+1$ singular quads that have $x$ as deep point. Hence, $v = |\Gamma_0(x) \cap H| + |\Gamma_1(x) \cap H| + |\Gamma_2(x) \cap H| + |\Gamma_3(x) \cap H| = 1 + q(q^2+q+1) + 0 + q^5 = q^5 + q^3 + q^2 + q + 1$.

In each of the four cases, the value of $l$ can be derived from Lemma 11. □

By Lemmas 10, 12 and 14, we have:

**Corollary 15.** *$v \in \{q^5+q^3+q^2+q+1, q^5+q^4+q^3+q^2+q+1, q^5+q^4+q^3+q^2+2q+1\}$.*

We see that if case (2) of Lemma 12 occurs for one singular quad $Q$, then case (2) occurs for all singular quads $Q$. A similar remark holds applies to case (4) of Lemma 12.

## 3.2 The case $v = q^5 + q^3 + q^2 + q + 1$

Let $Q^*$ denote a singular quad and $x^*$ its deep point.

**Lemma 16.** *If $v = q^5+q^3+q^2+q+1$, then $H$ is a semi-singular hyperplane of $DW(5,q)$ with deepest point $x^*$.*

**Proof.** If $v = q^5 + q^3 + q^2 + q + 1$, then case (4) of Lemma 12 occurs for the pair $(Q^*, x^*)$. So, we have that $x^{*\perp} \subseteq H$ and $\Gamma_2(x^*) \cap H = \emptyset$ (no deep quad through $x^*$). Since $\Gamma_2(x^*) \cap H = \emptyset$, every line at distance 2 from $x^*$ contains a unique point of $\Gamma_3(x^*) \cap H$. It follows that $H$ is a semi-singular hyperplane of $DW(5,q)$ with deepest point $x^*$. □

The following proposition was proved in De Bruyn and Vandecasteele [10, Corollary 6.3].

**Proposition 17.** *If $q$ is a prime power such that every ovoid of $Q(4,q)$ is classical, then $DW(5,q)$ does not have semi-singular hyperplanes.*

By Propositions 4 and 17, we have

**Corollary 18.** *If $q$ is prime, then $DW(5,q)$ has no semi-singular hyperplanes.*

We will now use hyperbolic sets of quads of $DW(5,q)$ to prove the nonexistence of semi-singular hyperplanes of $DW(5,q)$, $q$ even.

**Theorem 19.** *The dual polar space $DW(5,q)$, $q$ even, has no semi-singular hyperplanes.*

**Proof.** Suppose $H$ is a semi-singular hyperplane of $DW(5,q)$, $q$ even, and as before let $x^*$ denote the deepest point of $H$. Let $Q$ be a quad through $x^*$, let $G$ be a $(q+1) \times (q+1)$-subgrid of $\widetilde{Q}$ not containing $x^*$, let $L_1$ and $L_2$ be two disjoint lines of $G$ and let $Q_i$, $i \in \{1,2\}$, be a quad through $L_i$ distinct from $Q$. Then $Q_1$ and $Q_2$ are disjoint. Put

$\mathcal{H} = \mathcal{H}(Q_1, Q_2)$. Every $Q_3 \in \mathcal{H}$ intersects $Q$ in a line of $G$ and hence $x^* \notin Q_3$. It follows that every $Q_3 \in \mathcal{H}$ is ovoidal with respect to $H$. Suppose $Q_3 \in \mathcal{H} \setminus \{Q_1\}$ and $x_3 \in Q_3 \cap H$ such that $x_1 = \pi_{Q_1}(x_3) \in Q_1 \cap H$. Then the line $x_1x_3$ is contained in $H$ and hence $x^* \in x_1x_3$. But this is impossible, since no quad of $\mathcal{H}$ contains $x^*$. Hence, $\pi_{Q_1}(Q_3 \cap H)$ is disjoint from $Q_1 \cap H$. By Lemma 8, the set $\{\pi_{Q_1}(Q_3 \cap H) \,|\, Q_3 \in \mathcal{H}\}$ is a partition of $Q_1$ into ovoids. This is however impossible since the generalized quadrangle $Q(4, q)$, $q$ even, has no partition in ovoids by Payne and Thas [18, Theorem 1.8.5]. □

## 3.3 The case $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$

We suppose that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$ and $l = (q^2 + q + 1)(q^3 + 2)$. Recall that if $Q$ is a singular quad and $x$ is the deep point of $Q$, then case (2) of Lemma 12 occurs for the pair $(Q, x)$.

**Lemma 20.** *Let $Q$ be a singular quad, let $x$ be the deep point of $Q$, let $L$ be the line through $x$ not contained in $Q$ such that $x^\perp \cap H = (x^\perp \cap Q) \cup L$ and let $y$ be a point of $L \setminus \{x\}$. Then there are $q + 1$ lines $L_1, L_2, \ldots, L_{q+1}$ through $y$ different from $L$ that are contained in $H$. The $q + 2$ lines $L, L_1, L_2, \ldots, L_{q+1}$ form a hyperoval of the projective plane $Res(y) \cong \mathrm{PG}(2, q)$. (Hence, $q$ must be even.)*

**Proof.** The $q + 1$ quads $R_1, \ldots, R_{q+1}$ through $L$ determine a partition of the set of lines through $y$ different from $L$. Each of these quads is subquadrangular. Hence, $R_i$, $i \in \{1, 2, \ldots, q + 1\}$, contains a unique line $L_i \neq L$ through $y$ that is contained in $H$.

For all $i, j \in \{1, 2, \ldots, q + 1\}$ with $i \neq j$, the lines $L$, $L_i$ and $L_j$ are not contained in a quad since the quad $\langle L, L_i\rangle$ is subquadrangular. Suppose there exist mutually distinct $i, j, k \in \{1, 2, \ldots, q + 1\}$ such that $L_i$, $L_j$ and $L_k$ are contained in a quad $Q'$. Then $L$ is not contained in $Q'$ and hence $Q \cap Q' = \emptyset$. Since $L_i$, $L_j$ and $L_k$ are contained in $H$, $Q'$ is singular with deep point $y$. Let $z' \in Q' \setminus y^\perp$ and $z := \pi_Q(z')$. Since $z$ and $z'$ are not contained in $H$, the line $zz'$ contains a unique point $z'' \in H$. Let $Q''$ denote the unique quad through $z''$ intersecting $L$ in a point $u$. Then $Q'' \in \mathcal{H}(Q, Q')$. So, every point of $u^\perp \cap Q''$ is contained in a line joining a point of $y^\perp \cap Q'$ with a point of $x^\perp \cap Q$ and hence is contained in $H$. Since also $z'' \in H$, $Q'' \subseteq H$, contradicting the fact that there are no deep quads. □

**Lemma 21.** *There are four possible types of points in $H$:*

*($A$) points $x$ for which $\Lambda_H(x)$ is the union of a line of $Res(x)$ and a point of $Res(x)$ not belonging to that line;*

*($B$) points $x$ for which $\Lambda_H(x)$ is a hyperoval of $Res(x)$;*

*($C$) points $x$ for which $|\Lambda_H(x)| = 2$;*

*($D$) points $x$ for which $\Lambda_H(x)$ is empty.*

*Moreover, we have:*

*($i$) Every point of Type ($A$) has distance 1 from precisely $q^2 - 1$ points of Type ($A$), $q$ points of Type ($B$) and $q + 1$ points of Type ($C$).*

*($ii$) Every point of Type ($B$) has distance 1 from precisely $q + 2$ points of Type ($A$), $(q + 2)(q - 1)$ points of Type ($B$) and 0 points of Type ($C$).*

(*iii*) *Every point of Type* $(C)$ *has distance* 1 *from precisely* $2q$ *points of Type* $(A)$, 0 *points of Type* $(B)$ *and* 0 *points of Type* $(C)$.

**Proof.** Suppose $Q^*$ is a singular quad and $x^*$ is its deep point. Consider the collinearity graph $\Gamma$ of $DW(5,q)$ and let $\Gamma_H$ denote the subgraph of $\Gamma$ induced on the vertex set $H$. Suppose $x$ is a point of $H$ such that $x$ and $x^*$ belong to different connected components of $\Gamma_H$. We prove that $\Lambda_H(x)$ is empty. Suppose to the contrary that there exists a line $L$ through $x$ contained in $H$. If $L$ meets $Q^*$, then $L \cap Q^*$ must be contained in $x^{*\perp}$, contradicting the fact that $x^*$ and $x$ belong to different connected components of $\Gamma_H$. So, $L$ is disjoint from $Q^*$. Then $\pi_{Q^*}(L)$ meets $x^{*\perp}$ and hence $x^*$ and $x$ are connected by a path of $\Gamma_H$, again a contradiction.

Notice that by Lemma 14 and the fact that $v = q^5+q^4+q^3+q^2+2q+1$, $x^*$ is a point of Type (A). So, in order to prove the first part of the lemma, it suffices to verify that every vertex $x$ of Type $(X)$, $X \in \{A,B,C\}$, of $\Gamma_H$ is adjacent with only vertices of Type (A), (B) or (C). As a by-product of our verification, also the conclusions of the second part of the lemma will be obtained.

First, suppose that $x$ is a point of Type (A). Without loss of generality, we may suppose that $x = x^*$. Let $L^*$ denote the unique line through $x^*$ such that $x^{*\perp} \cap H = (x^{*\perp} \cap Q^*) \cup L^*$. By Lemma 20, every point of $L^* \setminus \{x^*\}$ has Type (B). Now, let $L$ be a line through $x^*$ contained in $Q^*$. Then $\langle L, L^* \rangle$ is a subquadrangular quad. Any quad through $L$ different from $\langle L, L^* \rangle$ and $Q^*$ is singular with deep point contained in $L \setminus \{x^*\}$. By Lemmas 12 and 14 and the fact that $v = q^5+q^4+q^3+q^2+2q+1$, every point of $L \setminus \{x^*\}$ is the deep point of at most 1 such singular quad. Hence, $q-1$ points of $L \setminus \{x^*\}$ have Type (A) and the remaining point of $L \setminus \{x^*\}$ has type (C).

Suppose $x$ is a point of Type (C). Let $L_1$ and $L_2$ denote the two lines through $x$ that are contained in $H$. Then $\langle L_1, L_2 \rangle$ is a subquadrangular quad. If $Q$ is a quad through $L_1$ distinct from $\langle L_1, L_2 \rangle$, then $Q$ is singular with deep point on $L_1 \setminus \{x\}$. By Lemmas 12 and 14 and the fact that $v = q^5+q^4+q^3+q^2+2q+1$, every point of $L_1 \setminus \{x\}$ is the deep point of at most 1 such singular quad. It follows that every point of $L_1 \setminus \{x\}$ has Type (A). In a similar way, one shows that every point of $L_2 \setminus \{x\}$ has Type (A).

Suppose $x$ is a point of Type (B). Let $L$ be an arbitrary line through $x$ contained in $H$. Every quad through $L$ is subquadrangular. It follows that through every point $u \in L$ there are precisely $q+2$ lines that are contained in $H$. If at least three of these lines are contained in a certain quad $R$, then $R$ is singular with deep point $u$ and hence $u$ is of type (A). Otherwise, $u$ is of type (B). By Lemma 20, there are two possibilities.

(1) $L$ contains a unique point of Type (A) and $q$ points of Type (B).

(2) $L$ contains $q+1$ points of Type (B).

We show that case (2) cannot occur. Suppose it does occur. Then $|\Gamma_0(L) \cap H| = q+1$ and $|\Gamma_1(L) \cap H| = (q+1)^2 q$. Each quad intersecting $L$ in a unique point is either ovoidal or subquadrangular and contributes $q^2$ to the value of $|\Gamma_2(L) \cap H|$. Since every point of $\Gamma_2(L)$ is contained in a unique quad that intersects $L$ in a unique point, $|\Gamma_2(L) \cap H| = (q+1)q^2 \cdot q^2$.

It follows that $|H| = |\Gamma_0(L)\cap H|+|\Gamma_1(L)\cap H|+|\Gamma_2(L)\cap H| = (q+1)+(q+1)^2q+(q+1)q^4 = q^5+q^4+q^3+2q^2+2q+1$, contradicting the fact that $|H| = q^5+q^4+q^3+q^2+2q+1$. $\square$

Now, let $n_A$, $n_B$, $n_C$ respectively $n_D$, denote the total number of points of $H$ of Type (A), (B), (C), respectively (D). Then by Lemma 21, we have $n_A \cdot q = n_B \cdot (q+2)$ and $n_A \cdot (q+1) = n_C \cdot 2q$. Hence,

$$n_B = \frac{n_A \cdot q}{q+2}, \tag{1}$$

$$n_C = \frac{n_A \cdot (q+1)}{2q}. \tag{2}$$

Now, counting in two different ways the number of pairs $(x, L)$, with $x \in H$ and $L$ a line through $x$ contained in $H$, we obtain

$$n_A \cdot (q+2) + n_B \cdot (q+2) + n_C \cdot 2 = l \cdot (q+1) = (q^2+q+1)(q+1)(q^3+2). \tag{3}$$

From equations (1), (2) and (3), we find $n_A = \frac{(q^2+q+1)(q^3+2)q}{2q+1}$, $n_B = \frac{(q^2+q+1)(q^3+2)q^2}{(q+2)(2q+1)}$ and $n_C = \frac{(q^2+q+1)(q^3+2)(q+1)}{2(2q+1)}$. If $q = 3$, then $n_A \notin \mathbb{N}$. If $q \geqslant 4$, then

$$\begin{aligned} n_A + n_B + n_C &= (q^2+q+1)(q^3+2) \cdot \frac{5q^2+7q+2}{2(q+2)(2q+1)} \\ &> (q^5+q^4+q^3+q^2+2q+1) \cdot 1 \\ &= v, \end{aligned}$$

a contradiction. Hence, the case $v = q^5+q^4+q^3+q^2+2q+1$ cannot occur.

## 3.4 The case $v = q^5 + q^4 + q^3 + q^2 + q + 1$

Suppose $v = q^5 + q^4 + q^3 + q^2 + q + 1$.

**Lemma 22.** *There are five possible types of points in $H$:*
(*A*) *points $x$ for which $|\Lambda_H(x)| = 1$;*
(*B*) *points $x$ for which $\Lambda_H(x)$ is a line of $Res(x)$;*
(*C*) *points $x$ for which $\Lambda_H(x)$ is the union of two distinct lines of $Res(x)$;*
(*D*) *points $x$ for which $\Lambda_H(x)$ is an oval of $Res(x)$;*
(*E*) *points $x$ for which $\Lambda_H(x)$ is empty.*

**Proof.** Suppose $Q^*$ is a singular quad and $x^*$ is its deep point. Consider the collinearity graph $\Gamma$ of $DW(5, q)$ and let $\Gamma_H$ denote the subgraph of $\Gamma$ induced on the vertex set $H$. Suppose $x$ is a point of $H$ such that $x$ and $x^*$ belong to different connected components of $\Gamma_H$. Then we prove that $\Lambda_H(x)$ is empty. Suppose to the contrary that there exists a line $L$ through $x$ contained in $H$. If $L$ meets $Q^*$, then $L \cap Q^*$ must be contained in $x^{*\perp}$, contradicting the fact that $x^*$ and $x$ belong to different connected components of $\Gamma_H$. So,

$L$ is disjoint from $Q^*$. Then $\pi_{Q^*}(L)$ meets $x^{*\perp}$ and hence $x^*$ and $x$ are connected by a path of $\Gamma_H$, again a contradiction.

By Lemmas 12 and 14 applied to the pair $(Q^*, x^*)$, $x^*$ is a point of Type (B) or (C). So, in order to prove the lemma, it suffices to prove that if $x$ is a point of Type $(X) \in \{(A), (B), (C), (D)\}$ and $y$ is a point of $H \setminus \{x\}$ collinear with $x$, then $y$ is of Type (A), (B), (C) or (D). Put $L := xy$. Since $x$ is of Type (A), (B), (C) or (D), one of the following two possibilities occurs:

(1) $L$ is contained in $q+1$ singular quads with deep point on $L$.

(2) $L$ is contained in a unique singular quad with deep point on $L$ and $q$ subquadrangular quads.

Observe that case (1) can only occur if $x$ has Type (A), (B) or (C), while case (2) can only occur if $x$ has Type (C) or (D).

Suppose case (1) occurs. Then $\Lambda_H(y)$ is the union of a number of lines of $Res(y)$ through a given point of $Res(y)$, union this point. Since every quad through $y$ is singular, subquadrangular or ovoidal, every line of $Res(y)$ intersects $\Lambda_H(y)$ in either 0, 1, 2 or $q+1$ points. Notice also that the point $y$ cannot be deep with respect to $H$, since otherwise Lemmas 12 and 14 applied to any singular quad through $y$ would yield that $v = q^5 + q^3 + q^2 + q + 1$, which is impossible. It follows that $y$ is of Type (A), (B) or (C).

If case (2) occurs, then there are two possibilities:

(2a) $\Lambda_H(y)$ is a line of $Res(y)$ + $q$ extra points. By Lemma 12, $y$ necessarily is a point of Type (C).

(2b) $|\Lambda_H(y)| = q+1$. If at least three of the points of $\Lambda_H(y)$ are collinear, then $\Lambda_H(y)$ is necessarily a line of $Res(y)$. But this is impossible since $y$ is not the deep point of a singular quad through $L$. So, no three points of $\Lambda_H(y)$ are collinear. This implies that $\Lambda_H(y)$ is an oval of $Res(y)$, i.e. $y$ is a point of Type (D). ☐

**Definition.** As we have already noticed in the proof of Lemma 22, every line $L \subseteq H$ must be contained in either $q+1$ singular quads or one singular quad and $q$ subquadrangular quads. If all quads on $L$ are singular, then $L$ is said to be *special*.

**Lemma 23.** *If $L$ is a special line, then $L$ contains only points of Type* (*A*), (*B*) *and* (*C*)*. Moreover, the number of points of Type* (*A*) *on $L$ equals the number of points of Type* (*C*) *on $L$.*

**Proof.** Since every quad through $L$ is singular, there are $(q+1)q$ lines contained in $H$ that meet $L$ in a unique point. Moreover, for every $y \in L$, $\Lambda_H(y)$ is the union of a number of lines of $Res(y)$, union the point of $Res(y)$ corresponding to $L$. It follows that every point of $L$ is of Type (A), (B) or (C). Let $n_1$, $n_2$, respectively $n_3$, denote the number of points of Type (A), (B), respectively (C), contained in $L$. Then $n_1 + n_2 + n_3 = q+1$ and $n_1 \cdot 0 + n_2 \cdot q + n_3 \cdot 2q = q(q+1)$. It follows that $n_1 = n_3$. ☐

The proof of the following lemma is straightforward.

**Lemma 24.** *Every point of Type (A) is contained in a unique special line. Every point of Type (C) is contained in a unique special line.*

Let $n_A$, $n_B$, $n_C$, $n_D$, respectively $n_E$, denote the total number of points of $H$ of Type (A), (B), (C), (D), respectively (E). The following is an immediate corollary of Lemmas 23 and 24.

**Corollary 25.** *We have* $n_C = n_A$.

**Lemma 26.** *We have* $n_E = 0$.

**Proof.** We count in two different ways the number of pairs $(x, L)$ with $x \in H$ and $L$ a line of $H$ through $x$. We find

$$n_A \cdot 1 + n_B \cdot (q+1) + n_C \cdot (2q+1) + n_D \cdot (q+1) + n_E \cdot 0 = l(q+1).$$

Using the facts that $n_A = n_C$ and $l = (q^2+q+1)(q^3+1) = v$, we find $n_A+n_B+n_C+n_D = v$. Hence, $n_E = 0$. □

**Lemma 27.** *We have* $n_D = \frac{2q^2}{q+1} n_A$.

**Proof.** We count in two different ways the number of pairs $(x, Q)$ where $Q$ is a singular quad and $x$ is its deep point. We find

$$Si = n_B + 2 \cdot n_C, \tag{4}$$

where $Si$ denotes the total number of singular quads. We count in two different ways the number of pairs $(x, Q)$ where $Q$ is a singular quad and $x$ is a point of $Q \cap H$ distinct from the deep point of $Q$. We find

$$(q+1)q \cdot Si = (q+1)n_A + q(q+1)n_B + (q-1)n_C + (q+1)n_D. \tag{5}$$

From (4) and (5) and the fact that $n_A = n_C$, it readily follows that $n_D = \frac{2q^2}{q+1} n_A$. □

Now, put $\delta := n_A$. Then we have $n_A = n_C = \delta$, $n_D = \frac{2q^2}{q+1} \cdot \delta$ and $n_B = (q^2+q+1)(q^3+1) - \frac{2(q^2+q+1)}{q+1} \cdot \delta$.

**Lemma 28.** *We have* $0 \leqslant \delta \leqslant \lfloor \frac{1}{2}(q+1)(q^3+1) \rfloor$.

**Proof.** This follows from the fact that $n_B \geqslant 0$. □

**Remark.** If $q \geqslant 4$ is even, then by De Bruyn [7], the dual polar space $DW(5, q)$ has up to isomorphism two hyperplanes not containing quads. The values of $\delta$ corresponding to these two hyperplanes are respectively equal to 0 and $\frac{q^3(q+1)}{2}$. If $q$ is odd, then by Cooperstein and De Bruyn [5], the dual polar space $DW(5, q)$ has up to isomorphism two hyperplanes not containing quads. The values of $\delta$ corresponding to these two hyperplanes are respectively equal to $\frac{1}{2}(q+1)(q^3-1)$ and $\frac{1}{2}(q+1)(q^3+1)$. So, the lower and upper bounds in Lemma 28 can be tight.

**Definition**. Recall that if $Q$ is a quad of $DW(5,q)$ then the points and lines of $DW(5,q)$ contained in $Q$ bijectively correspond to the points and lines of $\mathrm{PG}(4,q)$ that are contained in a given nonsingular parabolic quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. A *conic* of $Q$ is a set of $q+1$ points of $Q$ that corresponds to a nonsingular conic of $Q(4,q)$, i.e. with a set of $q+1$ points of $Q(4,q)$ contained in a plane $\pi$ of $\mathrm{PG}(4,q)$ intersecting $Q(4,q)$ in a nonsingular conic of $\pi$.

**Lemma 29.** *Let $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ such that $Q_1$ is ovoidal with respect to $H$ and $|\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)|\geqslant 2$. Then:*

(1) *$\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)$ is a conic of $Q_1$.*

(2) *The number of ovoidal quads of $\{Q_1,\ldots,Q_{q+1}\}$ is bounded above by $\frac{q+1}{2}$. If the number of these ovoidal quads is precisely $\frac{q+1}{2}$, then the remaining $\frac{q+1}{2}$ quads of $\{Q_1,\ldots,Q_{q+1}\}$ are subquadrangular with respect to $H$.*

**Proof.** We first prove that $\pi_{Q_1}(Q_2\cap H)\neq Q_1\cap H$. Suppose to the contrary that $\pi_{Q_1}(Q_2\cap H)=Q_1\cap H$. Let $u$ be a point of $Q_1\setminus H$, let $L$ be the unique line through $u$ meeting each quad of $\{Q_1,Q_2,\ldots,Q_{q+1}\}$, let $v$ denote the unique point of $L$ contained in $H$, and let $i$ be the unique element of $\{3,\ldots,q+1\}$ such that $v\in Q_i$. Now, since $Q_i\cap H$ contains $\pi_{Q_i}(Q_2\cap H)$ and the point $v\in Q_i\setminus\pi_{Q_i}(Q_2\cap H)$, we must have $Q_i\subseteq H$. This is however impossible since no quad is contained in $H$.

So, $\pi_{Q_1}(Q_2\cap H)\neq Q_1\cap H$. By Lemma 8, $\{\pi_{Q_1}(Q_i\cap H)\,|\,1\leqslant i\leqslant q+1\}$ is a pencil of hyperplanes of $\widetilde{Q_1}$. Let $\alpha_1$, $\alpha_2$, respectively $\alpha_3$, denote the number of quads of $\{Q_1,\ldots,Q_{q+1}\}$ that are ovoidal, singular, respectively subquadrangular, with respect to $H$. Put $\beta:=|\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)|\geqslant 2$. We prove that $\beta=q+1$.

If $\alpha_1=q+1$ and $\alpha_2=\alpha_3=0$, then $(q+1)(q^2+1)=|Q_1|=\beta+(q+1)(q^2+1-\beta)=(q+1)(q^2+1)-q\beta<(q+1)(q^2+1)$, a contradiction. So, without loss of generality, we may suppose that $Q_2$ is not ovoidal with respect to $H$. If $Q_2$ is subquadrangular with respect to $H$, then $\beta=|\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)|=q+1$. If $Q_2$ is singular with respect to $H$ with deep point $u$ such that $\pi_{Q_1}(u)\notin Q_1\cap H$, then also $\beta=|\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)|=q+1$. If $Q_1$ were singular with respect to $H$ with deep point $u$ such that $\pi_{Q_1}(u)\in Q_1\cap H$, then $\beta=|\pi_{Q_1}(Q_2\cap H)\cap(Q_1\cap H)|=1$, a contradiction. Hence, $\beta=q+1$ as claimed.

Now, we have $\alpha_1+\alpha_2+\alpha_3=q+1$ and $(q+1)(q^2+1)=|Q_1|=(q+1)+\alpha_1(q^2-q)+\alpha_2q^2+\alpha_3(q^2+q)=(q+1)+(q+1)q^2+q(\alpha_3-\alpha_1)$, i.e. $\alpha_1+\alpha_2+\alpha_3=q+1$ and $\alpha_1=\alpha_3$. Hence, $\alpha_1=\alpha_3\leqslant\frac{q+1}{2}$. Moreover, if $\alpha_1=\alpha_3=\frac{q+1}{2}$, then $\alpha_2=0$. This proves claim (2).

Now, $\alpha_2+\alpha_3\geqslant\frac{q+1}{2}$. So, $\alpha_2+\alpha_3\geqslant 2$. Without loss of generality, we may suppose that the quads $Q_2$ and $Q_3$ are singular or subquadrangular with respect to $H$.

The points and lines contained in $Q_1$ can be identified (in a natural way) with the points and lines lying on a given nonsingular parabolic quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. Now, each of $\pi_{Q_1}(Q_2\cap H)$ and $\pi_{Q_1}(Q_3\cap H)$ is either a singular hyperplane or a subgrid of $\widetilde{Q_1}$ and hence arises by intersecting $Q(4,q)$ with a hyperplane of $\mathrm{PG}(4,q)$. Since $\pi_{Q_1}(Q_2\cap$

$H) \cap \pi_{Q_1}(Q_3 \cap H) = \pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a set of $q+1$ mutually noncollinear points, $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ must be a conic of $Q_1$. □

**Lemma 30.** *If $Q_1$ is an ovoidal quad, then through every two points of $Q_1 \cap H$, there is a conic of $Q_1$ that is completely contained in $Q_1 \cap H$.*

**Proof.** Let $x_1$ and $x_2$ be two distinct points of $Q_1 \cap H$. By Lemmas 22 and 26, there exists a line $L_i$, $i \in \{1, 2\}$ through $x_i$ that is contained in $H$. Let $Q_2$ be a quad distinct from $Q_1$ that meets $L_1$ and $L_2$, and let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be the unique hyperbolic set of quads of $DW(5, q)$ containing $Q_1$ and $Q_2$. Since $\{x_1, x_2\} \subseteq \pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$, Lemma 29 applies. We conclude that $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a conic containing $x_1$ and $x_2$. □

**Lemma 31.** *For every quad $Q_1$ that is ovoidal with respect to $H$, there is a quad $Q_2$ disjoint from $Q_1$ that is singular with respect to $H$ such that $\pi_{Q_1}(u) \notin Q_1 \cap H$ where $u$ is the deepest point of the singular hyperplane $Q_2 \cap H$ of $\widetilde{Q_2}$.*

**Proof.** The number of points $x \in \Gamma_1(Q_1) \cap H$ for which $\pi_{Q_1}(x) \notin Q_1 \cap H$ is equal to $(|Q_1| - |Q_1 \cap H|) \cdot q^2 = q^3(q^2 + 1)$. Now, since $n_D = \frac{2q^2}{q+1}\delta \leqslant \frac{2q^2}{q+1} \cdot \frac{1}{2}(q + 1)(q^3 + 1) = q^2(q^3 + 1) < q^3(q^2 + 1)$, there exists a point $y \in \Gamma_1(Q_1) \cap H$ not of type (D) for which $\pi_{Q_1}(y) \notin Q_1 \cap H$. Let $L \subseteq H$ be a special line through $y$ and let $z$ denote the unique point of $L$ for which $\pi_{Q_1}(z) \in Q_1 \cap H$. By Lemma 22, there are at most two quads $R$ through $L$ for which $z$ is the deep point of the singular hyperplane $R \cap H$ of $\widetilde{R}$. Hence, there exists a quad $Q_2$ through $L$ for which the deep point $u$ of the singular hyperplane $Q_2 \cap H$ of $\widetilde{Q_2}$ is distinct from $z$. Since $u$ is not collinear with a point of $Q_1 \cap H$, $Q_1$ and $Q_2$ are disjoint. □

**Lemma 32.** *If $Q_1$ is ovoidal with respect to $H$, then $Q_1 \cap H$ is a classical ovoid of $\widetilde{Q_1}$.*

**Proof.** By Lemma 31, there exists a quad $Q_{q+1}$ disjoint from $Q_1$ that is singular with respect to $H$ such that $\pi_{Q_1}(u) \notin Q_1 \cap H$ where $u$ is the deep point of the singular hyperplane $Q_{q+1} \cap H$ of $\widetilde{Q_{q+1}}$. Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ denote the unique hyperbolic set of quads of $DW(5, q)$ containing $Q_1$ and $Q_{q+1}$. By Lemma 29, we then have:

(1) $X := \pi_{Q_1}(Q_{q+1} \cap H) \cap (Q_1 \cap H)$ is a conic of $Q_1$;

(2) the number $k$ of ovoidal quads of the set $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ is at most $\frac{q}{2}$.

Without loss of generality, we may suppose that $Q_1, \ldots, Q_k$ are the quads of $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ that are ovoidal with respect to $H$. Since $(q + 1) - \frac{q}{2} \geqslant 2$, $Q_q$ and $Q_{q+1}$ are not ovoidal with respect to $H$. By Lemmas 5 and 8, $\pi_{Q_1}(Q_q \cap H)$ and $\pi_{Q_1}(Q_{q+1} \cap H)$ are contained in a unique pencil of classical hyperplanes of $\widetilde{Q_1}$. Moreover, this pencil contains the hyperplanes $\pi_{Q_1}(Q_i \cap H)$, $i \in \{k+1, \ldots, q+1\}$. Let $A_1, \ldots, A_k$ denote the remaining elements of this pencil. Then $X \subseteq A_1 \cap \cdots \cap A_k$ and $A_1 \cup \cdots \cup A_k = \pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)$. Now, $|A_1 \cup \cdots \cup A_k| \geqslant |X| + k(q^2 + 1 - |X|) = (q + 1) + k(q^2 - q)$ and equality holds if and only if every $A_j$, $j \in \{1, \ldots, k\}$, is a classical ovoid of $\widetilde{Q_1}$. Now, since $|\pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)| = |X| + k(q^2 + 1 - |X|) = (q + 1) + k(q^2 - q)$, we can conclude that every $A_j$, $j \in \{1, \ldots, k\}$, is a classical ovoid of $\widetilde{Q_1}$.

Now, let $i \in \{1, \ldots, k\}$ and suppose there exists no $j \in \{1, \ldots, k\}$ such that $\pi_{Q_1}(Q_i \cap H) = A_j$. Then $X \subseteq \pi_{Q_1}(Q_i \cap H) \subseteq A_1 \cup \cdots \cup A_k$ and there exist two distinct $j_1, j_2 \in \{1, \ldots, k\}$ such that $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X) \neq \emptyset$ and $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X) \neq \emptyset$. Let $y_1$ be an arbitrary point of $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X)$ and let $y_2$ be an arbitrary point of $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X)$. By Lemma 30, there exists a conic $C$ through $y_1$ and $y_2$ that is completely contained in $\pi_{Q_1}(Q_i \cap H)$ and hence also in $A_1 \cup \cdots \cup A_k$. Since $|C| = q+1$ and $k \leqslant \frac{q}{2}$, there exists a $j_3 \in \{1, \ldots, k\}$ such that $|C \cap A_{j_3}| \geqslant 3$. Since $A_{j_3}$ is a classical ovoid of $\widetilde{Q_1}$, this necessarily implies that $C \subseteq A_{j_3}$, contradicting the fact that $y_1 \in A_{j_1} \setminus X$, $y_2 \in A_{j_2} \setminus X$ and $j_1 \neq j_2$. Hence, there exists a $j \in \{1, \ldots, k\}$ such that $\pi_{Q_1}(Q_i \cap H) = A_j$. This implies that the ovoid $Q_i \cap H$ of $\widetilde{Q_i}$ is classical. $\square$

**Corollary 33.** *The hyperplane $H$ is classical.*

**Proof.** This is an immediate corollary of Proposition 7 and Lemma 32. $\square$

**Remark.** With the terminology of Cooperstein & De Bruyn [5] and De Bruyn [7], the hyperplane $H$ is either a hyperplane of Type V or a hyperplane of Type VI.


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