# Hamilton saturated hypergraphs of essentially minimum size

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**Abstract**

For $1 \leqslant \ell < k$, an $\ell$-overlapping cycle is a $k$-uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of $k$ consecutive vertices and every two consecutive edges share exactly $\ell$ vertices. A $k$-uniform hypergraph $H$ is $\ell$-Hamiltonian saturated, $1 \leqslant \ell \leqslant k-1$, if $H$ does not contain an $\ell$-overlapping Hamiltonian cycle $C_n^{(k)}(\ell)$ but every hypergraph obtained from $H$ by adding one more edge does contain $C_n^{(k)}(\ell)$. Let $sat(n,k,\ell)$ be the smallest number of edges in an $\ell$-Hamiltonian saturated $k$-uniform hypergraph on $n$ vertices. Clark and Entringer proved in 1983 that $sat(n,2,1) = \lceil \frac{3n}{2} \rceil$. In this paper we prove that $sat(n,k,\ell) = \Theta(n^\ell)$ for $\ell = 1$ as well as for all $k \geqslant 5$ and $\ell \geqslant 0.8k$.

## 1 Introduction

The notion of a hypergraph cycle can be ambiguous. In this paper we are *not* concerned with the Berge cycles as defined by Berge in [1] (see also [12]). Instead, given integers $1 \leqslant \ell < k$, we define an *$\ell$-overlapping cycle* as a $k$-uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of $k$ consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly $\ell$ vertices. The notion of an *$\ell$-overlapping path* is defined similarly. Note that the number of edges of an $\ell$-overlapping cycle with $s$ vertices is $s/(k-\ell)$ (and thus, $s$ is divisible by $k-\ell$). The two extreme cases of $\ell = 1$ and $\ell = k-1$ are referred

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to as, respectively, *loose* and *tight* cycles (paths). We denote an *$\ell$-overlapping cycle* on $s$ vertices by $C_s^{(k)}(\ell)$.

*An $\ell$-overlapping Hamiltonian cycle* in a $n$-vertex $k$-graph $H$ is any subhypergraph of $H$ isomorphic to $C_n^{(k)}(\ell)$. If $H$ contains an $\ell$-overlapping Hamiltonian cycle then $H$ itself is called *$\ell$-Hamiltonian*. A tight Hamiltonian cycle was introduced in the seminal paper by Katona and Kierstad [16] under the name of *a Hamiltonian chain*. Since the appearence of [16], *$\ell$-Hamiltonian* cycles have been studied intensively in the context of Dirac-type properties (for a survey see [17]), Ramsey properties (e.g., in [13, 14]), random hypergraphs ([10, 6, 7]). However, the saturation problem for Hamiltonian cycles in hypergraphs is mentioned only in a survey paper by Katona [15].

Given a $k$-uniform hypergraph $H$ (or, shortly, a $k$-graph) and a $k$-element set $e \in H^c$, where $H^c$ is the complement of $H$, we denote by $H + e$ the hypergraph obtained from $H$ by adding $e$ to its edge set. A $k$-graph $H$ is *$\ell$-Hamiltonian saturated*, $1 \leqslant \ell \leqslant k-1$, if $H$ is not $\ell$-Hamiltonian but for every $e \in H^c$ the $k$-graph $H + e$ is such. The largest number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices, that is, the Turán number for the cycle $C_n^{(k)}(\ell)$, denoted by $ex(n, C_n^{(k)}(\ell))$, has been determined recently in [11]. It turned out that

$$ex(n, C_n^{(k)}(\ell)) = \binom{n-1}{k} + ex(n-1, P),$$

where $P = P(k, l)$ is the $(k-1)$-uniform, $(\ell-1)$-overlapping path with $\lfloor \frac{k}{k-\ell} \rfloor$ edges. In particular, for graphs ($k = 2$) the largest size of a Hamiltonian saturated graph is $\binom{n-1}{2}+1$. This value is realized by a unique graph consisting of a clique on $n-1$ vertices and a pedant vertex. Note that this is the only Hamiltonian saturated graph with minimum degree 1.

In this paper we are interested in the other extreme. For $n$ divisible by $k - \ell$, let $sat(n, k, \ell)$ be the *smallest* number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices. In the case of graphs, Clark and Entringer proved in 1983 that $sat(n, 2, 1) = \lceil \frac{3n}{2} \rceil$ for $n$ large enough.

For $k$-graphs with $k \geqslant 3$ it seems to be quite hard to obtain such precise results. Therefore, the emphasis is put on the order of magnitude of $sat(n, k, \ell)$. It was observed in [15] that $sat(n, k, k-1) = \Omega(n^{k-1})$. After some preliminary results in [8, 9], the second author showed recently that for $k \geqslant 2$, $sat(n, k, k-1) = \Theta(n^{k-1})$, see [18]. Here we extend that result to $\ell$-overlapping Hamiltonian cycles for several other values of $\ell$. Our main result is the following.

**Theorem 1.1.** *For all $k \geqslant 3$ and $\ell = 1$, as well as for all $\frac{4}{5}k \leqslant \ell \leqslant k-1$*

$$sat(n, k, \ell) = \Theta(n^{\ell}).$$

We conjecture that Theorem 1.1 holds for all $k$ and $1 \leqslant \ell \leqslant k-1$.

## 2 Preliminaries

The next two sections contain proofs of the upper bound in Theorem 1.1. Here we give a simple proof of the lower bound.

**Proposition 2.1.** *For all $k \geqslant 2$ and $1 \leqslant \ell \leqslant k-1$*

$$sat(n,k,\ell) = \Omega(n^\ell).$$

*Proof.* If $H$ is an $\ell$-saturated $k$-graph with $n$ vertices and $m$ edges then for every nonedge $e \in H^c$ there is an edge $f \in H$ such that $|e \cap f| = \ell$ (in fact, there are two such edges $f$, since $e$ has to close an $\ell$-overlapping cycle). But for every $f \in H$, the number of $k$-element subsets $e$ which satisfy $|e \cap f| = \ell$ is exactly

$$\binom{k}{\ell}\binom{n-k}{k-\ell}.$$

Thus, every $f \in H$ can intersect this way at most $\binom{k}{\ell}\binom{n-k}{k-\ell}$ nonedges $e$. Hence,

$$\left(\binom{n}{k} - m\right) \leqslant m \times \binom{k}{\ell}\binom{n-k}{k-\ell}$$

which implies that $m = \Omega(n^\ell)$. □

In the rest of the paper we assume that $G$ is a graph on the vertex set $\{1,\dots,n\}$. Let $c(G)$ denote the number of components of $G$. Given a subset $T \subseteq V(G)$, let $G[T]$ be the subgraph of $G$ induced by $T$.

**Fact 2.2.** *Let $k$, $\ell$, and $\Delta$ be constants. If $\Delta(G) \leqslant \Delta$ then the number of $k$-element subsets $T \subseteq V(G)$ with $c(G[T]) \leqslant \ell$ is $O(n^\ell)$.*

*Proof.* The number of $k$-element subsets $T \subseteq V(G)$ with $c(G[T]) \leqslant \ell$ is at most

$$\big(n(k-1)!\Delta^{k-1}\big)^\ell = O(n^\ell). \quad \square$$

Given a graph $G$ and an integer sequence $\mathbf{a} = (a_1,\dots,a_n)$, the $\mathbf{a}$-*blow-up* of $G$ is the $k$-graph $H$ with

$$V(H) = \bigcup_{i=1}^{n} U_i, \quad |U_i| = a_i,$$
$$H = \bigcup_{ij \in G} K^{(k)}(U_i \cup U_j)$$

where $K^{(k)}(U)$ is the complete $k$-graph on $U$ and the sets $U_i$ are pairwise disjoint. If $a_i = a$ for all $i = 1,\dots,n$, then we simply write $a$-blow-up instead of $\mathbf{a}$-blow-up. For a subset $S \subset V(H)$, let

$$tr(S) = \{i \in V(G) : U_i \cap S \neq \emptyset\}.$$

Furthermore, set

$$c(S) = c\left(G[tr(S)]\right).$$

The following result is an immediate corollary of Fact 2.2.

**Corollary 2.3.** *Let $a_1, \ldots, a_n$, $k$, $\ell$, and $\Delta$ be constants. If $\Delta(G) \leqslant \Delta$ and $H$ is an* **a***-blow-up of $G$ then the number of $k$-element subsets $S \subseteq V(H)$ with $c(S) \leqslant \ell$ is $O(n^\ell)$.* □

## 2.1 Hamiltonian cycle saturated graphs

The proofs of the upper bounds in Theorem 1.1 are constructive. The starting points of our constructions are sparse Hamiltonian saturated *graphs*, also known as maximally non-Hamiltonian graphs. Probably the best known Hamiltonian saturated graphs of minimumm size are Isaac's snarks $J_k$ which are 3-regular, connected, bridgeless graphs with chromatic index four, and the number of vertices $n = 4k$. In a series of papers Clark, Crane, Entringer and Shapiro [3, 4, 5] constructed Hamiltonian saturated graphs (by a modification of Isaac's snarks) with minimum possible size for all sufficiently large $n$.

**Theorem 2.4** ([5])**.** *For all even $n \geqslant 36$ as well as all odd $n \geqslant 53$ there exists a Hamiltonian saturated graph of order $n$ and size $\lceil 3n/2 \rceil$.*

In order to obtain the right order of magnitude for $sat(n, k, \ell)$ for all values of $\ell$ considered in the paper, we will need Hamiltonian saturated graphs with bounded maximum degree. (Due to the asymptotic nature of our result, the numerical value of the bound does not matter to us.) By analyzing the construction in [5] one can see that the Hamiltonian saturated graphs obtained there do have bounded maximum degree. An alternative way, which we prefer, is by combining Theorem 2.4 with the following result of Bondy.

**Theorem 2.5** ([2])**.** *Let $G$ be a Hamiltonian saturated graph with $n \geqslant 7$ vertices. If for some $0 \leqslant m \leqslant n$ the graph $G$ has $m$ vertices of degree 2, then $|E(G)| \geqslant (3n + m)/2$.*

**Corollary 2.6.** *For all $n \geqslant 52$ there exists a Hamiltonian saturated graph $G$ of order $n$ with $\Delta(G) \leqslant 5$.*

*Proof.* By Theorem 2.4 for all $n \geqslant 52$ there exists a Hamiltonian saturated graph $G$ with $n$ vertices and at most $(3n+1)/2$ edges. Clearly, $\delta(G) \geqslant 2$. Hence, by Theorem 2.5, there is at most one vertex of degree 2 in $G$, and, consequently, no vertex of degree greater than 5. □

# 3 The loose case : $\ell = 1$

In this Section we prove Theorem 1.1 for $\ell = 1$. We begin with a simple lemma.

**Lemma 3.1.** *If a graph $G$ is not Hamiltonian then the $(k-1)$-blow-up $H$ of $G$ is not 1-Hamiltonian.*

*Proof.* Suppose that $H$ contains a 1-Hamiltonian cycle $C_H = \{e_1, \ldots, e_n\}$. Define $f : C_H \to G$ by $f(e_s) = \{i, j\} \in G$, where $e_s \in H[U_i \cup U_j]$. Since $|H[U_i \cup U_j] \cap C_H| \leqslant 1$ for all $1 \leqslant i < j \leqslant n$, the mapping $f$ is one-to-one. Furthermore, $C_G = \{f(e_s), s = 1, \ldots, n\}$ is a connected, spanning subgraph of $G$. Moreover, $\delta(C_G) \geqslant 2$. Indeed, fix $i \in \{1, \ldots, n\}$, recall that $|U_i| = k - 1$, and observe that every subset of $k - 1$ vertices of $C_H$ intersects at least two edges of $C_H$. Thus, $C_G$ is a Hamiltonian cycle in $G$, a contradiction. □

In view of Proposition 2.1, in order to prove Theorem 1.1 for $\ell = 1$ it suffices to construct for every sufficiently large $N$ divisible by $k - 1$, a 1-Hamiltonian saturated $k$-graph $H$ with $N$ vertices and $O(N)$ edges. Let $H_1$ be a $(k-1)$-blow-up of a Hamiltonian saturated graph $G$ with $n$ vertices, $n = \frac{N}{k-1}$, and $\Delta(G) = O(1)$. (By Corollary 2.6 $G$ exists.) By Lemma 3.1, $H_1$ is not 1-Hamiltonian and $|V(H)| = N$. Set $V = V(H_1)$ and let

$$H_2 = \left\{ e \in \binom{V}{k} : tr(e) \text{ is a clique in } G \right\}.$$

Since for every $e \in H_1$ the set $tr(e)$ spans an edge of $G$, we have $H_1 \subseteq H_2$. Finally, let $H$ be a maximal $k$-graph on the vertex set $V$ such that $H_1 \subseteq H \subseteq H_2$ and $H$ is not 1-Hamiltonian. By Corollary 2.3, $|H| \leqslant |H_2| = O(N)$. The following lemma completes the proof of Theorem 1.1 in the case $\ell = 1$.

**Lemma 3.2.** *For every $e \in H^c$, $H + e$ is 1-Hamiltonian.*

*Proof.* By the maximality of $H$ we may restrict our attention to only those $e$ for which $tr(e)$ is not a clique. Fix one pair $\{i, j\} \notin G$ such that $tr(e) \supset \{i, j\}$. Without loss of generality (w.l.o.g.) we may assume that $i = 1$ and $j = 2$. Since $G$ is Hamiltonian saturated, $G + \{1, 2\}$ has a Hamiltonian cycle containing the edge $\{1, 2\}$. Let $C_G$ be a Hamiltonian cycle in $G + \{1, 2\}$ corresponding, w.l.o.g., to a cyclic ordering $(1, \ldots, n)$. Set $r_s = |e \cap U_s|$, $s = 1, 2, \ldots, n$. We build a 1-Hamiltonian cycle $C_H = \{e_1, \ldots, e_n\}$ in $H$ by 'tracing' $C_G$. In doing so, we make sure that the last vertex of each edge $e_i$ belongs to the set $U_{i+1}$ and that $U_{i+1} \subseteq e_1 \cup e_i \cup e_{i=1}$.

Formally, we construct $C_H$ as follows. (See Fig. 1 for an illustration.)

- Let $e_1 = e$ and choose $v_1 \in e_1 \cap U_1$ and $v_2 \in e_1 \cap U_2$.
- Further, let $e_2 \in H[U_2 \cup U_3]$ with $e_1 \cap e_2 = \{v_2\}$ and $|e_2 \cap U_2| = k - r_2$. Note that

  $$|e_2 \cap U_3| = k - |e_2 \cap U_2| = r_2 \leqslant k - 1 - r_3 = |U_3 \setminus e_1|$$

  and $U_2 \subset e_1 \cup e_2$. Choose $v_3 \in e_2 \cap U_3$.
- Subsequently, for $3 \leqslant t \leqslant n-1$, let $e_t \in H[U_t \cup U_{t+1}]$ with $e_{t-1} \cap e_t = \{v_t\}$, $e_t \cap e_1 = \emptyset$, and $|e_t \cap U_t| = k - \sum_{s=2}^{t} r_s$. Note that

  $$|e_t \cap U_{t+1}| = \sum_{s=2}^{t} r_s \leqslant k - 1 - r_{t+1} = |U_{t+1} \setminus e_1|,$$

  because $\sum_{s=1}^{n} r_s = k$ and $r_1 \geqslant 1$. Moreover, $U_t \subset e_1 \cup e_t \cup e_{t+1}$. Set $v_{t+1} \in e_t \cap U_{t+1}$.

- Finally, let $e_n \in H[U_n \cup U_1]$ with $e_{n-1} \cap e_n = \{v_n\}$, $|e_n \cap U_n| = k - \sum_{s=2}^{n} r_s$, and $e_n \setminus U_n = (U_1 \setminus e_1) \cup \{v_1\}$. Note that

$$|e_n \cap U_1| = \sum_{s=2}^{n} r_s = k - r_1 = 1 + |U_1 \setminus e_1|$$

and $U_1 \subset e_n \cup e_1$.

Thus, indeed, $C_H = \{e_1, \ldots, e_n\}$ is a 1-Hamiltonian cycle in $H$. □

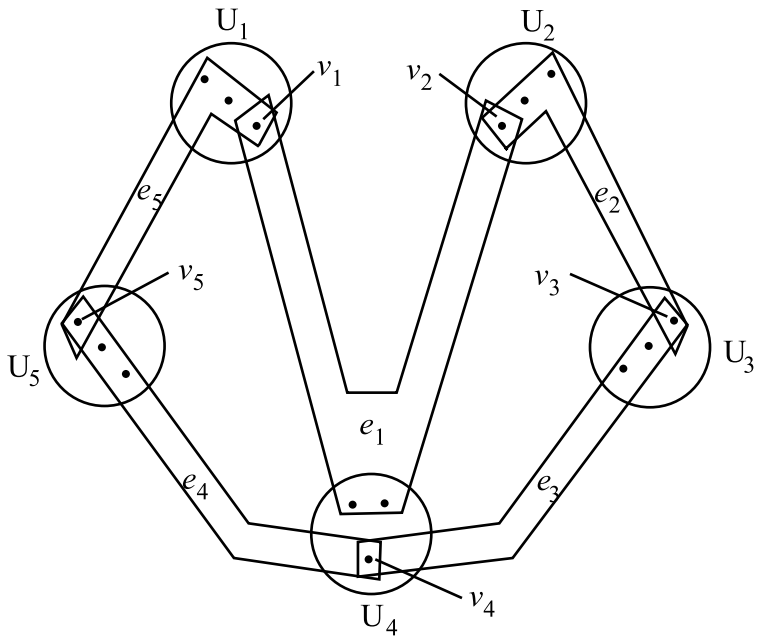


Figure 1: An illustration to the proof of Lemma 3.2: $n = 5$, $k = 4$

# 4 The case $\ell \geqslant 4k/5$

In this Section we prove our main result, that is, Theorem 1.1 for $\ell \geqslant \frac{4}{5}k$. Let $\mathbf{a} = (a_1, \ldots, a_n)$, where

$$2k - \ell + 1 \leqslant a_i \leqslant 4\ell - 2k + 1, \qquad i = 1, \ldots, n. \tag{1}$$

Note that under our assumption on $\ell$ we do have $2k - \ell + 1 \leqslant 4\ell - 2k + 1$, and that for all $1 \leqslant \ell \leqslant k - 1$,

$$a_i \leqslant 2\ell - 1. \tag{2}$$

Let $G$ be an $n$-vertex Hamiltonian saturated graph with $n$ sufficiently large and $\Delta(G) = O(1)$, guaranteed by Corollary 2.6, and let $H_1$ be the $\mathbf{a}$-blow-up $k$-graph of $G$ with

$$V = V(H_1) = \bigcup_{i=1}^{n} U_i, \text{ where } |U_i| = a_i, \quad i = 1, \ldots, n.$$

Observe that for each $e \in H_1$, the set $tr(e)$ is either a vertex or an edge of $G$ and thus $c(e) = 1$. Given a set $S \subseteq V$, let

$$\min(S) = \min\{i : i \in tr(S)\} = \min\{i : U_i \cap S \neq \emptyset\}.$$

Further, let $H_2$ be

$$\left\{ e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geqslant k - l + 1, c(e) \geqslant k - l + 1 \text{ and } |e \cap U_{\min(e)}| + c(e) \geqslant l + 2 \right\}.$$

Since for every $e \in H_2$ we have $c(e) \geqslant k - \ell + 1 \geqslant 2$, the $k$-graphs $H_1$ and $H_2$ are edge-disjoint.

**Lemma 4.1.** *$H_1 \cup H_2$ is not $\ell$-Hamiltonian.*

*Proof.* Suppose that $H_1 \cup H_2$ contains an $\ell$-Hamiltonian cycle $C_H$.

**Case 1.** Assume first that $C_H \subseteq H_1$ and define a subgraph $C_G$ of $G$ as the set of all *2-element* traces $tr(e)$ of the edges $e$ of $C_H$. Formally,

$$C_G = \{tr(e) : e \in C_H \text{ and } |tr(e)| = 2\}.$$

We are going to arrive at a contradiction by showing that $C_G$ is a Hamiltonian cycle in $G$. Since, clearly, $C_G$ is a connected, spanning subgraph of $G$, it is enough to prove that $C_G$ is 2-regular.

Let us fix $i \in \{1, \ldots, n\}$. As $C_H$ has to enter and leave the set $U_i$ at some point, there exist an edge $e \in C_H$ and an index $j \neq i$ such that $tr(e) = \{i, j\}$. Let $P$ be a longest $\ell$-overlapping path in $C_H$ (a segment of $C_H$) containing $e$ and with $\bigcup_{f \in P} tr(f) = \{i, j\}$. Further, let $e', e''$ be the two edges of $C_H$ which intersect $V(P)$ each in $\ell$ vertices and set $A' = e' \cap V(P)$ and $A'' = e'' \cap V(P)$ (see Fig. 2). Since on the one hand $tr(A') \subseteq \{i, j\}$ while, on the other hand, $A' \subset e'$ and $|tr(e') \cap \{i, j\}| = 1$ (and the same is true for $A''$) we have $|tr(A')| = |tr(A'')| = 1$. However, $tr(A') \neq tr(A'')$. Indeed, if, say, $A' \cup A'' \subseteq U_i$ then, by (2), we would have $A' \cap A'' \neq \emptyset$ and consequently

$$e \subseteq V(P) \subseteq A' \cup A'' \subseteq U_i,$$

a contradiction with the choice of $e$.

In conclusion, if for some $e \in C_H$ we have $tr(e) = \{i, j\}$ then there is a set $A \subset U_i$ with $|A| = \ell$ which on the cycle $C_H$ is connected to $e$ by an $\ell$-overlapping path consisting of vertices from $U_i \cup U_j$ only. Moreover, the edge, say $e'$, extending $A$ along $C_H$ in the opposite direction (away from $e$) satisfies $tr(e') = \{j', i\}$, where $j' \neq i, j$, and so $N_{C_G}(i) \supseteq \{j, j'\}$. To show that $C_G$ is indeed 2-regular, suppose to the contrary that there exist edges $e_1, e_2, e_3 \in C_H$ with $tr(e_s) = \{i, j_s\}$, $s = 1, 2, 3$, where $j_1, j_2, j_3$ are mutually distinct and different from $i$. Let $A_s$, $s = 1, 2, 3$, be the sets described above (with respect to $e_s$). Since $|A_s| = \ell$, again by (2), the sets $A_1, A_2, A_3$ intersect pairwise. Assume w.l.o.g that $A_1$ is located (along $C_H$) between $e_1$ and $e_2$. Then $A_3$ cannot intersect $A_1$, a contradiction. (See Fig. 3 for an illustration of this proof.)

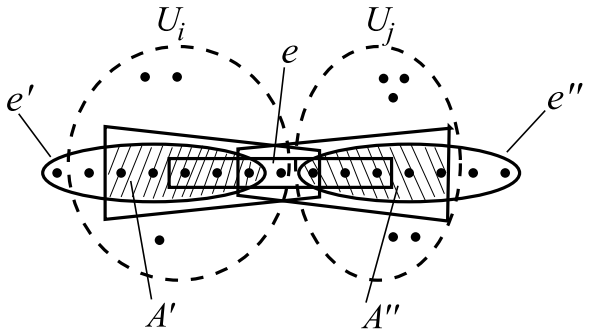


Figure 2: An illustration to the proof of Lemma 4.1: $k = 7$, $\ell = 5$, $|U_i| = |U_j| = 10$, the path $P$ consists of 3 "quadrangular" edges.

**Case 2.** Assume that $C_H$ contains a segment of more than $k^2$ consecutive edges from $H_2$ (that is, an $\ell$-overlapping path in $C_H \cap H_2$). Let $e_1, \ldots, e_{s-1}$, $s > k^2 + 1$, be such a segment. Recall that $|e \cap U_{\min(e)}| \geqslant k - \ell + 1$ for every $e \in H_2$, while $|e_t \cap e_{t+1}| = \ell$ for all $t = 1, \ldots, s-2$. These two facts imply that $\min(e_t) = \min(e_{t+1})$ for $t = 1, \ldots, s-2$, and so $|e_t \cap U_i| \geqslant k - \ell + 1$ for some $i \in [1, n]$ and all $t = 1, \ldots, s-1$. On the other hand, observe that a vertex can belong to at most $k$ edges of $C_H$. Hence,

$$|U_i| \geqslant |(e_1 \cup \cdots \cup e_{s-1}) \cap U_i| \geqslant \frac{1}{k}(s-1)(k-\ell+1) \geqslant \frac{2}{k}(s-1) > 2k > |U_i|,$$

a contradiction.

**Case 3.** Assume that $H_2 \cap C_H \neq \emptyset$ but the longest segment in $C_H$ of consecutive edges from $H_2$ has length at most $k^2$. Let $e_1, \ldots, e_{s-1}$, $2 \leqslant s \leqslant k^2 + 1$, be such a segment. Then $e_m \in H_1$ and $e_s \in H_1$. As in Case 2, $|e_t \cap U_i| \geqslant k - \ell + 1$ for some $i \in [1, n]$ and all $t = 1, \ldots, s-1$. Consequently, $e_m \cap U_i \neq \emptyset$ as well as $e_s \cap U_i \neq \emptyset$. By the definition of $H_1$, each of $tr(e_m)$ and $tr(e_s)$ is either the singleton $\{i\}$ or an edge of $G$ containing vertex $i$ and thus, $c(e_m) = c(e_s) = 1$. In view of this and the inequality $c(e_1) \geqslant k - \ell + 1$ we have $(e_1 \setminus e_m) \cap U_i = \emptyset$. Analogously, $(e_{s-1} \setminus e_s) \cap U_i = \emptyset$. Moreover, in fact $c(e_1) = c(e_{s-1}) = k - \ell + 1$. Therefore, by the third criterion in the definition of $H_2$,

$$|U_i \cap (e_1 \cap e_m)| = |U_i \cap e_1| \geqslant \ell + 2 - (k - \ell + 1) = 2\ell - k + 1$$

and, similarly,

$$|U_i \cap (e_{s-1} \cap e_s)| \geqslant 2\ell - k + 1.$$

Observe that for large $n$, $e_m \cap e_s = \emptyset$. Indeed, if $e_m \cap e_s \neq \emptyset$, then, necessarily $e_1 \subseteq e_s \cup e_m$, and consequently, $c(e_1) = 1$ – a contradiction with the definition of $H_2$. Hence, we have

$$|U_i| \geqslant |U_i \cap (e_1 \cap e_m)| + |U_i \cap (e_{s-1} \cap e_s)| \geqslant 4\ell - 2k + 2,$$

a contradiction with (1). □

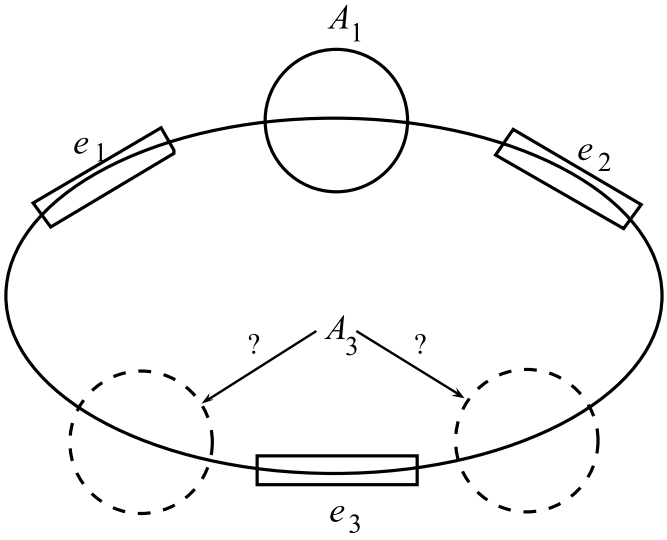


Figure 3: An illustration to the proof of Lemma 4.1

Let

$$H_3 = \left\{ e \in \binom{V}{k} : c(e) \leqslant \ell \right\}.$$

Recall that for all $e \in H_1$ we have $c(e) = 1$, while for all $e \in H_2$ we have $c(e) \leqslant \ell$. Thus, $H_1 \cup H_2 \subseteq H_3$. Finally, let $H$ be a maximal $k$-graph on the vertex set $V$ such that $H_1 \cup H_2 \subseteq H \subseteq H_3$ and $H$ is not $\ell$-Hamiltonian. By Corollary 2.3,

$$|H| \leqslant |H_3| = O(N^\ell). \tag{3}$$

We will next show that $H$ is $\ell$-Hamiltonian saturated.

**Lemma 4.2.** *For every $e \in H^c$, $H + e$ is $\ell$-Hamiltonian.*

*Proof.* By the definition of $H$ the thesis holds for each $e$ with $c(e) \leqslant \ell$. Hence, we may assume that $c(e) \geqslant \ell + 1$. We will build an $\ell$-overlapping Hamiltonian cycle

$$C_H = (e_1, \ldots, e_m) = (u_1, \ldots, u_N), \qquad m = \frac{N}{k-\ell},$$

in $H + e$ using the Hamiltonian saturation of $G$. As the general proof is a bit complicated, we will first assume that $\ell = k-1$, in which case the construction can be simplified. This way, avoiding tedious details, we will be able to exhibit the main ideas quite clearly.

**The tight case : $\ell = k-1$.** We have $k+2 \leqslant a_j = |U_j| \leqslant 2k-3$ for all $j = 1, \ldots, n$. Since $c(e) \geqslant k$, the set $tr(e)$ is, in fact, an independent $k$-element set in $G$. Let

$$tr(e) = \{i < j_{k-1} < \cdots < j_1\}$$

and $e = (u_1, \ldots, u_k)$, where $u_1 \in U_i$ and $u_{1+t} \in U_{j_t}$ for $t = 1, \ldots, k-1$. We construct first a tight path $P \subseteq H_2 + e$ extending $e$ in both directions, so that the two ends $A$ and $B$ of $P$ are $(k-1)$-tuples contained in, respectively, $U_i$ and $U_{j_{k-1}}$. To do so, let $u_{k+t}$, $t = 1, \ldots, k-2$, be any vertices of $U_{j_{k-1}}$ different from $u_k$, whereas $u_{N-t}$, $t = 0, 1, \ldots, k-3$, be any vertices of $U_i$ different from $u_1$. Then

$$P = (u_{N-k+3}, \ldots, u_N, u_1, \ldots, u_{2k-2}).$$

(See Fig. 4 for an illustration of this construction.)

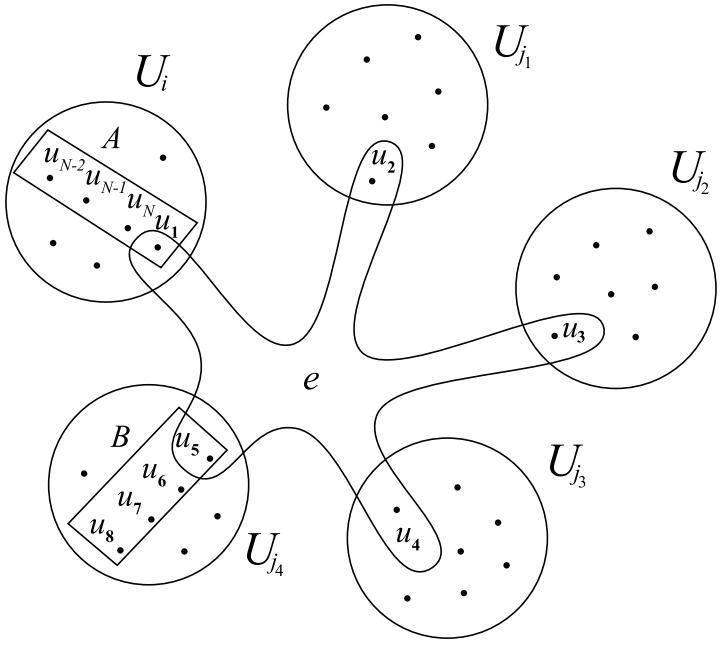


Figure 4: The construction of $P$ in the tight case: $k = 5$, all $|U_j| = 7$.

To see that $P$ is a tight path in $H + e$ with ends $A = (u_{N-k+3}, \ldots, u_N, u_1)$ and $B = (u_k, \ldots, u_{2k-2})$, note that for each $q = 2, \ldots, k-1$ the edge $e_q = (u_q, \ldots, u_{q+k-1})$ satisfies: $\min(e_q) = j_{k-1}$, $|e_q \cap U_{j_{k-1}}| = q$, $c(e_q) = k-q+1$, and thus $e_q \in H_2$. Similarly, for $q = 0, \ldots, k-3$, the edges $e_{m-q} = (u_{N-q}, \ldots, u_N, u_1, \ldots, u_{k-q-1})$, for which $\min(e_{m-q}) = i$, also belong to $H_2$.

Recall that $ij_{k-1} \notin G$ and thus, by the Hamiltonian saturation of $G$ there is a Hamiltonian path $Q$ from $i$ to $j_{k-1}$ in $G$. We connect the ends of $P$, that is, the sets $A$ and $B$, by a tight path $P'$ in $H_1 \subseteq H$, tracing the path $Q$ in $G$ in such a way that every time $Q$ visits a vertex $v$ of $G$ we add to $P'$ all vertices of $U'_v = U_v \setminus V(P)$. Since, $|U'_v| \geqslant |U_v| - 1 \geqslant k-1$ (with some margin), we can always do so by using only the edges of $H_1$.

**General case.** For $\ell \leqslant k-2$ the situation becomes more complicated and the above simple construction of the $\ell$-overlapping path $P$ fails. For instance, if $u_{k-1}$ and $u_k$ are in the same component of $G[tr(e)]$ and $k-2$ is divisible by $k-\ell$, then $c(e_{(k-2)/(k-\ell)+1}) = 1$,

and so $e_{(k-2)/(k-\ell)+1} \notin H_2$. Nevertheless we manage to follow the same idea by slightly modifying the above construction.

Recall that $c(e) \geqslant \ell + 1$. Let $j_1 > j_2 > \cdots > j_\ell > i = \min(e)$ be some $\ell + 1$ elements of $tr(e)$, belonging to different components of $G[tr(e)]$ and including $i = \min(e)$. Further, let $e_1 = e = (u_1, \ldots, u_k)$, where $u_1 \in U_i$ and $u_{k-\ell+t} \in U_{j_t}$, $t = 1, \ldots, \ell$, while $u_2, \ldots, u_{k-\ell}$ remain unspecified.

Our plan is, again, first to construct a path $P \subseteq H_2 + e$ extending $e$ in both directions (Part 1), and then to complete $C_H$ by connecting the ends of $P$ by a path $P' \subseteq H_1$ (Part 2). The path $P'$ will follow a Hamiltonian path $Q$ in $G$ which together with the pair $\{i, j_\ell\}$ forms a Hamiltonian cycle in $G + \{i, j_\ell\}$.

**Part 1.** Let integers $q$ and $r$ be defined by

$$(q+1)(k-\ell) + r = k, \quad 1 \leqslant r \leqslant k - \ell. \tag{4}$$

The $\ell$-path $P$ will consist of $3q + 5$ edges, $e_{m-2q}, \ldots, e_{q+4}$, and thus, of $k + (3q+4)(k-\ell)$ vertices, $u_{N-k+2r-\ell+1}, \ldots, u_N, u_1, \ldots, u_{4k-2\ell-r}$. The edges are determined by the vertices as they begin at every $(k-\ell)$th vertex. (Note that $k + \ell - 2r = (2q+1)(k-\ell)$, and thus $e_1$, the $(2q+2)$nd edge of $P$ does coincide with $e$.)

We now list all the vertices of $P$, that is, for each index $x$ we specify the set $U_j$ from which we (arbitrarily) select a vertex $u_x$.

1. For $N - k + 2r - \ell + 1 \leqslant x \leqslant N - k + 2r$ we select $u_x \in U_i$; thus, $P$ begins with $\ell$ vertices of $U_i$; we denote their set by $I_1$.

2. For $N - k + 2r + 1 \leqslant x \leqslant N - k + \ell$ we select $u_x \in U_{j_t}$, where $t = x - (N-k) - 1$; this segment of $P$ has exactly one vertex from each set $U_{j_{2r+t}}$, $t = 0, \ldots, \ell - 2r - 1$; we denote this set by $M_1$ ("M" like in mixed).

3. For $N - k + \ell + 1 \leqslant x \leqslant N$ we select $u_x \in U_i$; thus, $P$ returns to $U_i$ for $k - \ell$ steps; we denote this set *enlarged* by $u_1$, the first vertex of $e_1$, by $I_2$.

4. The next $k$ vertices of $P$ are the vertices of $e = e_1$, namely $u_1, \ldots, u_k$; we set $X = \{u_2, \ldots, u_{k-\ell}\}$ and $M_2 = \{u_{k-\ell+1}, \ldots, u_{k-1}\}$ (we know nothing about the elements of $X$).

5. For $k + 1 \leqslant x \leqslant 2k - \ell$ we select $u_x \in U_{j_\ell}$; thus, $P$ traverses through some $k - \ell$ vertices of $U_{j_\ell}$; we denote this set *enlarged* by $u_k$, the last vertex of $e_1$, by $L_1$.

6. For $2k - \ell + 1 \leqslant x \leqslant 4k - 3\ell - r$ we select $u_x \in U_{j_t}$, where $t = x - 2k + \ell$; this segment of $P$ has exactly one vertex from each set $U_{j_t}$, $t = 1, \ldots, 2k - 2\ell - r$; we denote this set by $M_3$.

7. For $4k - 3\ell - r + 1 \leqslant x \leqslant 4k - 2\ell - r$ we select $u_x \in U_{j_\ell}$; thus, $P$ ends with $\ell$ vertices from $U_{j_\ell}$; we denote their set by $L_2$.

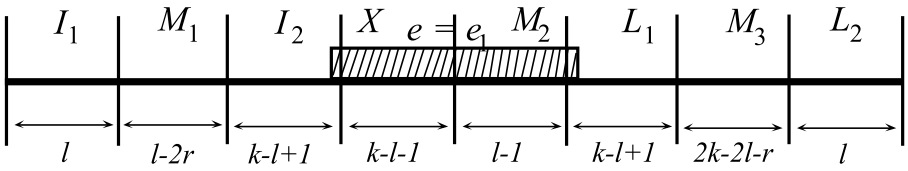


Figure 5: The construction of the $\ell$-path $P$

The construction of the path $P$ is illustrated in Fig. 5.

Let us now estimate how many vertices of each set $U_j$ are used by the above constructed path $P$. Set $r_j = |V(P) \cap U_j|$, $j = 1, \ldots, n$.

**Fact 4.3.** *$r_i \leqslant 2k - \ell$, $r_{j_\ell} \leqslant 2k - \ell$ and $r_j \leqslant k - \ell + 2$ for all $j \notin \{i, j_\ell\}$.*

*Proof.* Note that since $c(e_1) \geqslant \ell + 1$, $|e_1 \cap U_j| \leqslant k - \ell$ for each $j \in [1, n]$. In addition, $P$ uses $\ell + (k - \ell) = k$ vertices of both, $U_i$ and $U_{j_\ell}$, and at most two vertices of each set $U_{j_t}$, $t = 1, \ldots, \ell - 1$. □

**Fact 4.4.** *Every edge of $P$ belongs to $H_2 + e$.*

*Proof.* Let us split the edges of $P$ into those appearing "before $e$" (b.e.) and "after $e$" (a.e.) Formally, set

$$P = B_e \cup \{e\} \cup A_e,$$

where $B_e = \{e_{m-2q}, \ldots, e_m\}$ and $A_e = \{e_2, \ldots, e_{q+4}\}$ (recall that $e_1 = e$). We will give the proof first for the a.e. edges and then for the b.e. edges. Set

$$I = I_1 \cup I_2 \qquad M = M_1 \cup M_2 \cup M_3 \qquad L = L_1 \cup L_2.$$

Let $f \in P - e$.

**Case $f \in A_e$:** In this case, $f \cap X = \emptyset$ and $\min(f) = j_\ell$. Consequently, $f \cap U_{\min(f)} = f \cap L$ and our first goal (c.f. the definition of $H_2$) is to show that

$$|f \cap L| \geqslant k - \ell + 1. \tag{5}$$

Observe that either $f \supset L_1$, in which case (5) is true, or $f \subset L \cup M_3$ and so,

$$|f \cap L| = k - |M_3| = k - (2k - 2\ell - r) = 2\ell - k + r \geqslant k - \ell + 1,$$

because $r \geqslant 1$ and $\ell \geqslant \frac{2}{3}k$. Thus (5) holds again.

As a next step we will show that

$$c(f) \geqslant k - \ell + 1. \tag{6}$$

Note that

$$|f \cap M| = |f \cap (M_2 \cup M_3)| \leqslant k - |L_1| = k - (k - \ell + 1) = \ell - 1.$$

Hence, the elements of $f \cap M_2$ and $f \cap M_3$ come from different sets $U_j$, $j \in \{1, \ldots, \ell-1\}$ and, consequently, $c(f) = |f \cap M| + 1$, where we add 1 because of $U_{j_\ell}$ (recall that the set $\{j_1 \ldots, j_\ell\}$ is independent in $G$). If $f \supset M_3$ then

$$c(f) \geqslant |M_3| + 1 = (2k - 2\ell - r) + 1 \geqslant k - \ell + 1,$$

since $r \leqslant k - \ell$. Otherwise, $f \cap L_2 = \emptyset$ and

$$c(f) = k - |L_1| + 1 = \ell \geqslant k - \ell + 1$$

and (6) holds again. Since, clearly,

$$|f \cap U_{\min(f)}| + c(f) = |f| + 1 = k + 1 \geqslant \ell + 2,$$

$f \in H_2$, and so $A_e \subseteq H_2$.

**Case** $f \in B_e$**:** In this case, $\min(f) = i$. Consequently, $f \cap U_{\min(f)} = f \cap I$ and our first goal is to show that

$$|f \cap I| \geqslant k - \ell + 1. \tag{7}$$

Observe that the first (that is, with the smallest index) vertex of $f$ coincides with, or is to the left of the first vertex of $I_2$. Thus, $f \supset I_2$ or

$$|f \cap I| \geqslant k - |M_1| = k - \ell + 2r \geqslant k - \ell + 2,$$

and in either case (7) holds.

As a next step we will prove that

$$c(f) \geqslant k - \ell + 1. \tag{8}$$

Note that

$$|f \cap M| = |f \cap (M_1 \cup M_2)| \leqslant k - |I_2| - |X| = k - 2(k - \ell) \leqslant \ell - 1.$$

Hence, the elements of $f \cap M_1$ and $f \cap M_2$ come from different sets $U_j$, $j \in \{1, \ldots, \ell-1\}$ and, again, $c(f) = |f \cap M| + 1$. If $f \cap I_1 = \emptyset$ then

$$|f \cap M| \geqslant k - |I_2| - |X| = k - 2(k - \ell) \geqslant k - \ell,$$

because $\ell \geqslant \frac{2}{3}k$. If $f \supset M_1$ then

$$|f \cap M| \geqslant |M_1| = \ell - 2r \geqslant k - \ell,$$

because $r \leqslant k - \ell$ and $\ell \geqslant \frac{3}{4}k$. Otherwise, that is, when $f \cap I_1 \neq \emptyset$ but $f \not\supset M_1$,

$$|f \cap M| = k - |f \cap I_1| \geqslant k - \ell.$$

Thus, (8) holds in all cases.

It remains to prove that

$$|f \cap U_{\min(f)}| + c(f) \geqslant \ell + 2. \tag{9}$$

Recall that $\min(f) = i$ and $|f \cap U_i| = |f \cap I|$, while $c(f) = |f \cap M| + 1$. Since, clearly,

$$|f \cap I| + |f \cap M| + |f \cap X| = |f| = k,$$

we have

$$|f \cap U_{\min(f)}| + c(f) \geqslant k + 1 - |f \cap X| \geqslant k + 1 - (k - \ell - 1) = \ell + 2.$$

Hence, $f \in H_2$ and, consequently, $B_e \subseteq H_2$. This completes the proof of Fact 4.4. □

**Part 2.** Recall that $\{i, j_l\}$ is not an edge of $G$. Hence, by the Hamiltonian saturation property of $G$, there is a Hamiltonian path $Q$ from $j_\ell$ to $i$ in $G$. As in the loose ($\ell = 1$) and tight ($\ell = k - 1$) cases treated earlier, we build the rest of $C_H$ by 'tracing' $Q$. Each time we visit a vertex $x \in V(Q)$ we consecutively include to $C_H$ all vertices from $U_x \setminus V(P)$ (in any order). This way we create an $\ell$-path $P'$ consisting of $k$-tuples $e_{q+5}, \ldots, e_{m-2q-1}$.

Note that by Fact 4.3 and the lower bound in (1), we have

$$|U_x \setminus V(P)| = |U_x| - r_x \geqslant (2k - \ell + 1) - (k - \ell + 2) = k - 1 \tag{10}$$

for each $x \in V(Q) \setminus \{i, j_\ell\}$. Hence, $|tr(e_j)| \leqslant 2$, for all $j = q+5, \ldots, m-2q+1$. Moreover, for each such $j$ with $|tr(e_j)| = 2$ the pair $tr(e_j)$ is an edge of $G$. Therefore, $e_j \in H_1$, for $j = q + 5, \ldots, m - 2q + 1$. In conclusion, $C_H = P \cup P'$ is an $\ell$-Hamiltonian path in $H_1 \cup H_2 + e \subseteq H + e$, which completes the proof of Lemma 4.2. □

**The conclusion of the proof of Theorem 1.1.** In order to prove Theorem 1.1 for $\ell \geqslant \frac{4}{5}k$ we need to construct, for every sufficiently large $N$ divisible by $k - \ell$, an $\ell$-Hamiltonian saturated $k$-graph $H$ with $N$ vertices and $O(N^\ell)$ edges. Assume first that $\ell > \frac{4}{5}k$. As then $2k - \ell + 2 \leqslant 4\ell - 2k + 1$ we may use as the sizes $a_i = |U_i|$ both numbers, $2k - \ell + 1$ and $2k - \ell + 2$. It is well known that every number $N \geqslant N_0 = N_0(k, \ell)$ (the Frobenius number) can be expressed as a sum of these two numbers. For an $N$ divisible by $k - \ell$, let us fix one such partition

$$N = a_1 + \cdots + a_n, \qquad 2k - \ell + 1 \leqslant a_i \leqslant 2k - \ell + 2,$$

and let $H$ be as in Lemma 4.2. Then, by (3), $H$ indeed is an $\ell$-Hamiltonian saturated $k$-graph with $N$ vertices and $O(N^\ell)$ edges.

In the critical case $\ell = \frac{4}{5}k$, we need to refine our previous estimates a bit. Assume that for some integer $p \geqslant 1$, we have $k = 5p$ and $\ell = 4p$. Then, by (4), $r = p$, and so, $2r = 2p > 2k - 2\ell - r = p$. Thus, every index $j \in \{j_1, \ldots, j_{\ell-1}\}$ appears at most once in the set $M_1 \cup M_3$, and consequently, we can improve the bound on $r_j$ from Fact 4.3 down to $k - \ell + 1$. This implies, in turn, that the crucial estimate $|U_x| - r_x \geqslant k - 1$ from Part 2 of the construction of the cycle $C_H$ in the proof of Lemma 4.2 (see (10)) remains valid

even for sets $U_x$ with $|U_x| = 2k - \ell$. Note that the lower bound in (1) was not used in any other part of the proof. We may thus complete the proof as before, expressing $N$ this time as

$$N = a_1 + \cdots + a_n, \qquad 2k - \ell \leqslant a_i \leqslant 2k - \ell + 1.$$

□

# 5 Remarks and open problems

Note that in the case $\ell = k - 1$ our Theorem 1.1, as stated, covers only $k \geqslant 5$. However, in the proof of Lemma 4.2 we could have $k \leqslant a_j = |U_j| = k + 1$. Indeed, then we still have $|U_j'| \geqslant |U_j| - 1 \geqslant k - 1$, while the punch-line inequality in the proof of Lemma 4.1, that is, $|U_i| \leqslant k + 1 \leqslant 4\ell - 2k + 1 = 2k - 3$ holds already for $k \geqslant 4$. So, in fact, our proof of Theorem 1.1 works also in the case $k = 4, \ell = 3$. Moreover, for $k = 3$, by fixing $|U_j| = 3$ for all $j$, the proofs of both lemmas, Lemma 4.1 and Lemma 4.2, go through and yield that $sat(3n, 3, 2) = \Theta(n^2)$. As we mentioned in the Introduction, it has been proved in [18], via a different construction, that $sat(n, k, k - 1) = \Theta(n^{k-1})$ for all $k \geqslant 3$.

A big open problem is to extend our result to all $1 \leqslant \ell \leqslant k - 1$, that is, to prove the following conjecture.

**Conjecture 5.1.** *For all* $1 \leqslant \ell \leqslant k - 1$, $k \geqslant 2$, $sat(n, k, \ell) = \Theta(n^\ell)$.

The smallest open case is $k = 4, \ell = 2$.


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