Congruences for q-Lucas numbers

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Abstract

For $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ and $\nu=(\alpha,\beta,\gamma,\delta)$, the q-Fibonacci numbers are given by

$$F_0^{\nu}(q)=0,\ F_1^{\nu}(q)=1 \text{ and } F_{n+1}^{\nu}(q)=q^{\alpha n-\beta}F_n^{\nu}(q)+q^{\gamma n-\delta}F_{n-1}^{\nu}(q) \text{ for } n\geqslant 1.$$

And define the q-Lucas number $L_n^{\nu}(q)=F_{n+1}^{\nu}(q)+q^{\gamma-\delta}F_{n-1}^{\nu_*}(q)$, where $\nu_*=(\alpha,\beta-\alpha,\gamma,\delta-\gamma)$. Suppose that $\alpha=0$ and γ is prime to n, or $\alpha=\gamma$ is prime to n. We prove that

$$L_n^{\nu}(q)\equiv(-1)^{\alpha(n+1)} \pmod{\Phi_n(q)}$$

for $n\geqslant 3$, where $\Phi_n(q)$ is the n-th cyclotomic polynomial. A similar congruence for q-Pell-Lucas numbers is also established.

Keywords: q-Lucas number; q-congruence; cyclic layered match partition

1 Introduction

The Fibonacci numbers F_n are given by

$$F_0=0,\ F_1=1 \text{ and } F_{n+1}=F_n+F_{n-1} \text{ for } n\geqslant 1.$$

The Fibonacci numbers have rich arithmetic properties. For example, if p is a prime, then we have

$$F_p\equiv\left(\frac{5}{p}\right) \pmod p,\ \text{ and } F_{p-\left(\frac{5}{p}\right)}\equiv 0 \pmod p, \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol.

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The Lucas numbers L_n are given by

$$L_0 = 2,\ L_1 = 1 \text{ and } L_{n+1} = L_n + L_{n-1} \text{ for } n \geqslant 1.$$

It is easy to check that $L_n = F_{n+1} + F_{n-1}$ for $n \geqslant 1$. So in view of (1.1), we get

$$L_p \equiv 1 \pmod p \tag{1.2}$$

for each prime p.

Suppose that $\alpha, \beta, \gamma, \delta$ are integers and $\nu = (\alpha, \beta, \gamma, \delta)$. Define the q-Fibonacci numbers $F_n^\nu(q)$ as

$$F_0^\nu(q) = 0,\ F_1^\nu(q) = 1 \text{ and } F_{n+1}^\nu(q) = q^{\alpha n-\beta} F_n^\nu(q) + q^{\gamma n-\delta} F_{n-1}^\nu(q) \text{ for } n \geqslant 1.$$

In [1, 2], Andrews showed that $F_n^{(0,0,1,2)}(q)$ and $F_n^{(0,0,1,1)}(q)$ are respectively correspond to two kinds of the Rogers-Ramanujan identities. Furthermore, Goyt and Sagan [5] found that $F_n^{(1,0,1,1)}(q)$ is related to some set partitions statistics.

Andrews [1] also proved that for prime $p \equiv \pm 2 \pmod 5$,

$$F_{p+1}^{(0,0,1,2)}(q) \equiv 0 \pmod{[p]_q},$$

where $[p]_q = (1-q^p)/(1-q)$ and the above congruence is considered in the polynomial ring $\mathbb{Z}[q]$. This is a partial q-analogue of (1.1). The complete q-analogues of (1.1) for $F_n^{(0,0,1,2)}(q)$ and $F_n^{(0,0,1,1)}(q)$ are given in [6].

Define the q-Lucas numbers to be

$$L_n^\nu(q) = F_{n+1}^\nu(q) + q^{\gamma-\delta} F_{n-1}^{\nu_*}(q),$$

where $\nu_* = (\alpha, \beta-\alpha, \gamma, \delta-\gamma)$. In this note, we shall establish a q-analogue of (1.2).

Theorem 1.1. *Let $\alpha, \beta, \gamma, \delta$ be integers and $\nu = (\alpha, \beta, \gamma, \delta)$. Suppose that $\alpha = 0$ and γ is prime to n, or $\alpha = \gamma$ is prime to n. Then*

$$L_n^\nu(q) \equiv (-1)^{\alpha(n+1)} \pmod{\Phi_n(q)} \tag{1.3}$$

for $n \geqslant 3$, where $\Phi_n(q)$ is the n-th cyclotomic polynomial.

In the next section, we shall prove Theorem 1.1 via purely combinatorial techniques. And in the third section, a similar result for q-Pell-Lucas numbers will be given.

2 The combinatorics of q-Lucas numbers

In fact, before Leonardo Fibonacci, the combinatorics of Fibonacci numbers had been discussed by the ancient indian scholars, arising from a problem on rhythmic patterns [4]. Here we shall use the language of set partitions to describe such a combinatorial interpretation.

For $[n] = \{1, 2, 3, \dots, n\}$, we say $\pi = B_1/B_2/\cdots/B_k$ is a partition of $[n]$, provided that $[n]$ is the disjoint union of $B_1, B_2, \dots, B_k$. In particular, we may assume that

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

A partition $\pi = B_1/\cdots/B_k$ of $[n]$ is called *layered*, if π is of the form

$$[1, a_1]/[a_1 + 1, a_2]/[a_2 + 1, a_3]/\cdots/[a_{k-2}, a_k]/[a_{k-1} + 1, n].$$

For example, 123/45/678/9 is a layered partition of [9]. On the other hand, we say a partition $B_1/\cdots/B_k$ is *match*, provided $|B_i| \leqslant 2$ for all i. Then the Fibonacci number F_{n+1} is the number of all layered match partitions of $[n]$. In fact, for $s = 1, 2$, it is easy to see that the the number of layered match partitions of $[n]$ with $|B_k| = s$ equals to the the number of layered match partitions of $[n - s]$. Thus by induction on n, $[n]$ has exactly $F_{n+1} = F_n + F_{n-1}$ layered match partitions.

We say a partition $\pi = B_1/\cdots/B_k$ of $[n]$ is *cyclic layered*, if π is either layered, or of the form

$$([a_{k-1} + 1, n] \cup [1, a_1])/[a_1 + 1, a_2]/\cdots/[a_{k-2}, a_{k-1}].$$

For example, 912/34/5/678 is a cyclic layered partition of [9]. Then for $n \geqslant 3$, the Lucas number L_n is the number of all cyclic layered match partitions of $[n]$. In fact, the number of all layered match partitions of $[n]$ is F_{n+1}. And the number of the cyclic layered match partitions of $[n]$ with $|B_1| = \{n, 1\}$ is exactly F_{n-1}. Hence $[n]$ has $L_n = F_{n+1} + F_{n-1}$ cyclic layered match partitions.

For $\nu = (\alpha, \beta, \gamma, \delta)$ and a cyclic layered match partition $\pi = B_1/\cdots/B_k$ of $[n]$, define

$$\sigma_\nu(\pi) = \sum_{\substack{|B_i|=1 \\ B_i=\{a_i\}}} (\alpha a_i - \beta) + \sum_{\substack{|B_i|=2 \\ a_i=\max B_i}} (\gamma a_i - \delta),$$

where if $B_1 = \{n, 1\}$, we set $\max B_1 = 1$, rather than n. Let $\mathfrak{L}_n$ and $\mathfrak{C}_n$ respectively denote the sets of all layered match partitions and all cyclic layered match partitions of $[n]$. Suppose that $\pi = B_1/\cdots/B_k \in \mathfrak{L}_n$. If $|B_k| = 1$, i.e., $B_k = \{n\}$, then

$$\sigma_\nu(\pi) = \sigma(B_1/\cdots/B_{k-1}) + (\alpha n - \beta).$$

And if $|B_k| = 2$, i.e., $B_k = \{n - 1, n\}$, then

$$\sigma_\nu(\pi) = \sigma_\nu(B_1/\cdots/B_{k-1}) + (\gamma n - \delta).$$

Hence in view of the recurrence relation of $F_n^\nu(q)$, we get

$$F_{n+1}^\nu(q) = \sum_{\pi \in \mathfrak{L}_n} q^{\sigma_\nu(\pi)}. \tag{2.1}$$

Consider $\pi = B_1/\cdots/B_k \in \mathfrak{C}_n \setminus \mathfrak{L}_n$, i.e., $B_1 = \{n, 1\}$. Then

$$\sigma_\nu(\pi) = \sigma_\nu(B_2/\cdots/B_{k-1}) + (\gamma - \delta) = \sigma_{\nu_*}(B_2'/\cdots/B_k') + (\gamma - \delta),$$

where $B_j' = \{i-1 : i \in B_j\}$. Thus for $n \geqslant 3$,

$$L_n^\nu(q) = F_{n+1}^\nu(q) + q^{\gamma-\delta} F_{n-1}^{\nu_*}(q) = \sum_{\pi\in\mathfrak{L}_n} q^{\sigma_\nu(\pi)} + q^{\gamma-\delta} \sum_{\pi'\in\mathfrak{L}_{n-2}} q^{\sigma_{\nu_*}(\pi')} = \sum_{\pi\in\mathfrak{C}_n} q^{\sigma_\nu(\pi)}.$$

Our proof of Theorem 1.1 will follow the way of Sagan [7]. The case $n = 2$ can be verified directly:

$$L_2^\nu(q) = q^{3\alpha-2\beta} + q^{2\gamma-\delta} + q^{\gamma-\delta} \equiv (-1)^{3\alpha} \pmod{1+q},$$

by noting that $1+q$ divides $1+q^\gamma$ if γ is odd. Below suppose that $n \geqslant 3$. Let $G = \mathbb{Z}/n\mathbb{Z}$ and think of G as $\{1, 2, \dots, n\}$. For an integer x, define $(x)_n \in \{1, 2, \dots, n\}$ such that $x \equiv (x)_n \pmod{n}$. For $g \in G$ and $\pi = B_1/\cdots/B_k \in \mathfrak{C}_n$, define

$$g\pi = (B_1+g)/(B_2+g)/\cdots(B_k+g),$$

where $B_j + g = \{(i+g)_n : i \in B_j\}$. It is easy to see that this is a group action of G on $\mathfrak{C}_n$. For $\pi \in \mathfrak{C}_n$, define the orbit of π

$$\mathcal{O}_\pi = \{g\pi : g \in G\},$$

and the stabilizer of π

$$\mathcal{S}_\pi = \{g \in G : g\pi = \pi\}.$$

Clearly $\mathfrak{C}_n$ can be partitioned into the disjoint union of the orbits. And let $\mathfrak{O}_n$ denote the set of all those orbits. Then

$$L_n^\nu(q) = \sum_{\mathcal{O}\in\mathfrak{O}_n} \sum_{\pi\in\mathcal{O}} q^{\sigma_\nu(\pi)}.$$

Below we shall show that for $\pi \in \mathfrak{C}_n$,

$$\sum_{\tau\in\mathcal{O}_\pi} q^{\sigma_\nu(\tau)} \equiv 0 \pmod{\Phi_n(q)},$$

provided $|\mathcal{O}_\pi| > 1$. Clearly $|\mathcal{O}_\pi| > 1$ if and only if $|\mathcal{S}_\pi|$ is a proper divisor of n. Suppose that $d > 1$ is a divisor of n. Then $d \in \mathcal{S}_\pi$ if and only if π is of the form

$$B_1/\cdots/B_k/(B_1+d)/\cdots/(B_k+d)/\cdots/(B_1+n-d)/\cdots/(B_k+n-d).$$

Let

$$\mathcal{A}_\pi = \{a : B_i = \{a\} \text{ for some } i\}, \quad \mathcal{B}_\pi = \{a : B_i = \{(a-1)_n, a\} \text{ for some } i\}.$$

Note that for $1 \leqslant g \leqslant d$,

$$\begin{aligned}
\sigma_\nu(g\pi) &= \sum_{a\in\mathcal{A}_\pi} (\alpha(a+g)_n - \beta) + \sum_{a\in\mathcal{B}_\pi} (\gamma(a+g)_n - \delta) \\
&\equiv \sum_{\{a\}\in\mathcal{A}_\pi} (\alpha(a+g) - \beta) + \sum_{a\in\mathcal{B}_\pi} (\gamma(a+g) - \delta) \\
&\equiv \sigma_\nu(\pi) + \alpha g|\mathcal{A}_\pi| + \gamma g|\mathcal{B}_\pi| \pmod{n}.
\end{aligned}$$

Hence if $\mathcal{S}_\pi$ is generated by d,

$$\sum_{\tau\in\mathcal{O}_\pi} q^{\sigma_\nu(\tau)} = \sum_{g=1}^{d} q^{\sigma_\nu(g\pi)} \equiv q^{\sigma_\nu(\pi)} \sum_{g=1}^{d} q^{g(\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|)}$$
$$= q^{\sigma_\nu(\pi)+(\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|)} \cdot \frac{q^{d(\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|)}-1}{q^{\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|}-1} \pmod{[n]_q}.$$

Since $|\mathcal{B}_\pi| > 1$ now, we have $|\mathcal{A}_\pi| + |\mathcal{B}_\pi| < n$. Hence under the assumptions of Theorem 1.1, we always have $\alpha|\mathcal{A}_\pi| + \gamma|\mathcal{B}_\pi|$ is not divisible by n. Furthermore both $|\mathcal{A}_\pi|$ and $|\mathcal{B}_\pi|$ are multiples of n/d. Hence

$$\frac{q^{d(\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|)}-1}{q^{\alpha|\mathcal{A}_\pi|+\gamma|\mathcal{B}_\pi|}-1} \equiv 0 \pmod{\Phi_n(q)}.$$

On the other hand, it is easy to see that the unique $\pi \in \mathfrak{C}_n$ with $|\mathcal{O}_\pi| = 1$ is $1/2/\cdots/n$. So

$$L_n^\nu(q) \equiv q^{\sigma_\nu(1/\cdots/n)} q^{\alpha\binom{n+1}{2}-\beta n} \pmod{\Phi_n(q)}.$$

Finally, let us explain why

$$q^{\alpha\binom{n+1}{2}-\beta n} \equiv (-1)^{\alpha(n+1)} \pmod{\Phi_n(q)}.$$

Clearly $\alpha\binom{n+1}{2}$ is divisible by n, if n is odd or α is even. And when n is even and α is odd, we have

$$q^{\alpha\binom{n+1}{2}-\beta n} \equiv q^{n/2} = -1 + 1 + q^{n/2} = -1 + \frac{1-q^n}{1-q^{n/2}} \equiv -1 \pmod{\Phi_n(q)}.$$

The proof of Theorem 1.1 is concluded. □

Suppose that $\gamma = 0$ and α is prime to n. Then n divides $\alpha|\mathcal{A}_\pi| + \gamma|\mathcal{B}_\pi|$ if and only if $|\mathcal{A}_\pi| = 0$, i.e., n is even and $\pi = 12/34/\cdots/(n-1)n$ or $\pi = n1/23/\cdots/(n-2)(n-1)$. So if n is even, then

$$L_n^\nu(q) \equiv q^{\sigma_\nu(1/\cdots/n)} + q^{\sigma_\nu(12/\cdots/(n-1)n)} + q^{\sigma_\nu(n1/\cdots/(n-2)(n-1))}$$
$$= q^{\alpha\binom{n+1}{2}-\beta n} + q^{-\delta n/2} + q^{-\delta n/2} \equiv (-1)^{\alpha(n+1)} + 2(-1)^\delta \pmod{\Phi_n(q)}.$$

Hence we have

Theorem 2.1. *Suppose that α, β, δ are integers and α is prime to n. Then*

$$L_n^{(\alpha,\beta,0,\delta)}(q) \equiv \begin{cases} 1 \pmod{\Phi_n(q)}, & \text{if } n \text{ is odd,} \\ (-1)^\alpha + 2(-1)^\delta \pmod{\Phi_n(q)}, & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

for $n \geqslant 2$.

3 q-Pell-Lucas numbers

The Pell numbers P_n are given by

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geqslant 1.$$

And the Pell-Lucas numbers Q_n are given by

$$Q_0 = 2,\ Q_1 = 2 \text{ and } Q_{n+1} = 2Q_n + Q_{n-1} \text{ for } n \geqslant 1.$$

Similar to the Fibonacci numbers and the Lucas numbers, we have $Q_n = P_{n+1} + P_{n-1}$ and

$$Q_p \equiv 2 \pmod{p}. \tag{3.1}$$

Define a marked layered match partition of $[n]$ to be a layered match partition $\pi = B_1/\cdots/B_k$ of $[n]$ such that some B_i with $|B_i| = 1$ may be marked. For example, $\overline{1}/23/\overline{4}/5$ and $\overline{1}/23/4/\overline{5}$ are two different marked layered match partitions of [5], where $\overline{a}$ means the part $B_i = \{a\}$ are marked. It is not difficult to see that the Pell number P_{n+1} is the number of all marked layered match partitions of $[n]$. For example, $P_3 = 5$ and all marked layered match partitions of [2] are $1/2$, $\overline{1}/2$, $1/\overline{2}$, $\overline{1}/\overline{2}$, 12. Similarly, the Pell-Lucas number Q_n is the number of all cyclic marked layered match partitions of $[n]$ for $n \geqslant 3$.

Now for $\nu = (\alpha, \beta, \gamma, \delta)$, define

$$P_0^{\nu}(q) = 0,\ P_1^{\nu}(q) = 1 \text{ and } P_{n+1}^{\nu}(q) = (1 + q^{\alpha n - \beta})P_n^{\nu}(q) + q^{\gamma n - \delta} P_{n-1}^{\nu}(q) \text{ for } n \geqslant 1.$$

In [8], Santos and Sills showed $P_n^{(1,1,1,2)}(q)$ and $P_n^{(1,1,1,1)}(q)$ respectively correspond to two identities of Lebesgue. And the combinatorics of $P_n^{(1,1,1,2)}(q)$ has been studied by Briggs, Little and Sellers [3], in which they used the notion of tilings.

Let

$$Q_n^{\nu}(q) = P_{n+1}^{\nu}(q) + q^{\gamma - \delta} P_{n-1}^{\nu_*}(q),$$

where $\nu_* = (\alpha, \beta - \alpha, \gamma, \delta - \gamma)$. For a marked layered match partition $\pi = B_1/\cdots/B_k$ of $[n]$, define

$$\overline{\mathcal{A}}_\pi = \{a :\ B_i = \{a\} \text{ for some } i \text{ and } B_i \text{ is marked}\}.$$

And define

$$\overline{\sigma}_\nu(\pi) = \sum_{a \in \overline{\mathcal{A}}_\pi} (\alpha a - \beta) + \sum_{a \in \mathcal{B}_\pi} (\gamma a - \delta),$$

where $\mathcal{B}_\pi$ is the set of all those a such that $\{(a-1)_n, a\} = B_i$ for some i. Let $\overline{\mathfrak{L}}_n$ and $\overline{\mathfrak{C}}_n$ respectively denote the set of all marked layered match partitions and all marked cyclic layered match partitions of $[n]$. Then we have

$$P_{n+1}^{\nu}(q) = \sum_{\pi \in \overline{\mathfrak{L}}_n} q^{\overline{\sigma}_\nu(\pi)}.$$

Similarly, for $n \geqslant 3$,

$$Q_n^{\nu}(q) = \sum_{\pi \in \overline{\mathfrak{C}}_n} q^{\overline{\sigma}_\nu(\pi)}.$$

Suppose that $\alpha = \gamma$ is prime to n. Clearly,

$$Q_2^\nu(q) = 1 + q^{3\alpha-2\beta} + q^{2\alpha-\beta} + q^{\alpha-\beta} + q^{2\gamma-\delta} + q^{\gamma-\delta} \equiv 1 + (-1)^{3\alpha} \pmod{1+q}$$

if both α and γ are odd. So we may assume that $n \geqslant 3$. Consider the same action of $G = \mathbb{Z}/n\mathbb{Z}$ on $\overline{\mathfrak{C}}_n$ as we used before. Evidently for $\pi \in \overline{\mathfrak{C}}_\pi$, $|\mathcal{O}_\pi| = 1$ if and only if $\pi = 1/2/\cdots/n$ or $\overline{1}/\overline{2}/\cdots/\overline{n}$. We shall prove that

$$\sum_{\tau \in \mathcal{O}_\pi} q^{\overline{\sigma}_\nu(\tau)} \equiv 0 \pmod{\Phi_n(q)}$$

for those $\pi \in \overline{\mathfrak{C}}_n$ with $|\mathcal{O}_\pi| > 1$. Suppose that $d > 1$ is a divisor of n and $\mathcal{S}_\pi = \{d, 2d, \ldots, n\}$. Then for $1 \leqslant g \leqslant d$,

$$\begin{aligned}\overline{\sigma}_\nu(g\pi) &\equiv \sum_{a \in \overline{\mathcal{A}}_\pi} (\alpha(a+g) - \beta) + \sum_{a \in \mathcal{B}_\pi} (\gamma(a+g) - \delta)\\ &\equiv \overline{\sigma}_\nu(\pi) + \alpha g|\overline{\mathcal{A}}_\pi| + \gamma g|\mathcal{B}_\pi| \pmod{n}.\end{aligned}$$

Hence we have

$$\sum_{\tau \in \mathcal{O}_\pi} q^{\overline{\sigma}_\nu(\tau)} = \sum_{g=1}^{d} q^{\overline{\sigma}_\nu(g\pi)} \equiv \sum_{g=1}^{d} q^{\overline{\sigma}_\nu(\pi) + \alpha g|\overline{\mathcal{A}}_\pi| + \gamma g|\mathcal{B}_\pi|} \equiv 0 \pmod{\Phi_n(q)},$$

by noting that n/d divides $|\overline{\mathcal{A}}_\pi|, |\mathcal{B}_\pi|$ and n doesn't divide $\alpha|\overline{\mathcal{A}}_\pi| + \gamma|\mathcal{B}_\pi|$. Thus we obtain

Theorem 3.1. *Let α, β, δ be integers and $\nu = (\alpha, \beta, \alpha, \delta)$. If α is prime to n, then*

$$Q_n^\nu(q) \equiv 1 + (-1)^{\alpha(n+1)} \pmod{\Phi_n(q)} \tag{3.2}$$

for $n \geqslant 2$.

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