Labeling the regions of the type C_n Shi arrangement

Karola Mészáros*

Department of Mathematics
Cornell University
Ithaca, NY 14853

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Abstract

The number of regions of the type C_n Shi arrangement in $\mathbb{R}^n$ is $(2n+1)^n$. Strikingly, no bijective proof of this fact has been given thus far. The aim of this paper is to provide such a bijection and use it to prove more refined results. We construct a bijection between the regions of the type C_n Shi arrangement in $\mathbb{R}^n$ and sequences $a_1a_2\ldots a_n$, where $a_i \in \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$, $i \in [n]$. Our bijection naturally restrict to bijections between special regions of the arrangement and sequences with a given number of distinct elements.

Keywords: type C_n Shi arrangements, sequences, posets, nonnesting partitions.

1 Introduction

A **hyperplane arrangement** $\mathcal{A}$ is a finite set of affine hyperplanes in $\mathbb{R}^n$. The **regions** of $\mathcal{A}$ are the connected components of the space $\mathbb{R}^n \backslash \cup_{H\in\mathcal{A}} H$. In this paper we study the **type** C_n **Shi arrangement**, which is an affine hyperplane arrangement whose hyperplanes are parallel to reflecting hyperplanes of the type C_n Coxeter group. The closely related type A_{n-1} Shi arrangement has been much studied before. Shi [7] proved the beautiful result that the number of regions of the type A_{n-1} Shi arrangement is $(n+1)^{n-1}$. This statement is clearly deserving of a combinatorial proof; two different bijections proving this result were provided by Stanley and Pak [9, 10] and Athanasiadis and Linusson [4]. Our type C_n results can be considered a generalization of the Athanasiadis-Linusson bijection. In their work on parking spaces [2], Armstrong, Reiner and Rhoades provide another generalization of the Athanasiadis-Linusson bijection.

We now review the definitions necessary to state our results.

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The C_n **Coxeter arrangement** $\mathrm{Cox^C(n)}$ in $\mathbb{R}^n$ is defined as follows.

$$\mathrm{Cox^C(n)} = \{x_i - x_j = 0, x_i + x_j = 0, 2x_k = 0 \mid 1 \leqslant i < j \leqslant n, k \in [n]\}.$$

The regions of the arrangements $\mathrm{Cox^C(n)}$ naturally correspond to type C_n permutations $w \in \mathfrak{S}_n^C$. Recall that $\mathfrak{S}_n^C$ is the group of all bijections w of the set $[\pm n] = \{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\}$ onto itself such that

$$w(-i) = -w(i),$$

for all $i \in [\pm n]$ and composition as group operation. The notation $w = [a_1, \ldots, a_n]$ means $w(i) = a_i$, for $i \in [n]$, and is called the **window** of w. In **line notation** this same $w = -a_n \ -a_{n-1} \ldots - a_1 \ a_1 \ldots a_{n-1} \ a_n$.

Let $C^C \subset \mathbb{R}^n$ be our distinguished **cone** of $\mathrm{Cox^C(n)}$ corresponding to the type C_n identity permutation:

$$C^C = \{\mathbf{x} \in \mathbb{R}^n \mid -x_n > -x_{n-1} > \cdots > -x_2 > -x_1 > x_1 > x_2 > \cdots > x_{n-1} > x_n\}.$$

Let

$$wC^C = \{\mathbf{x} \in \mathbb{R}^n | x_{w(-n)} > x_{w(-n+1)} > \cdots > x_{w(-1)} > x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)}\},$$

where $\{x_1, \ldots, x_n\}$ are the standard coordinate functions on $\mathbb{R}^n$ and $x_{-i} = -x_i$ for $i < 0$. It follows that the number of regions of $\mathrm{Cox^C(n)}$ is $|\mathfrak{S}_n^C| = 2^n n!$.

The **type C_n Shi arrangement $\mathcal{S}_n^C$** [7] is:

$$\mathcal{S}_n^C = \mathrm{Cox^C(n)} \cup \{x_i - x_j = 1, x_i + x_j = 1, 2x_k = 1 \mid 1 \leqslant i < j \leqslant n, k \in [n]\}.$$

We construct a bijection between the regions of the type C_n Shi arrangement $\mathcal{S}_n^C$ in $\mathbb{R}^n$ and sequences in the set

$$\mathcal{A}^C(n) = \{(a_1, a_2, \ldots, a_n) | a_i \in \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}, i \in [n]\}.$$

Theorem 1. *The map ϕ, which is defined in Section 3, is a bijection between the regions of $\mathcal{S}_n^C$ and sequences in the set $\mathcal{A}^C(n)$.*

Athanasiadis and Linusson [4, Section 4, Question 3] were the first to ask for the construction of such bijection in their paper dealing with the type A_{n-1} case. The properties of our bijection yield Theorem 2. To state it we need a little more terminology. A hyperplane H is a **wall** of a region R if it is the affine span of a codimension-1 face of R. A wall H is called a **floor** if H does not contain the origin and R and the origin lie in opposite half-spaces defined by H. Denote by $f(R)$ the number of floors of R. A wall H is called a **ceiling** if H does not contain the origin and R and the origin lie in the same half-spaces defined by H. Denote by $c(R)$ the number of ceilings of R. Denote by $R(\mathcal{H})$ the set of regions of the hyperplane arrangement $\mathcal{H}$.

Theorem 2.
$$\sum_{R\in R(\mathcal{S}_n^C)} q^{c(R)} = \sum_{R\in R(\mathcal{S}_n^C)} q^{f(R)} = \sum_{\mathbf{a}\in\mathcal{A}^C(n)} q^{n-d^C(\mathbf{a})},$$
where $d^C(\mathbf{a})$ is the number of distinct absolute values of the nonzero numbers appearing in $\mathbf{a}$.

The outline of the paper is as follows. In Section 2 we explain the connection between the regions of the type C_n Shi arrangements and the poset of nonnesting C_n-partitions. In Section 3 we build on this connection to prove Theorems 1 and 2. Section 4 reiterates the basic thoughts of the paper on the level of posets and sequences.

2 Posets and the regions of $\mathcal{S}_n^C$

In this section we explain how to label a region R of $\mathcal{S}_n^C$ by the set of its ceilings and the permutation $w \in \mathfrak{S}_n^C$, if R is in the cone wC^C. We will see that the set of ceilings can be encoded as certain antichains of a special poset. Such an approach is inspired by a correspondence developed by Stanley in [9, Section 5] between the antichains of a family of posets and regions of the type A_{n-1} Shi arrangement. Our bijection between the regions of $\mathcal{S}_n^C$ and sequences (as presented in Section 3) will grow out from an extension of Stanley's correspondence to the type C_n Shi arrangement. For basic definitions about posets see [11, Chapter 3].

Pick a region R of $\mathcal{S}_n^C$ in the cone wC^C of $\mathrm{Cox^C(n)}$, $w \in \mathfrak{S}_n^C$. The set of hyperplanes of $\mathcal{S}_n^C$ that intersect wC^C is
$$\mathcal{H}_w^C = \mathcal{H}_w^+ \cup \mathcal{H}_w^-$$
where
$$\mathcal{H}_w^- = \{x_{w(i)} - x_{w(j)} = 1 \mid i<j, 0<w(i)<w(j)\}$$
and
$$\mathcal{H}_w^+ = \{x_{w(i)} - x_{w(j)} = 1 \mid i<j, w(j)<0<w(i)\}.$$

Taking into consideration that $x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)}$ since $w(i) = -w(-i)$ and $x_{-i} = -x_i$ for all $i \in [\pm n]$, it follows that
$$\mathcal{H}_w^C = \{x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)} = 1 \mid i<j, 0<w(i)\leqslant |w(j)|\}.$$

Partial order on the hyperplanes. If $x_{w(a)} - x_{w(b)} = 1$, $a<b$, and $x_{w(a')} - x_{w(b')} = 1$, $a'<b'$, belongs to $\mathcal{H}_w^C$ and R is on the same side of the hyperplane $x_{w(a)} - x_{w(b)} = 1$ as the origin and $a' \leqslant a < b \leqslant b'$, then R is also on the same side of the hyperplane $x_{w(a')} - x_{w(b')} = 1$ as the origin, since $x_{w(a')} - x_{w(b')} \leqslant x_{w(a)} - x_{w(b)} < 1$. Considering all such implications among the hyperplanes of $\mathcal{H}_w$ we arrive to a partial order (there are two choices of partial order, pick one) on the hyperplanes. Note that if $x_{w(a)} - x_{w(b)} = 1$ is a ceiling of R, then $x_{w(a')} - x_{w(b')} = 1$ cannot be its wall, so cannot be its ceiling either. We will make the convention that the hyperplane $x_{w(a)} - x_{w(b)} = 1$ is **bigger** than the hyperplane $x_{w(a')} - x_{w(b')} = 1$ in some poset of hyperplanes, which we formalize below. By the above observations two ceilings are always incomparable in this order.

A poset based on the partial order on the hyperplanes. The following poset could be defined on the set of hyperplanes directly, but for ease of representation we do it otherwise. Define the poset Q_w^C containing both (i,j) and $(-j,-i)$ subject to constraints below:

$$Q_w^C = \{(i,j), (-j,-i) \mid i,j \in [\pm n], i < j, 0 < w(i) \leqslant |w(j)|\} \tag{1}$$

with the partial ordering inherited from the hyperplanes:

$$(i,j) \leqslant (r,s) \text{ if } r \leqslant i < j \leqslant s. \tag{2}$$

See Figure 1 for an example.

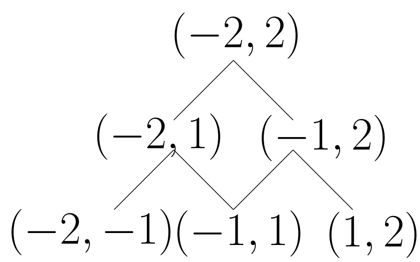

Figure 1: Poset Q_w^C for $w = [-2,-1]$.

Note that when $(i,j) \in Q_w^C$, then $(-j,-i) \in Q_w^C$, and these two elements are incomparable unless $i = -j$. In our informal thinking, these two elements stand for the same hyperplane: $x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)} = 1$. The reason we include both of these elements in Q_w^C, as opposed to just one of them, is that we do not want to miss any implication among the hyperplanes as explained in the above paragraph"Partial order on the hyperplanes." For example, if $w(1) = 1, w(4) = 5, w(2) = -3, w(3) = -2$, then both $(1,4)$ and $(2,3)$ are in Q_w^C, $(2,3) < (1,4)$, while if we only included one of (i,j) and $(-j,-i)$ then we could have missed this relation.

The antichains of Q_w^C with the property that if the element (i,j) is in the antichain, then so is $(-j,-i)$, $i,j \in [\pm n]$, correspond to nonnesting C_n-partitions if we think of $(k,l) \in Q_w^C$ as an arc in a partition of $[\pm n]$. In the rest of the paper we call the property that if the element (i,j) is in an antichain, then so is $(-j,-i)$, $i,j \in [\pm n]$, **property P**. Recall that a **nonnesting C_n-partition** of $[\pm n]$ can be thought of as a nonnesting diagram of arcs, which are drawn over the ground set $-n, -n+1, \ldots, -2, -1, 1, 2, \ldots, n-1, n$ (in this order) such that if there is an arc between i and j, for $i,j \in [\pm n]$, then there is also an arc between $-j$ and $-i$ (there are no multiple arcs). See Figure 3 for an example.

The antichains of Q_w^C with the property P are of interest to us, since they encode the sets of the ceilings of the regions. We can think of mapping a region of $\mathfrak{S}_n^C$ to the set of its ceilings (or its floors) (more precisely, when talking of a ceiling $x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)} = 1$, we are talking of the elements (i,j) and $(-j,-i)$ in Q_w^C) to obtain antichains with property P, see Figure 2 and its caption. Whether we consider the set of ceilings or floors, Theorem 3 follows. For a related bijection between the positive chambers of the Shi arrangement and order ideals of the root poset of corresponding type see [1, Theorem 5.1.13] and [5].

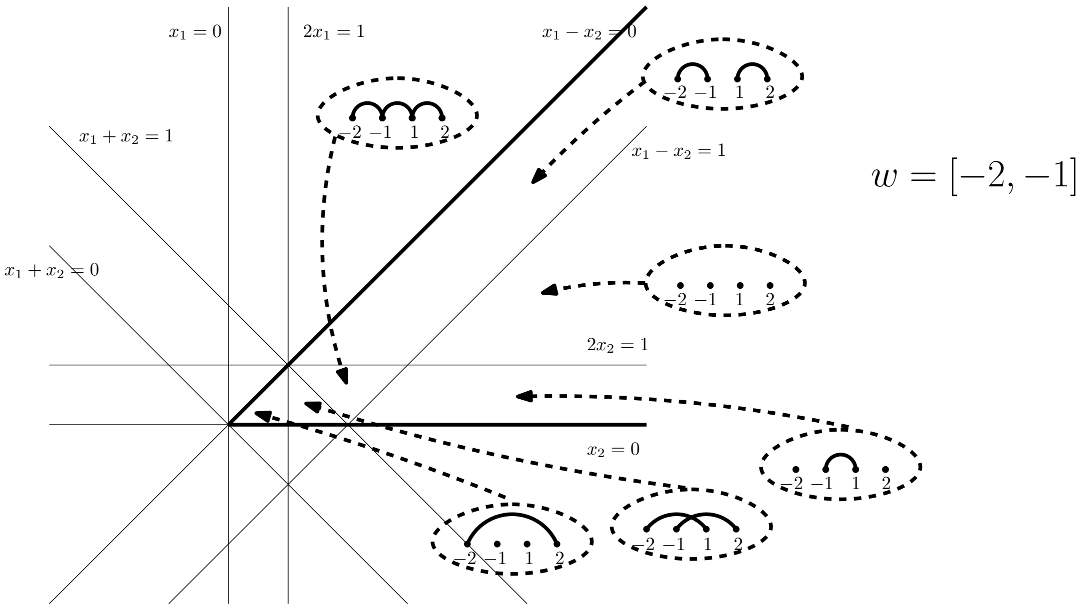

Figure 2: We label each region of the type C_n Shi arrangement $\mathcal{S}_n^C$ by a nonnesting C_n-partition and a type C_n permutation w. The permutation w is specified by the cone. Consider the set of ceilings of the regions of $\mathcal{S}_n^C$. Draw the arcs (i, j) and $(-j, -i)$ for the ceilings $x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)} = 1$ obtaining a nonnesting C_n-partition for each region. We drew in these partitions for the cone defined by $x_1 > x_2 > -x_2 > -x_1$, corresponding to the permutation $w = 1\ 2\ -2\ -1$. Note that these partitions exactly correspond to the antichains of $Q^C_{[-2,-1]}$ (see Figure 1) possessing property P.

Theorem 3. *The regions of $\mathcal{S}_n^C$ contained in wC^C are in bijection with the antichains of Q_w^C possessing property P. In particular,*

$$|R(\mathcal{S}_n^C)| = \sum_{w \in \mathfrak{S}_n^C} j_P(Q_w^C),$$

where $j_P(Q_w^C)$ denotes the number of antichains of the poset Q_w^C possessing property P.

Proof. It is clear from the above that there is an injective map from the regions of $\mathcal{S}_n^C$ to the multiset of the antichains of the posets Q_w^C possessing property P, $w \in \mathfrak{S}_n^C$. Since it is known that $|R(\mathcal{S}_n^C)| = (2n+1)^n$ [8] and $\sum_{w \in \mathfrak{S}_n^C} j_P(Q_w^C) = (2n+1)^n$ can be proved without reference to $\mathcal{S}_n^C$ (see Corollary 13 in Section 4) the map also has to be surjective and Theorem 3 follows. $\square$

Labeling the regions of $\mathcal{S}_n^C$. Reiterating from above, we label each region of the type C_n Shi arrangement $\mathcal{S}_n^C$ by a nonnesting C_n-partition and a type C_n permutation w. The permutation w is specified by the cone wC^C in which the region lies. The nonnesting C_n-partition is obtained in the following way. Consider the set of all ceilings of a region of $\mathcal{S}_n^C$. Draw the arcs (i,j) and $(-j,-i)$ for the ceilings $x_{w(i)} - x_{w(j)} = x_{w(-j)} - x_{w(-i)} = 1$ obtaining a nonnesting C_n-partition for each region. These partitions exactly correspond to the antichains of Q_w^C possessing property P. See Figure 2.

3 Sequences and Shi arrangements in type C_n

In this section we construct a bijection between the regions of $\mathcal{S}_n^C$ and the set of sequences $\mathcal{A}^C(n) = \{a_1 \dots a_n | a_i \in [\pm n] \cup \{0\}, i \in [n]\}$. Our proof yields enumeration of regions by the ceiling and floor statistic, which we express in a generating function form.

The **type** of a C_n-partition π is the integer partition λ whose parts are the sizes of the nonzero blocks of π, including one part for each pair of blocks $\{B, -B\}$. The zero block is a block B such that $B = -B$. Figure 3 shows a nonnesting C_5-partition with blocks $\{2\}, \{-2\}, \{-1,-4\}, \{1,4\}, \{-5,-3,3,5\}$. The last block is a zero block, and so the type of this partition is $(2,1)$.

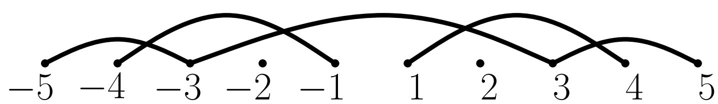

Figure 3: A type $(2,1)$ nonnesting C_5-partition.

Recall from Section 2 that we label each region of $\mathcal{S}_n^C$ by the nonnesting C_n-partition corresponding to an antichain of Q_w^C, $w \in \mathfrak{S}_n^C$, possessing property P, and a permutation $w \in \mathfrak{S}_n^C$. While we generally think of π as on the vertices $-n, -n+1, \dots, -1, 1, 2, \dots, n-1, n$, in this order, the C_n-partition π also has w**-labels** $w(-n), w(-n+1), \dots, w(-1), w(1), \dots, w(n-1), w(n)$. Given a block $B = \{v_1, \dots, v_k\}$ of π the set of w-labels of B is $\{w(v_1), \dots, w(v_k)\}$.

Lemma 4. *Given a nonnesting C_n-partition π, let S_B be a set of size $|B|$ for each block B of π such that the sets S_B's are disjoint, their union is $[\pm n]$ and $S_B = -S_{-B}$, where $-S_{-B} = \{-a \mid a \in S_{-B}\}$. Then there exists a unique w such that π is an antichain in Q_w^C possessing property P and the set of w-labels of each block B are equal to S_B.*

Proof. Recall that

$$Q_w^C = \{(i,j), (-j,-i) \mid i,j \in [\pm n], i < j, 0 < w(i) \leqslant |w(j)|\}$$

with the partial ordering:

$$(i,j) \leqslant (r,s) \text{ if } r \leqslant i < j \leqslant s.$$

Suppose that nonnesting C_n-partition π is an antichain in Q_w^C possessing property P. Then, for a block $B = \{v_1, v_2, \ldots, v_k\}$ of π with $v_1 < v_2 < \cdots < v_k$, it must be that for all $l \in [k-1]$ either

$$(0 < w(v_l) \leqslant |w(v_{l+1})|) \text{ or } (w(v_{l+1}) < 0 \text{ and } |w(v_{l+1})| \leqslant |w(v_l)|). \tag{3}$$

This follows since by the definition of Q_w^C either (v_l, v_{l+1}) satisfies $0 < w(v_l) \leqslant |w(v_{l+1})|$ or $(-v_{l+1}, -v_l)$ satisfies $0 < w(-v_{l+1}) \leqslant |w(-v_l)|$. Note that (3) implies that the w-labels of B are such that all the positive w-labels come first followed by all the negative w-labels and the absolute values of the w-values read from left to right form a unimodal sequence. Since there is a unique way to arrange a set of numbers in this manner, and when arranged so π is an antichain in Q_w^C possessing property P, it follows that w is unique. See Figure 4 for an example. $\square$

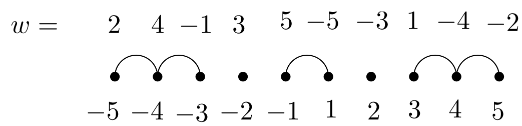

Figure 4: The blocks of the nonnesting C_n-partition π presented above are $B_0 = \{-1, 1\}$, $B_1 = \{-2\}$, $B_2 = \{-3, -4, -5\}$. For sets $S_{B_0} = \{-5, 5\}$, $S_{B_1} = \{3\}$ and $S_{B_2} = \{-1, 2, 4\}$, as described in Lemma 4, the unique w for which π is an antichain of Q_w^C is $w = [-5, -3, 1, -4, -2]$ as shown on the figure in line notation (and its construction explained in the proof of Lemma 4).

Lemma 5. *In the labeling described in Section 2 the number of regions of $\mathcal{S}_n^C$ labeled by the nonnesting C_n-partition π of type λ (and some permutation) is equal to*

$$\binom{n}{\lambda_1, \ldots, \lambda_d, n - |\lambda|} \prod_{i=1}^{d} 2^{\lambda_i}. \tag{4}$$

Proof. In this proof we count the number of signed permutations $w \in \mathfrak{S}_n^C$ such that π is an antichain in the poset Q_w^C possessing property P, since the latter is equal to the number of regions of $\mathcal{S}_n^C$ labeled by the nonnesting C_n-partition π. By Lemma 4 if we have a collection of sets S_B of size $|B|$ for each block B of π such that the sets S_B's are disjoint, their union is $[\pm n]$ and $S_B = -S_{-B}$, where $-S_{-B} = \{-a \mid a \in S_{-B}\}$, then there is a unique w for which π is an antichain in Q_w^C possessing property P and the set of w-labels of each block B are equal to S_B. Moreover, note that for any w for which π is an antichain in the poset Q_w^C possessing property P the sets of w-labels of the blocks have to satisfy the above criteria for the S_B's. Thus, number of regions of $\mathcal{S}_n^C$ containing the nonnesting C_n-partition π of type λ is equal to the number of collections of sets S_B's satisfying the above criteria. There are $\binom{n}{\lambda_1, \ldots, \lambda_d, n-|\lambda|}$ ways of choosing the absolute values of the elements of the sets S_B's and there are $\prod_{i=1}^{d} 2^{\lambda_i}$ ways of choosing the signs for the sets corresponding to the nonzero blocks of π. Thus, the total number of collections of sets S_B's satisfying the above criteria is $\binom{n}{\lambda_1,\ldots,\lambda_d,n-|\lambda|}\prod_{i=1}^{d}2^{\lambda_i}$. $\square$

The following theorem is based on a bijection of Fink and Iriarte [6] between noncrossing and nonnesting C_n-partitions which preserves type and a bijection of Athanasiadis [3] between noncrossing C_n-partitions and pairs (S,g), where S is a set and g is a function subject to the conditions stated below. We invite the interested reader to learn about these bijections from the original papers themselves, as their description would take up considerable amount of space in the current paper and as such it is omitted.

Theorem 6. *There is a bijection between the set of type $\lambda=(\lambda_1,\ldots,\lambda_d)$ nonnesting C_n-partitions and pairs (S,g), where S is a d-subset of $[n]$ and the map $g:S\to\{\lambda_1,\ldots,\lambda_d\}$ is such that $|g^{-1}(i)|=\#\{j\mid\lambda_j=i,j\in[d]\}$, $0<i$.*

Proof. [6, Theorem 2.4] establishes a type-preserving bijection b_1 between nonnesting and noncrossing C_n-partitions, and the proof of [3, Theorem 2.3] provides a bijection b_2 between the set of type $\lambda=(\lambda_1,\ldots,\lambda_d)$ noncrossing C_n-partitions and pairs (S,g), where S is a d-subset of $[n]$ and the map $g:S\to\{\lambda_1,\ldots,\lambda_d\}$ is such that $|g^{-1}(i)|=\#\{j\mid\lambda_j=i,j\in[d]\}$, $0<i$. $\square$

Given a type $\lambda=(\lambda_1,\ldots,\lambda_d)$ nonnesting C_n-partition π, denote by S_π the set and g_π the function from Theorem 6. Let M_π be the multiset consisting of $n-|\lambda|$ 0's, and λ_i copies of each element of $g_\pi^{-1}(\lambda_i)$, for each part in the set (not multiset!) $\{\lambda_1,\ldots,\lambda_d\}$. A **marked permutation** of M_π is a permutation of the elements of the multiset M_π such that each nonzero entry has a $\pm$ sign in addition. For example the marked permutations of the multiset $\{\{0,1,1\}\}$ are $011,101,110,0-1-1,-10-1,-1-10,01-1,10-1,1-10,0-11,-101,-110$ (we omitted the $+$ signs).

Given two blocks B_1 and B_2 in a partition, block B_1 is smaller than B_2 in the **order** σ if the smallest vertex that B_1 contains is smaller than the smallest vertex that B_2 contains. By convention, if for a block $B\neq -B$, we consider block B smaller than block $-B$ in the order σ.

Theorem 7. *There is a bijective map ϕ between the regions of $\mathcal{S}_n^C$ labeled by the nonnesting C_n-partition π of type λ and marked permutations of the multiset M_π.*

Proof. There are multiple ways to set up the map ϕ. We present one of these ways here, and based on it the interested reader can device several others (though one will of course suffice for all we need it).

Given the nonnesting C_n-partition π of type $\lambda=(\lambda_1,\ldots,\lambda_d)$ first we define a map f from the pairs of blocks $(\{B,-B\})$ of π to the set underlying M_π. If π has a zero block $B=-B$, then let $f(B)=0$. Let $\{B_1,-B_1,B_2,-B_2,\ldots,B_d,-B_d\}$ be all the nonzero blocks of π, where $|B_i|=\lambda_i$, $i\in[d]$, such that if $\lambda_j=\lambda_{j+1}$ for some $j\in[d-1]$, then B_j is smaller than B_{j+1} in the order σ.

Order the nonzero numbers in the multiset M_π so that the numbers with bigger multiplicites come first. Among the numbers with the same multiplicity order them according to the natural order on integers. Note that by construction the sequence we get as a result has the form $a_1^{\lambda_1}a_2^{\lambda_2}\dots a_d^{\lambda_d}$, where $a_i^{\lambda_i}$ denotes the sequence of λ_i a_i's. Let $f(B_i) = a_i$ for $i \in [d]$.

Recall that given a region R of $\mathcal{S}_n^C$ it is labeled by the nonnesting C_n-partition π of type λ and a signed permutation w for which π is an antichain in Q_w^C possessing property P. For such a region R construct the marked permutation $\phi(R) = c_1 \dots c_n$ of the multiset M_π by the following rule: if vertex v is in block B_k, $k \in \{0\} \cup [d]$, then $c_{|w(v)|} = sign(w(v))f(B_k)$, where $sign(a) = -1$ if $a < 0$ and $sign(a) = 1$ if $a > 0$.

To show that ϕ is bijective, we exhibit its inverse. Given a marked permutation $c_1 \dots c_n$ of the multiset M_π, we trivially obtain the underlying multiset and from that we can obtain the set-function pair (S_π, g_π) from which the multiset was constructed. From these, by Theorem 6 we can recover the (unique) C_n-partition π associated to the region(s). Now, knowing the blocks of π and the multiset M_π, we can reconstruct the function f. Once we know f, the marked permutation $c_1 \dots c_n$ specifies the set of w-labels on each block of π and by Lemma 4 that uniquely specifies w. Thus, there is a unique region - namely the one labeled by π and w - which maps to $c_1 \dots c_n$ under ϕ. □

Extend the map ϕ defined in the proof of Theorem 7 to a map between all regions of $\mathcal{S}_n^C$ and the set of sequences $\mathcal{A}^C(n) = \{a_1 \dots a_n | a_i \in [\pm n] \cup \{0\}, i \in [n]\}$, to obtain the following corollaries.

Theorem 8. *The map $\phi : R(\mathcal{S}_n^C) \to \mathcal{A}^C(n)$ is a bijection.*

Corollary 9.

$$\sum_{\lambda \vdash n} \frac{n!}{m_\lambda (n-d)!} \binom{n}{\lambda_1, \dots, \lambda_d, n - |\lambda|} \prod_{i=1}^{d} 2^{\lambda_i} = (2n+1)^n,$$

where $m_\lambda = \prod_{i=1}^n r_i!$, if r_i denotes the number of parts of λ equal to i.

Proof. Athanasiadis [3] proved that the number of nonnesting C_n-partitions of type λ is

$$\frac{n!}{m_\lambda (n-d)!},$$

which together with Lemma 5 and Theorem 8 imply the above equality. □

Theorem 2 is a corollary of the proofs of Theorems 6, 7 and 8. For further details see Section 4, and in particular Theorem 10.

Theorem 2.

$$\sum_{R \in R(\mathcal{S}_n^C)} q^{c(R)} = \sum_{R \in R(\mathcal{S}_n^C)} q^{f(R)} = \sum_{\mathbf{a} \in \mathcal{A}^C(n)} q^{n - d^C(\mathbf{a})}.$$

4 Posets and sequences in type C_n

In this section we revisit the type C_n world of posets Q_w^C, $w \in \mathfrak{S}_n^C$, and sequences in $\mathcal{A}^C(n)$ and state their relation explicitly.

Recall that

$$Q_w^C = \{(i,j), (-j,-i) \mid i,j \in [\pm n], i < j, 0 < w(i) \leqslant |w(j)|\}$$

is partially ordered by

$$(i,j) \leqslant (r,s) \text{ if } r \leqslant i < j \leqslant s.$$

We will prove refinements of the equation

$$\sum_{w \in \mathfrak{S}_n^C} j_P(Q_w^C) = (2n+1)^n, \tag{5}$$

without reference to arrangements. Here $j_P(Q_w^C)$ denotes the number of antichains of Q_w^C possessing property P.

Partition $\mathcal{A}^C(n)$ according to the number of nonzero absolute values in the set $\{a_1, a_2 \ldots, a_n\}$, denoted by $d^C(\mathbf{a})$ for $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. Let $A_k^C(n) = \{(a_1, a_2, \ldots, a_n) \in \mathcal{A}^C(n) : d^C(\mathbf{a}) = k\}$. Then

$$\mathcal{A}^C(n) = \bigcup_{k=0}^{n} A_k^C(n).$$

Let $\mathcal{M}^C(n)$ be the multiset of antichains of Q_w^C possessing property P, $w \in \mathfrak{S}_n^C$. Given an antichain $\mathbf{x} \in \mathcal{M}^C(n)$ it naturally corresponds to a nonnesting C_n-partition $\pi_{\mathbf{x}}$ obtained by simply drawing an arc (a,b) for each $(a,b) \in \mathbf{x}$. Partition the multiset $\mathcal{M}^C(n)$ according to the number of pairs of nonzero blocks in the corresponding nonnesting C_n-partition. Denote by $b(\mathbf{x})$ the number of pairs of nonzero blocks in $\pi_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{M}^C(n)$. Let $M_k^C(n) = \{\{\mathbf{x} \in \mathcal{M}^C(n) | b(\mathbf{x}) = k\}\}$. Then

$$\mathcal{M}^C(n) = \bigcup_{k=0}^{n} M_k^C(n).$$

Theorem 10.

$$|A_k^C(n)| = |M_k^C(n)|, \ k \in \{0\} \cup [n].$$

We prove Theorem 10 by providing a bijection between the sets $A_k^C(n)$ and $M_k^C(n)$, $k \in \{0\} \cup [n]$. Before proceeding to the proof of Theorem 10 we partition the sets $A_k^C(n)$ and $M_k^C(n)$, $k \in \{0\} \cup [n]$, further.

Partition $A_k^C(n)$, $k \in \{0\} \cup [n]$, according to the k distinct nonzero absolute values of the numbers appearing in the sequence and the number of times they appear. If

$$\{|a_1|, |a_2|, \ldots, |a_n|\} \backslash \{0\} = \{c_1 < c_2 < \cdots < c_k\}$$

and c_i appears o_i times in $(|a_1|, |a_2|, \ldots, |a_n|)$, $i \in [k]$, let $A_k^{C^{\mathbf{c},\mathbf{o}}}(n)$ equal

$$\left\{(a_1, a_2, \ldots, a_n) \in A_k^C(n) \middle| \{\{|a_1|, |a_2|, \ldots, |a_n|\}\} = \cup_{i=1}^{k} \cup_{j=1}^{o_i} \{\{c_i\}\} \cup_{i=1}^{n-\sum_{j=1}^{k} o_j} \{\{0\}\}\right\},$$

where $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^k o_i \leqslant n$.

For an antichain $\mathbf{x} \in M_k^C(n)$, let $(S_{\pi_{\mathbf{x}}}, g_{\pi_{\mathbf{x}}})$ be the pair of k-set and function corresponding to $\pi_{\mathbf{x}}$ under the bijection described in Theorem 6. Let

$$S_{\pi_{\mathbf{x}}} = \{c_1 < \ldots < c_k\} \text{ and } o_i = g_{\pi_{\mathbf{x}}}(c_i), i \in [k].$$

Denote $c(\mathbf{x}) = (c_1 < \ldots < c_k)$ and $o(\mathbf{x}) = (o_1, \ldots, o_k)$.

Partition the multiset $M_k^C(n)$, $k \in \{0\} \cup [n]$, according to $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^k o_i \leqslant n$, as described above. Let

$$M_k^{C^{\mathbf{c},\mathbf{o}}}(n) = \{\{\mathbf{x} \in M_k^C(n) | c(\mathbf{x}) = \mathbf{c}, o(\mathbf{x}) = \mathbf{o}\}\},$$

where $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^k o_i \leqslant n$.

Lemma 11. *The vectors $c(\mathbf{x}) = \mathbf{c}$ and $o(\mathbf{x}) = \mathbf{o}$, where $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $k \in \{0\} \cup [n]$, $o_i > 0$, for $i \in [k]$, $\sum_{i=1}^k o_i \leqslant n$, uniquely determine the antichain $\mathbf{x}$.*

Proof. Lemma 11 follows readily since Theorem 6 establishes a bijection. □

Theorem 12.

$$|A_k^{C^{\mathbf{c},\mathbf{o}}}(n)| = |M_k^{C^{\mathbf{c},\mathbf{o}}}(n)| = \binom{n}{o_1, \ldots, o_k, n - \sum_{j=1}^k o_j} 2^{\sum_{j=1}^k o_j},$$

where $k \in \{0\} \cup [n]$, $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $o_i > 0$, for $i \in [k]$, and $\sum_{i=1}^k o_i \leqslant n$.

Proof. A bijective proof of the first equality can be given using Theorem 6 and the ideas of Theorem 7. The enumeration is in Lemma 5. Note that arrangements do not enter any of the proofs. □

Proof of Theorem 10. Straightforward corollary of Theorem 12, since

$$A_k^C(n) = \sum_{\mathbf{c},\mathbf{o}} A_k^{C^{\mathbf{c},\mathbf{o}}}(n) = \sum_{\mathbf{c},\mathbf{o}} M_k^{C^{\mathbf{c},\mathbf{o}}}(n) = M_k^C(n),$$

where $\mathbf{c} = (c_1 < \ldots < c_k)$, $\mathbf{o} = (o_1, \ldots, o_k)$, $k \in [n]$, $o_i > 0$, for $i \in [k]$, $\sum_{i=1}^k o_i \leqslant n$. □

Corollary 13.

$$\sum_{w \in \mathfrak{S}_n^C} j_P(Q_w^C) = (2n+1)^n.$$

Proof. Theorems 12 and 10 extend to a bijection between

$$\mathcal{M}^C(n) = \cup_{k=0}^n \cup_{\mathbf{c},\mathbf{o}} M_k^{C^{\mathbf{c},\mathbf{o}}}(n) \text{ and } \mathcal{A}^C(n) = \cup_{k=0}^n \cup_{\mathbf{c},\mathbf{o}} A_k^{C^{\mathbf{c},\mathbf{o}}}(n),$$

the cardinalities of which are $\sum_{w \in \mathfrak{S}_n^C} j_P(Q_w^C)$ and $(2n+1)^n$, respectively. □

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