Orders induced by segments in floorplans and (2-14-3, 3-41-2)-avoiding permutations

Andrei Asinowski

Department of Mathematics, Technion – Israel Institute of Technology, Haifa, Israel
and Institut für Informatik, Freie Universität Berlin, Germany
asinowski@mi.fu-berlin.de

Gill Barequet

Department of Computer Science, Technion – Israel Institute of Technology, Haifa, Israel
and Department of Computer Science, Tufts University, Medford, MA, USA
barequet@cs.technion.ac.il

Mireille Bousquet-Mélou

CNRS, LaBRI, UMR 5800, Université de Bordeaux, France
mireille.bousquet@labri.fr

Toufik Mansour

Department of Mathematics, University of Haifa, Haifa, Israel
tmansour@univ.haifa.ac.il

Ron Y. Pinter

Department of Computer Science, Technion – Israel Institute of Technology, Haifa, Israel
pinter@cs.technion.ac.il

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Abstract

A floorplan is a tiling of a rectangle by rectangles. There are natural ways to order the elements – rectangles and segments – of a floorplan. Ackerman, Barequet and Pinter studied a pair of orders induced by neighborhood relations between rectangles, and obtained a natural bijection between these pairs and (2-41-3, 3-14-2)-avoiding permutations, also known as (reduced) Baxter permutations.

In the present paper, we first perform a similar study for a pair of orders induced by neighborhood relations between *segments* of a floorplan. We obtain a natural bijection between these pairs and another family of permutations, namely (2-14-3, 3-41-2)-avoiding permutations.

Then, we investigate relations between the two kinds of pairs of orders – and, correspondingly, between (2-41-3, 3-14-2)- and (2-14-3, 3-41-2)-avoiding permutations. In particular, we prove that the superposition of both permutations gives a *complete* Baxter permutation (originally called *w-admissible* by Baxter and Joichi in the sixties). In other words, (2-14-3, 3-41-2)-avoiding permutations are the hidden part of complete Baxter permutations. We enumerate these permutations. To our knowledge, the characterization of these permutations in terms of forbidden patterns and their enumeration are both new results.

Finally, we also study the special case of the so-called *guillotine* floorplans.

1 Introduction

A *floorplan*[1] is a partition of a rectangle into interior-disjoint rectangles such that no point belongs to the boundary of four rectangles (Fig. 1). We call *segment* of the floorplan any straight line, not included in the boundary of the partitioned rectangle, that is the union of some rectangle sides, and is maximal for this property. For example, each of the floorplans of Fig. 1 has four horizontal and four vertical segments. Since four rectangles of a floorplan never meet, the segments do not cross, and a meeting of segments has one of the following forms: ⊣, ⊥, ⊢, ⊤ (but not +).

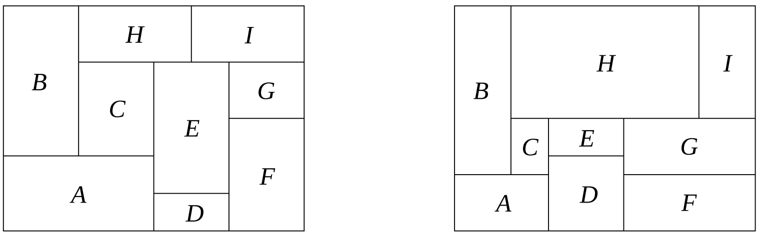


Figure 1: Two R-equivalent floorplans.

An easy induction shows that the number of segments in a floorplan is smaller than the number of rectangles by 1. Throughout the paper, for a given floorplan P, the number of segments in P is denoted by n; accordingly, $n+1$ is the number of rectangles in P. We say that P has *size* $n+1$. For instance, the floorplans in Fig. 1 have size 9.

Many papers have appeared about floorplans, not only in combinatorial but also in computational geometry literature [14, 28, 31]. The interest in floorplans is motivated, in particular, by the fact that their generation is a critical stage in integrated circuit layout [26, 27, 34, 40, 41], in architectural designs [8, 17, 22, 36, 37], etc.

The present paper is combinatorial in nature, and describes the relationship between a pair of natural orders defined on the segments of a floorplan and certain pattern avoiding permutations. It parallels a previous paper of Ackerman, Barequet and Pinter in which a similar study was carried out for a pair of orders defined on *rectangles* of a floorplan, in connection with the so-called *Baxter permutations* [1]. These permutations

[1]Sometimes called *mosaic floorplan* in the literature.

were introduced in the sixties, and were originally called *reduced* Baxter permutations, as opposed to *complete* Baxter permutations [6, 7, 12]. A complete Baxter permutation π has an odd size, say $2n+1$, and is completely determined by its values at odd points, $\pi(1), \pi(3), \dots, \pi(2n+1)$. After normalization, these values give the associated *reduced* Baxter permutation π_o (the subscript 'o' stands for odd; we denote by π_e the permutation obtained by normalizing the list $\pi(2), \pi(4), \dots, \pi(2n)$). Our paper provides the even part of the theory initiated in [1]: we prove that the permutation associated with the segments and the permutation associated with the rectangles are respectively the even and odd parts π_e and π_o of the same complete Baxter permutation π. We also characterize the even parts of Baxter permutations in terms of forbidden patterns and enumerate them.

One of our motivations for studying segments of floorplans was the observation that many questions on floorplans deal with segments rather than rectangles. An interesting example is the *rectangulation conjecture* by Ackerman et al. [2, Conj. 7.1] about embeddings of point sets in floorplans, recently proved by Felsner [19].

In order to present our results in greater detail, we first need to describe the related results obtained for rectangles in [1]. In that paper, the authors study a representation of two order relations between rectangles in floorplans by means of permutations. These order relations are induced by *neighborhood relations*, which are defined as follows. A rectangle A is a *left-neighbor* of B (equivalently, B is a *right-neighbor* of A) if there is a vertical segment in the floorplan that contains the right side of A and the left side of B. (Note that the right side of A and the left side of B may be disjoint.) Now, the relation *"A is to the left of B"* (equivalently, *"B is to the right of A"*), denoted by $A \leftarrow B$, is defined as the transitive closure of the relation "A is a left-neighbor of B." Finally, the relation $\twoheadleftarrow$ is the reflexive closure of $\leftarrow$. The terms *A is a below-neighbor of B* (equivalently, *B is an above-neighbor of A*) and *A is below B* (equivalently, *B is above A*) are defined similarly, as well as the notation $A \downarrow B$ for "A is below B,"[2] and $A \twoheaddownarrow B$ for "$A = B$ or $A \downarrow B$." It is easy to see that the relations $\twoheadleftarrow$ and $\twoheaddownarrow$ are partial orders. In both floorplans of Fig. 1, we have, among other relations, $A \downarrow I$ (because A is a below-neighbor of C, and C is a below-neighbor of I), $A \leftarrow G$, and $B \leftarrow F$.

The following results are proved in [1]. Let P be a floorplan of size $n+1$. Two distinct rectangles A and B of P are in *exactly one* of the relations $A \leftarrow B$, $B \leftarrow A$, $A \downarrow B$, or $B \downarrow A$. It follows that the relations $\swarrow$ and $\nwarrow$ between rectangles of P defined by

$$\begin{array}{lll} A \swarrow B & \textit{if} & A = B, \textit{ or } A \textit{ is to the left of } B, \textit{ or } A \textit{ is below } B, \\ A \nwarrow B & \textit{if} & A = B, \textit{ or } A \textit{ is to the left of } B, \textit{ or } A \textit{ is above } B, \end{array}$$

are *linear* orders (the signs $\swarrow$ and $\nwarrow$ are intended to resemble the inequality sign $\leqslant$). Each of these orders can be used to label the rectangles of P by $1, 2, \dots, n+1$. In the $\swarrow$ order, the rectangle in the lower left corner is labeled 1, and the rectangle in the upper right corner $n+1$. In the $\nwarrow$ order, the rectangle in the upper left corner is labeled 1,

[2]Hence, $A \downarrow B$ should be understood as $\begin{smallmatrix} B \\ \downarrow \\ A \end{smallmatrix}$; and similarly for $\twoheaddownarrow$.

and the rectangle in the lower right corner $n+1$. Let $R(P)$ be the sequence $a_1\,a_2\,\ldots\,a_{n+1}$, where, for all $1 \le i \le n+1$, a_i is the label in the $\swarrow$ order of the rectangle which is labeled i in the $\nwarrow$ order. It is clear that $R(P)$ is a permutation of $[n+1] := \{1,2,\ldots,n+1\}$; we call it the *R-permutation* of P. Loosely speaking, $R(P)$ is obtained by labeling the rectangles according to the $\swarrow$ order, and then reading these labels while passing the rectangles according to the $\nwarrow$ order. Fig. 2 shows the construction of the R-permutation of a floorplan P; the right part of the figure is the graph of $\rho = R(P)$, that is, the point set $\{(i,\rho(i)) :\ 1 \le i \le n+1\}$.

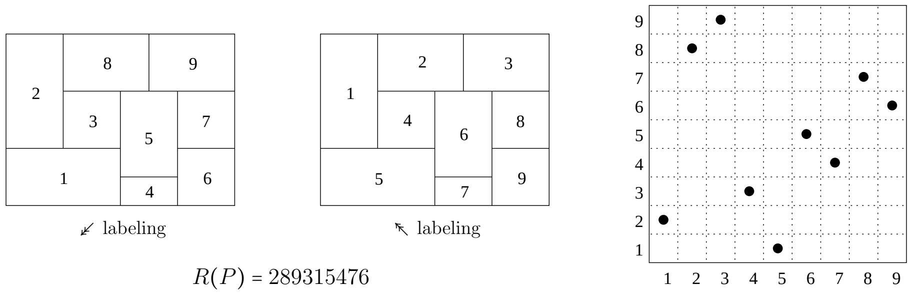


Figure 2: Constructing the R-permutation of a floorplan P.

Two floorplans P_1 and P_2 of size $n+1$ are said to be *R-equivalent*[3] if there exists a labeling of the rectangles of P_1 by $A_1, A_2, \ldots, A_{n+1}$ and a labeling of the rectangles of P_2 by $B_1, B_2, \ldots, B_{n+1}$ such that for all $k, m \in [n+1]$, the rectangles A_k and A_m exhibit the same neighborhood relation as B_k and B_m. The two floorplans in Fig. 1 are R-equivalent: this follows from the labeling presented in this figure (in fact, $A, B, C, \ldots I$ is the $\swarrow$-order). It is easy to prove that two floorplans are R-equivalent if and only if they have the same R-permutation.

The main results of [1] state that the R-permutation of any floorplan P is a (2-41-3, 3-14-2)-avoiding permutation[4], originally called *(reduced) Baxter permutation*; moreover, R is a bijection between R-equivalence classes of floorplans and (2-41-3, 3-14-2)-avoiding permutations. Through this bijection, the size of a floorplan becomes the size of the permutation, and the order relations between rectangles in P can be easily read from $R(P)$.

We can now describe our results in greater detail.

In the first part of the paper, we develop for *segments* a theory that parallels the theory developed for *rectangles* in [1]. We define two order relations between segments (Section 2), which leads to the notion of S-equivalent floorplans[5]. Then we use these orders to construct a permutation $S(P)$ called the *S-permutation* of P. In Section 3 we prove that S-permutations coincide with (2-14-3, 3-41-2)-avoiding permutations, and that S, regarded as a function from S-equivalence classes to (2-14-3, 3-41-2)-avoiding permutations, is a bijection. The description of S and S^{-1} are fairly simple (both can be

[3]In [1], two R-equivalent floorplans are actually treated as two representations of the same floorplan.
[4]This notation is explained in Section 3.3.
[5]In the notion of R-equivalence and S-equivalence, R stands for *rectangles* and S for *segments*.

constructed in linear time), but as often, the proof remains technical, despite our efforts to write it carefully.

In the second part of the paper, we super-impose our theory with the analogous theory developed for rectangles in [1]. In Section 4 we show that the R-equivalence of floorplans implies their S-equivalence (this means that the R-equivalence refines the S-equivalence), and explain how $S(P)$ can be constructed directly from $R(P)$. This construction shows that $S(P)$ and $R(P)$, combined together, form the so-called *complete Baxter permutation* associated with $R(P)$, as defined in the seminal papers on Baxter permutations [6, 7, 12]. We also describe in terms of R when two floorplans give the same S-permutation. This is another difficult proof, but we need this result to express the number of $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$-avoiding permutations in terms of the number of Baxter permutations (Section 5).

To finish, in Section 6 we characterize and enumerate S-permutations corresponding to the so-called *guillotine* floorplans; a similar study was carried out in [1] for R-permutations. We end in Section 7 with a few remarks.

2 Orders between segments of a floorplan

In this section we define neighborhood relations between segments of a floorplan, use them to define two partial orders (denoted $\twoheadleftarrow$ and $\twoheaddownarrow$) and two linear orders (denoted $\swarrow$ and $\nwarrow$), and prove several facts about these orders. Most of them are analogous to facts about the orders on rectangles mentioned in the introduction, and proved in [1].

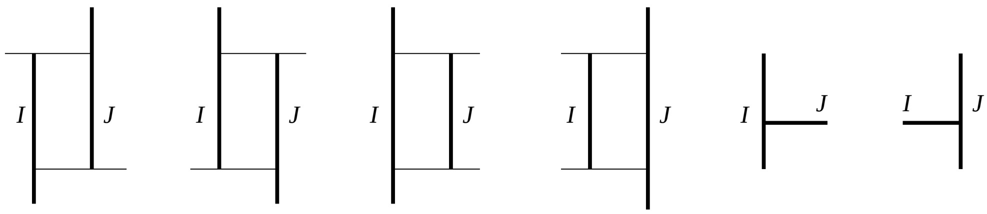


Figure 3: The segment I is a left-neighbor of the segment J.

Definition 2.1. Let I and J be two segments in a floorplan P. We say that I is a *left-neighbor* of J (equivalently, J is a *right-neighbor* of I) if one of the following holds:

- I and J are vertical, and there is *exactly one* rectangle A in P such that the left side of A is contained in I and the right side of A is contained in J;
- I is vertical, J is horizontal, and the left endpoint of J lies in I; or
- I is horizontal, J is vertical, and the right endpoint of I lies in J.

The terms "*I is a below-neighbor of J*" (equivalently, "*J is an above-neighbor of I*") are defined similarly.

Typical examples are shown in Fig. 3. Note that a horizontal segment I has at most one left-neighbor and at most one right-neighbor (no such neighbor(s) when the corresponding endpoint(s) of I lie on the boundary), which are both vertical segments. In contrast, a vertical segment may have several left- and right-neighbors, which may be horizontal or vertical (see Fig. 4).

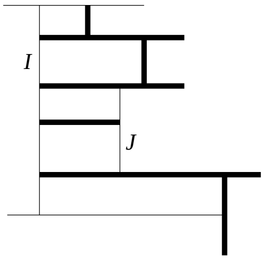


Figure 4: The right-neighbors of a vertical segment I (thick segments). Note that the vertical segment J is *not* a right-neighbor of I.

Definition 2.2. The relation *"I is to the left of J"* (equivalently, *J is to the right of I*), denoted by $I \leftarrow J$, is the transitive closure of the relation "I is a left-neighbor of J." The relation $\twoheadleftarrow$ is the reflexive closure of $\leftarrow$. The relations $I \downarrow J$ ("I *is below* J") and $I \downarrow\!\!\!\!\!- J$ (for "$I = J$ or $I \downarrow J$") are defined similarly.

Observation 2.3. *The relations $\twoheadleftarrow$ and $\downarrow\!\!\!\!\!-$ are partial orders.*

Proof. We prove the claim for the relation $\twoheadleftarrow$. Reflexivity and transitivity follow from the definition. For antisymmetry, note that $I \leftarrow J$ and $J \leftarrow I$ cannot hold simultaneously because if $I \leftarrow J$, then any interior point of I has a smaller abscissa than any interior point of J. $\square$

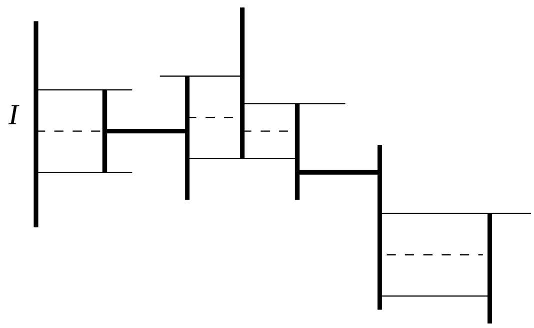


Figure 5: A chain in the $\twoheadleftarrow$ order (thick segments), and the corresponding traversing edges (dashed lines).

The following observation may help to understand the $\twoheadleftarrow$ order. If I and J are vertical segments and right-left neighbors, let us create a horizontal edge, called *traversing edge*, in the rectangle A that lies between them. Fig. 5 shows a chain of neighbors in the $\twoheadleftarrow$ order, starting from a segment I, and the corresponding traversing edges (dashed lines).

Observation 2.4. *Assume $I \twoheadleftarrow J$. Then any point of J lies weakly to the right of any point of I (that is, its abscissa is at least as large).*

Let x (respectively, y) be a point of minimal (respectively, maximal) abscissa on I (respectively, J). Then there exists a polygonal line from x to y formed of vertical and horizontal sections, such that

- *each vertical section is part of a vertical segment of the floorplan P,*
- *each horizontal section is an (entire) horizontal segment of P, or a traversing edge of P, visited from left to right,*
- *if I (respectively, J) is horizontal, it is entirely included in the polygonal line.*

It suffices to prove these properties when J is a right-neighbor of I, and they are obvious in this case (see Fig. 3).

Lemma 2.5. *In the $\twoheadleftarrow$ order, J covers I if and only if J is a right-neighbor of I. A similar statement holds for the $\downarrow$ order.*

Proof. Since $\twoheadleftarrow$ is constructed as the transitive closure of the left-right neighborhood relation, every covering relation is a neighborhood relation.

Conversely, let us prove that any neighborhood relation is a covering relation. Equivalently, this means that the right-neighbors of any segment I form an antichain. If I is horizontal, it has at most one right-neighbor, and there is nothing to prove. Assume I is vertical (as in Fig. 4), and that two of its neighbors, J_1 and J_2, satisfy $J_1 \leftarrow J_2$. By the first part of Observation 2.4, J_2 cannot be horizontal (its leftmost point would then lie on I, leaving no place for J_1). Hence J_2 is vertical. The possible configurations of I and J_2 are depicted in the first four cases of Fig. 3. Let x (resp. y) be a point of I (resp. J_2). By Observation 2.4, there exists a polygonal line from x to y that visits a point of J_1. This rules out the third and fourth cases of Fig. 3 (the line would be reduced to a single traversing edge). By symmetry we can assume that I and J_2 are as in the first case of Fig. 3. Then the polygonal line, which is not a single traversing edge, has to leave I at a point that lies lower than the lowest point of J_2, and to reach J_2 at a point that lies higher than the highest point of I: this means that it crosses two horizontal segments, which is impossible given the description of this line. $\square$

Lemma 2.6. *Let I and J be two different segments in a floorplan P. Then exactly one of the relations: $I \leftarrow J$, $J \leftarrow I$, $I \downarrow J$, or $J \downarrow I$, holds.*

Proof. Assume without loss of generality that I is a horizontal segment. Construct the *NE-sequence* $K_1, K_2, \ldots$ of I as follows (see Fig. 6 for an illustration): K_1 is the right-neighbor of I, K_2 the above-neighbor of K_1, K_3 the right-neighbor of K_2, and so on, until the boundary is reached. Construct similarly the SE-, NW-, and SW-sequences of I. These sequences partition the rectangle into four regions (or fewer, if some endpoints of I lie on the boundary); each segment of P (except I and those belonging to either of the sequences) lies in exactly one of them. Also, if J is in the interior of a region, then its neighbors are either in the same region, or in one of the sequences that bound the region.

It is not hard to see that the vertical segments of the NE-sequence are to the right of I, while horizontal segments are above I. A horizontal segment K_{2i} cannot be to the left of I, since it ends to the right of I. Let us prove that K_{2i} cannot be the right of I either. Assume this is the case, and consider the polygonal line going from the leftmost point of I to the rightmost point of K_{2i}, as described in Observation 2.4. The last section of this line is K_{2i}. Hence the line has points in the interior of the region comprised between the NW- and NE-sequences. But since the line always goes to the right, and follows entirely every horizontal segment it visits, it can never enter the interior of this region. Thus K_{2i} cannot be to the right of I. Thus, its only relation to I is $I \downarrow K_{2i}$. Similar arguments apply for vertical segments of the NE-sequence, and for the other three sequences.

Consider now a segment J that lies, for instance, in the North region (that is the region bounded by the NE-sequence, the NW-sequence, and the boundary; the case of other regions is similar). Then I is below J: if we consider the below-neighbors of J, then their below-neighbors, and so on, then we necessarily reach one of the horizontal segments of the NW- or NE-sequence, which, as we have seen, are above I (we cannot reach a vertical segment of the sequences without reaching a horizontal segment first).

Hence, we have that $I \downarrow J$; it remains to prove that the other three relations are impossible. First, $J \downarrow I$ is impossible since the relation $\downarrow$ is antisymmetric. To prove that J cannot be to the right of I, we argue as we did for K_{2i}: the polygonal line from I to J starting from the leftmost point of I cannot enter the North region. Symmetrically, J cannot be to the left of I. This completes the proof. $\square$

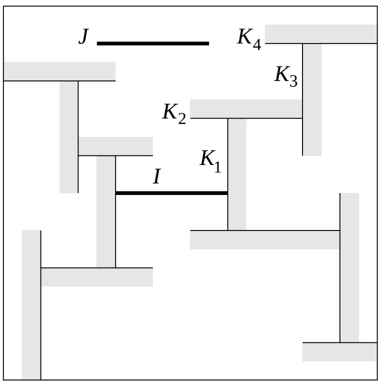


Figure 6: Four regions determining the relationship between I and other segments.

Definition 2.7. The relations $\swarrow$ and $\nwarrow$ between segments of a floorplan are defined by:

$$\begin{array}{lll} I \swarrow J & \textit{if} & I = J, \textit{ or } I \textit{ is to the left of } J, \textit{ or } I \textit{ is below } J, \\ I \nwarrow J & \textit{if} & I = J, \textit{ or } I \textit{ is to the left of } J, \textit{ or } I \textit{ is above } J. \end{array}$$

We also write $I \swarrow J$ when $I \swarrow J$ and $I \neq J$; and $I \nwarrow J$ when $I \nwarrow J$ and $I \neq J$.

Proposition 2.8. *The relations $\swarrow$ and $\nwarrow$ are linear orders.*

Proof. We prove the claim for the relation $\twoheadswarrow$. Reflexivity follows from the definition. Antisymmetry follows from the fact that $\twoheadleftarrow$ and $\twoheaddownarrow$ are order relations, and from Lemma 2.6.

For transitivity, assume that $I \swarrow J$ and $J \swarrow K$. If $I \leftarrow J$ and $J \leftarrow K$ (respectively, $I \downarrow J$ and $J \downarrow K$), then we have $I \leftarrow K$ (respectively, $I \downarrow K$) by the transitivity of $\leftarrow$ (respectively, $\downarrow$). Assume now that $I \leftarrow J$ and $J \downarrow K$ (the case $I \downarrow J$ and $J \leftarrow K$ is proven similarly). By Lemma 2.6, $I = K$ is impossible, and we have either $I \leftarrow K$, $K \leftarrow I$, $I \downarrow K$, or $K \downarrow I$. However, the combination of $K \leftarrow I$ and $I \leftarrow J$ yields $K \leftarrow J$, contradicting the assumption that $J \downarrow K$ (by Lemma 2.6). Similarly, combining $K \downarrow I$ with $J \downarrow K$ yields $J \downarrow I$, contradicting the assumption that $I \leftarrow J$. Therefore, we have either $I \leftarrow K$ or $I \downarrow K$; in particular, $I \swarrow K$.

Linearity follows from Lemma 2.6. $\square$

Observation 2.9. *The orders $\twoheadleftarrow$ and $\twoheaddownarrow$ can be recovered from $\twoheadswarrow$ and $\twoheadnwarrow$. Indeed, $I \twoheadleftarrow J$ if and only if $I \twoheadswarrow J$ and $I \twoheadnwarrow J$; moreover, $I \twoheaddownarrow J$ if and only if $I \twoheadswarrow J$ and $J \twoheadnwarrow I$.*

Throughout the paper, the ith segment in the $\twoheadnwarrow$ order ($1 \leq i \leq n$) will be denoted by I_i. See Fig. 10 for examples.

We now explain how to determine I_{i+1} among the neighbors of I_i. By Lemma 2.5, I_{i+1} is either a right- or below-neighbor of I_i. There are several cases depending on the existence of these neighbors and the relations between them. For a horizontal segment I, we denote by $\mathrm{R}(I)$ the right-neighbor of I (when it exists). By Lemma 2.5, the below-neighbors of I form an antichain of the $\twoheaddownarrow$ order. Since $\twoheadnwarrow$ is a linear order, they are totally ordered for the $\twoheadleftarrow$ order. By the first part of Observation 2.4, the leftmost is also the smallest, denoted $\mathrm{LB}(I)$. Thus $\mathrm{LB}(I)$ is either $\mathrm{LVB}(I)$ (the leftmost *vertical* below-neighbor of I) or $\mathrm{LHB}(I)$ (the leftmost *horizontal* below-neighbor of I). Similarly, for a vertical segment I, we denote by $\mathrm{B}(I)$ the below-neighbor of I; by $\mathrm{UR}(I)$ the highest[6] right-neighbor of I, and by $\mathrm{UHR}(I)$ (respectively, $\mathrm{UVR}(I)$) the highest horizontal (respectively, vertical) right-neighbor of I. Fig. 7 illustrates the following observation (it is assumed that all candidates for I_{i+1} are depicted. The dashed lines belong to the boundary).

Observation 2.10. *Let I_i be a segment in a floorplan P of size $n+1$. If I_i is horizontal, then I_{i+1} is either* $\mathrm{R}(I_i)$ *or* $\mathrm{LB}(I_i)$. *More precisely,*

1. *If none of* $\mathrm{R}(I_i)$ *and* $\mathrm{LB}(I_i)$ *exists, then I_i is the last segment in the $\twoheadnwarrow$ order (that is, $i = n$).*

2. *If exactly one of* $\mathrm{R}(I_i)$ *and* $\mathrm{LB}(I_i)$ *exists, then I_{i+1} is this segment.*

3. *If* $\mathrm{LVB}(I_i)$ *exists, then $I_{i+1} = \mathrm{LB}(I_i)$. This segment is* $\mathrm{LHB}(I_i)$ *if it exists, and otherwise* $\mathrm{LVB}(I_i)$.

4. *If* $\mathrm{LVB}(I_i)$ *does not exist but* $\mathrm{LHB}(I_i)$ *and* $\mathrm{R}(I_i)$ *exist, then*

   - *If the join of* $\mathrm{LHB}(I_i)$ *and* $\mathrm{R}(I_i)$ *is of type $\dashv$, then $I_{i+1} = \mathrm{LHB}(I_i)$.*

[6] The letter U stands for *up*.

- *If the join of* $\mathrm{LHB}(I_i)$ *and* $\mathrm{R}(I_i)$ *is of type* $\perp$*, then* $I_{i+1} = \mathrm{R}(I_i)$*.*

If I_i *is vertical, then* I_{i+1} *is either* $\mathrm{B}(I_i)$*,* $\mathrm{UHR}(I_i)$*, or* $\mathrm{UVR}(I_i)$ *(the details are similar to those in the case of a horizontal segment).*

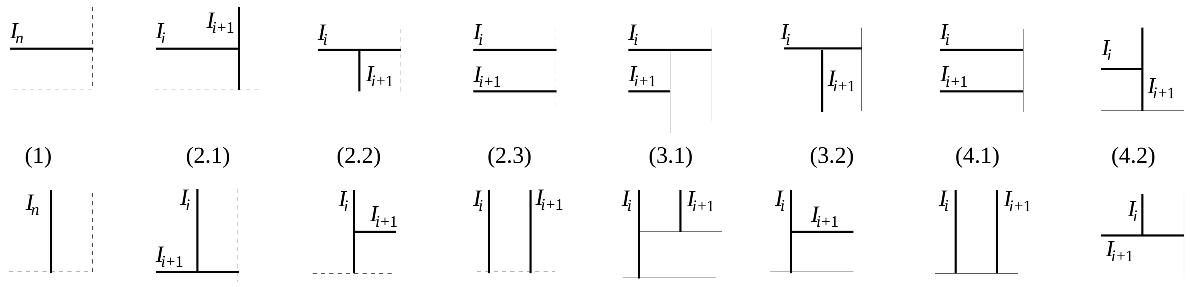


Figure 7: The segment I_{i+1} follows I_i in the $\nwarrow$ order (Top: I_i is horizontal. Bottom: I_i is vertical).

One can in fact construct in a single pass the labeling of rectangles and segments.

Proposition 2.11. *Let* P *be a floorplan of size* $n+1$*, and let* A_k *denote the rectangle labeled* k *in the* $\nwarrow$ *order. For* $1 \le k \le n$*, the following property, illustrated in Fig. 8, holds:*

- *If the segments forming the SE-corner of* A_k *have a* $\perp$ *join, let* J_k *be the segment containing the right side of* A_k*. Then* A_{k+1} *is the highest rectangle whose left side is contained in* J_k*.*

- *If the segments forming the SE-corner of* A_k *have a* $\dashv$ *join, let* J_k *be the segment containing the lower side of* A_k*. Then* A_{k+1} *is the leftmost rectangle whose upper side is contained in* J_k*.*

In both cases, J_k *is the* kth *segment in the* $\nwarrow$ *order of segments, denoted so far by* I_k*.*

Proof. By definition of the $\nwarrow$ order, A_{k+1} is either a right-neighbor or a below-neighbor of A_k. If there is a $\perp$ join in the SE-corner of A_k, then all the right-neighbors of A_k are above all its below-neighbors. Therefore, A_{k+1} is the topmost among them. If there is a

Figure 8: The rectangle A_{k+1} follows A_k in the $\nwarrow$ order.

$\dashv$ join in the SE-corner of A_k, then all the below-neighbors of A_k are to the left of all its right-neighbors. Therefore, A_{k+1} is the leftmost among them.

To prove the second statement, we observe it directly for $k = 1$, and proceed by induction. One has to examine several cases, depending on whether the segments in the SE-corners of A_k and of A_{k+1} have $\perp$ or $\dashv$ joins. In all cases, J_{k+1} is found to be the immediate successor of J_k in the $\nwarrow$ order, as described in Observation 2.10. See Fig. 9 for several typical situations. $\square$

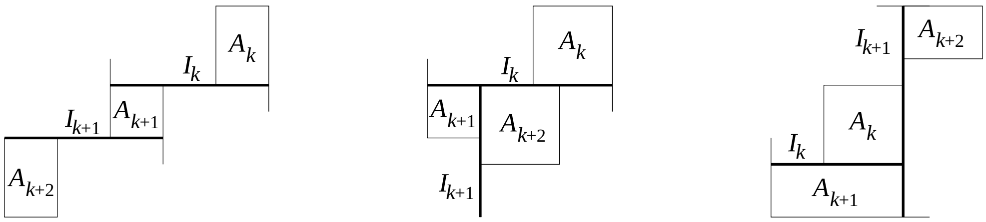


Figure 9: Successors of segments and rectangles for the $\nwarrow$ orders.

The group of symmetries of the square acts on floorplans (when floorplans are drawn in a square). It is thus worth examining how the orders are transformed when applying such symmetries. As this symmetry group is generated by two generators, for instance the reflections in the first diagonal and across a horizontal line, it suffices to study these two transformations. The following proposition easily follows from the description of the neighborhood relations of Fig. 3.

Proposition 2.12. *Let P be a (square) floorplan, and let P' be obtained by reflecting P in the first diagonal. If I is a segment of P, and I' the corresponding segment of P', then*

$$\begin{array}{lcl} I \twoheadleftarrow J & \Leftrightarrow & I' \twoheaddownarrow J', \\ I \twoheaddownarrow J & \Leftrightarrow & I' \twoheadleftarrow J', \\ I \swarrow J & \Leftrightarrow & I' \swarrow J', \\ I \nwarrow J & \Leftrightarrow & J' \nwarrow I'. \end{array}$$

If instead P' is obtained by reflecting P in a horizontal line,

$$\begin{array}{lcl} I \twoheadleftarrow J & \Leftrightarrow & I' \twoheadleftarrow J', \\ I \twoheaddownarrow J & \Leftrightarrow & J' \twoheaddownarrow I', \\ I \swarrow J & \Leftrightarrow & I' \nwarrow J', \\ I \nwarrow J & \Leftrightarrow & I' \swarrow J'. \end{array}$$

One consequence of this proposition is that a half-turn rotation of P reverses all four orders. We shall also use the fact that, if P' is obtained by applying a clockwise quarter-turn rotation to P, then $I \nwarrow J \Leftrightarrow J' \swarrow I'$.

3 A bijection between S-equivalence classes of floorplans and (2-14-3, 3-41-2)-avoiding permutations

In this section we define S-equivalence of floorplans and construct a map S from floorplans to permutations. We show that S induces an injection from S-equivalence classes to permutations. We then characterize the class of permutations obtained from floorplans in terms of (generalized) patterns.

3.1 S-equivalence

Definition 3.1. Two floorplans P_1 and P_2 of size $n+1$ are *S-equivalent* if it is possible to label the segments of P_1 by $I_1, I_2, \dots, I_n$ and the segments of P_2 by $J_1, J_2, \dots, J_n$ so that for all $k, m \in [n]$, the segments I_k and I_m exhibit the same neighborhood relation as J_k and J_m.

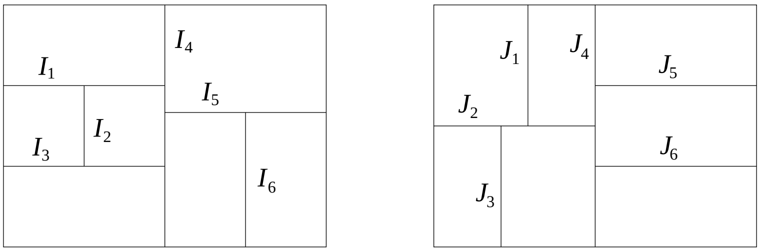


Figure 10: Two S-equivalent (but not R-equivalent) floorplans.

Fig. 10 shows two S-equivalent floorplans: in both cases, the left-right *neighborhood* relations are $1 \leftarrow 4$, $2 \leftarrow 4$, $3 \leftarrow 4$, $4 \leftarrow 5$, $4 \leftarrow 6$, and the below-above *neighborhood* relations are $2 \downarrow 1$, $3 \downarrow 2$, $6 \downarrow 5$. These floorplans are not R-equivalent, as can be seen by constructing their R-permutations. We will prove in Section 4 that, conversely, R-equivalence implies S-equivalence.

3.2 S-permutations

Let P be a floorplan of size $n+1$. There are n segments in P. Let $S(P)$ be the sequence $b_1, b_2, \dots, b_n$, where b_i is the label in the $\swarrow$ order of the segment labeled i in the $\nwarrow$ order, for all $1 \leq i \leq n$. Then $S(P)$ is a permutation of $[n]$; we call it the *S-permutation* of P and denote it by $S(P)$. Equivalently, if $I_1, \dots, I_n$ is the list of segments in the $\nwarrow$ order, then $I_{\sigma^{-1}(1)}, \dots, I_{\sigma^{-1}(n)}$ is the list of segments in the $\swarrow$ order, with $\sigma = S(P)$. Since the $\nwarrow$- and $\swarrow$-orders on segments can be determined in linear time (Proposition 2.11), the S-permutation is also constructed in linear time. An example is shown in Fig. 11.

Thus, we assign a permutation to a floorplan in a way similar to that used in [1], but this time we use order relations between segments rather than rectangles. Note that $S(P)$ is a permutation of $[n]$, while $R(P)$ is a permutation of $[n+1]$.

By definition of $S(P)$, if a segment of P is labeled i in the $\nwarrow$ order and j in the $\swarrow$ order, then $S(P)(i) = j$. In other words, the graph of $S(P)$ contains the point (i,j), which will be denoted by N_i.

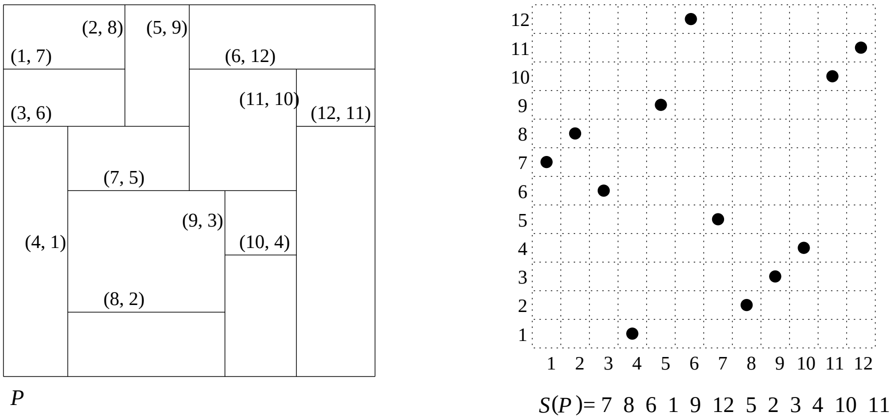


Figure 11: A floorplan P, with segments labeled (i,j), where i (respectively, j) is the label according to the $\nwarrow$ (respectively, $\swarrow$) order, and the corresponding S-permutation.

It follows from Proposition 2.12 that S is well-behaved with respect to symmetries.

Proposition 3.2. *Let P be a (square) floorplan, and P' be obtained by reflecting P in the first diagonal. Let $\sigma = S(P)$ and $\sigma' = S(P')$. Then σ' is obtained by reading σ from right to left or equivalently, by reflecting the graph of σ in a vertical line.*

If instead P' is obtained by reflecting P in a horizontal line, then $\sigma' = \sigma^{-1}$. Equivalently, σ' is obtained by reflecting the graph of σ in the first diagonal.

Proof. The following statements are equivalent:

- $\sigma(i) = j$,
- there exists a segment of P that has label i in the $\nwarrow$-order and j in the $\swarrow$-order (by definition of S),
- there exists a segment of P' that has label $n+1-i$ in the $\nwarrow$-order and j in the $\swarrow$-order (by Proposition 2.12),
- $\sigma'(n+1-i) = j$.

This proves the first result. The proof of the second result is similar. □

Since the two reflections of Proposition 3.2 generate the group of symmetries of the square, we can describe what happens for the other elements of this group: applying a rotation to P boils down to applying the same rotation to $S(P)$, and reflecting P in Δ, a symmetry axis of the bounding square, boils down to reflecting $S(P)$ in Δ', a line obtained by rotating Δ of 45° in counterclockwise direction. These properties will be extremely useful to decrease the number of cases we have to study in certain proofs.

We will now prove that $S(P)$ characterizes the S-equivalence class of P. Clearly, two S-equivalent floorplans give rise to the same orders, and thus to the same S-permutation. Conversely, let us define neighborhood relations between points in the graph of a permutation σ as follows. Let $N_i = (i, \sigma(i))$, $N_j = (j, \sigma(j))$ be two points in the graph of σ. If $i < j$ and $\sigma(i) < \sigma(j)$, then the point N_j is *to the NE* of the point N_i. If, in addition, there is no i' such that $i < i' < j$ and $\sigma(i) < \sigma(i') < \sigma(j)$, then N_j is a *NE-neighbor* of N_i. In a similar way we define when N_j is *to the SE / SW / NW* of N_i, and when the point N_j is a *SE- / SW- / NW-neighbor* of N_i. For example, in the graph of Fig. 11, the points $(1,7)$, $(2,8)$, $(3,6)$, $(5,9)$ and $(6,12)$ are to the NW of $N_7 = (7,5)$; among them, $(3,6)$, $(5,9)$ and $(6,12)$ are NW-neighbors of N_7.

The neighborhood relations between segments of P correspond to the neighborhood relations in the graph of $S(P)$ in the following way.

Observation 3.3. *Let P be a floorplan, and let I_i and I_j be two segments in P.*

The segment I_j is to the right of I_i if and only if the point N_j lies to the NE of N_i. Consequently, I_j is a right-neighbor of I_i if and only if N_j is a NE-neighbor of N_i.

Similar statements hold for the other directions: left (respectively, above, below) neighbors in segments correspond to SW- (respectively, NW-, SE-) neighbors in points.

Proof. The segment I_j is to the right of I_i if and only if $I_i \swarrow I_j$ and $I_i \nwarrow I_j$. By construction of $\sigma = S(P)$, this means that $i < j$ and $\sigma(i) < \sigma(j)$. Equivalently, N_j lies to the NE of N_i. $\square$

Remark. An analogous fact holds for *rectangles* of a floorplan and points in the graph of the corresponding *R-permutation.* It is not stated explicitly in [1], but follows directly from the definitions in the same way as Observation 3.3 does.

Since the neighborhood relations characterize the S-equivalence class, we have proved the following result.

Corollary 3.4. *Two floorplans are S-equivalent if and only if they have the same S-permutation.*

3.3 (2-14-3, 3-41-2)-avoiding permutations

In this section we first discuss the *dash notation* and *bar notation* for pattern avoidance in permutations, and then prove several facts about (2-14-3, 3-41-2)-avoiding permutations. We will prove later that these are precisely the S-permutations obtained from floorplans.

In the classical notation, a permutation $\pi = a_1 a_2 \dots a_n$ *avoids* a permutation (*a pattern*) $\tau = b_1 b_2 \dots b_k$ if there are no $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $a_{i_1} a_{i_2} \dots a_{i_k}$ (a subpermutation of π) is order isomorphic to τ ($b_x < b_y$ if and only if $a_{i_x} < a_{i_y}$).

The dash notation and the bar notation generalize the classical notation and provide a convenient way to define more classes of restricted permutations [38].

In the dash notation, some letters corresponding to those from the pattern τ may be required to be adjacent in the permutation π, in the following way. If there is a dash between two letters in τ, the corresponding letters in π may occur at any distance from each other; if there is no dash, they must be adjacent in π. For example, $\pi = a_1 a_2 \dots a_n$ avoids 2-14-3 if there are no $1 \le i < j < \ell < m$ such that $\ell = j + 1$ and $a_j < a_i < a_m < a_\ell$.

In the bar notation, some letters of τ may have bars. A permutation π avoids a barred pattern τ if every occurrence of the unbarred part of τ is a sub-occurrence of τ (with bars removed). For example, $\pi = a_1 a_2 \dots a_n$ avoids $21\bar{3}54$ if for any $1 \le i < j < \ell < m$ such that $a_j < a_i < a_m < a_\ell$, there exists k such that $j < k < \ell$ and $a_i < a_k < a_m$ (any occurrence of the pattern 2154 is a sub-occurrence of the pattern 21354).

A *(reduced) Baxter permutation* is a permutation of $[n]$ such that

There are no $i, j, \ell, m \in [n]$ satisfying $i < j < \ell < m$, $\ell = j + 1$, such that
either $\pi(j) < \pi(m) < \pi(i) < \pi(\ell)$ and $\pi(i) = \pi(m) + 1$,
or $\pi(\ell) < \pi(i) < \pi(m) < \pi(j)$ and $\pi(m) = \pi(i) + 1$.

In the dash notation, Baxter permutations are those avoiding (2-41-3, 3-14-2), and in the bar notation, Baxter permutations are those avoiding $(41\bar{3}52, 25\bar{3}14)$ (see [24] or [38, Sec. 7]). As proved in [1], the permutations that are obtained as R-permutations are precisely the Baxter permutations. It turns out that the permutations that are obtained as S-permutations may be defined by similar conditions, given below in Proposition 3.6. As in the Baxter case, these conditions can be defined in three different ways.

Lemma 3.5. *Let π be a permutation of $[n]$. The following conditions are equivalent:*

1. *There are no $i, j, \ell, m \in [n]$ such that $i < j < \ell < m$, $\ell = j + 1$, $\pi(j) < \pi(i) < \pi(m) < \pi(\ell)$, $\pi(m) = \pi(i) + 1$.*

2. *In the dash notation, π avoids* 2-14-3.

3. *In the bar notation, π avoids* $21\bar{3}54$.

Fig. 12 illustrates these three conditions. The rows (respectively, columns) marked by dots in parts (1) and (2) denote adjacent rows (respectively, columns). The shaded area in part (3) does not contain points of the graph.

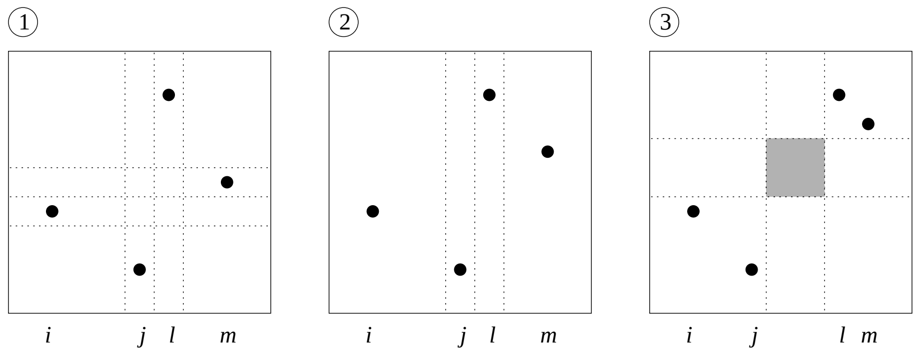


Figure 12: Three ways to define permutations avoiding 2-14-3.

Proof. It is clear that $3 \Rightarrow 2 \Rightarrow 1$: the four points displayed in Fig. 12(1) form an occurrence of the pattern of Fig. 12(2), and the four points displayed in Fig. 12(2) form an occurrence of the pattern of Fig. 12(3).

Conversely, let us prove that if a permutation π contains the pattern $21\bar{3}54$, then there exist i', j', ℓ', m' as in the first condition. Assume that there are $i < j < \ell < m$ such that $\pi(j) < \pi(i) < \pi(m) < \pi(\ell)$, and there is no k such that $j < k < \ell$ and $\pi(i) < \pi(k) < \pi(m)$. Let j' be the maximal number for which $j \leq j' < \ell$ and $\pi(j') < \pi(i)$. Let $\ell' = j' + 1$. Then $\pi(\ell') > \pi(m)$, and we have a pattern 2-14-3 with i, j', ℓ', m.

Furthermore, let i' be the number satisfying $i' < j'$ and $\pi(i) \leq \pi(i') < \pi(m)$, for which $\pi(i')$ is the maximal possible. Let $m' = \pi^{-1}(\pi(i') + 1)$. Then $m' > \ell'$ and $\pi(m') = \pi(i') + 1$, and, thus, the first condition holds with i', j', ℓ', m'. $\square$

A similar result holds for permutations that avoid 3-41-2. Therefore, the following proposition holds.

Proposition 3.6. *Let σ be a permutation of $[n]$. The following statements are equivalent:*

1. *There are no $i, j, \ell, m \in [n]$ satisfying $i < j < \ell < m$, $\ell = j + 1$, such that*
   either $\sigma(j) < \sigma(i) < \sigma(m) < \sigma(\ell)$ and $\sigma(m) = \sigma(i) + 1$,
   or $\sigma(\ell) < \sigma(m) < \sigma(i) < \sigma(j)$ and $\sigma(i) = \sigma(m) + 1$.

2. *In the dash notation, σ avoids* 2-14-3 *and* 3-41-2*.*

3. *In the bar notation, σ avoids* $21\bar{3}54$ *and* $45\bar{3}12$*.*

Corollary 3.7. *The group of symmetries of the square leaves invariant the set of* (2-14-3, 3-41-2)*-avoiding permutations.*

Proof. The second description in Proposition 3.6 shows that the set of (2-14-3, 3-41-2)-avoiding permutations is closed under reading the permutations from right to left. The first (or third) description shows that it is invariant under taking inverses, and these two transformations generate the symmetries of the square. $\square$

We shall also use the following fact.

Lemma 3.8. *Let σ be a* (2-14-3, 3-41-2)*-avoiding permutation of $[n]$. Then no point in the graph of σ has several NW-neighbors* and *several NE-neighbors. Similar statements hold for other pairs of adjacent diagonal directions.*

Proof. Assume that $N_i = (i, \sigma(i))$ has several NW-neighbors and several NE-neighbors. Let i' be the maximal number for which $N_{i'}$ is a NW-neighbor of N_i, and let N_j be another NW-neighbor of N_i. Then we have $j < i'$ and $\sigma(i) < \sigma(j) < \sigma(i')$. We conclude that $i' = i - 1$: otherwise $\sigma(i' + 1) < \sigma(i)$ and, therefore, $j, i', i' + 1, i$ form the forbidden pattern 3-41-2, which is a contradiction.

Similarly, if i'' is the minimal number for which $N_{i''}$ is a NE-neighbor of N_i, then $i'' = i + 1$. Let N_k be another NE-neighbor of N_i. We have $\sigma(i) < \sigma(k) < \sigma(i + 1)$.

Assume without loss of generality that $\sigma(i - 1) < \sigma(i + 1)$. Now, if $\sigma(j) < \sigma(k)$, then $j, i, i+1, k$ form the forbidden pattern 2-14-3; and if $\sigma(k) < \sigma(j)$, then $j, i-1, i, k$ form the forbidden pattern 3-41-2, which is, again, a contradiction. $\square$

3.4 S-permutations coincide with (2-14-3, 3-41-2)-avoiding permutations

By Corollary 3.4, the map S induces an injection from S-equivalence classes of floorplans to permutations. Here, we characterize the image of S.

Theorem 3.9. *The map S induces a bijection between S-equivalence classes of floorplans of size $n+1$ and* (2-14-3, 3-41-2)*-avoiding permutations of size n.*

The proof involves two steps: In Proposition 3.11 we prove that all S-permutations are (2-14-3, 3-41-2)-avoiding. Then, in Proposition 3.12, we show that for any (2-14-3, 3-41-2)-avoiding permutation σ of $[n]$, there exists a floorplan P such that $S(P) = \sigma$.

Recall that a horizontal segment has at most one left-neighbor and at most one right-neighbor, and a vertical segment has at most one below-neighbor and at most one above-neighbor. This translates as follows in terms of S-permutations.

Observation 3.10. *Let I_i be a segment in a floorplan P, and let N_i be the corresponding point in the graph of $S(P)$. If I_i is a horizontal segment, then the point N_i has at most one NE-neighbor and at most one SW-neighbor. Similarly, if I_i is a vertical segment, then N_i has at most one SE-neighbor and at most one NW-neighbor.*

Proposition 3.11. *Let P be a floorplan. Then $S(P)$ avoids* 2-14-3 *and* 3-41-2*.*

Proof. By Proposition 3.2, the image of S is invariant by all symmetries of the square. Hence it suffices to prove that $\sigma = S(P)$ avoids 2-14-3.

Assume that σ contains 2-14-3. By Lemma 3.5, there exist $i < j < \ell < m$, $\ell = j+1$ such that $\sigma(j) < \sigma(i) < \sigma(m) < \sigma(\ell)$ and $\sigma(m) = \sigma(i) + 1$ (see Fig. 13(1)). We claim that the four segments I_i, I_j, I_ℓ, I_m are vertical.

Consider I_j. The point N_ℓ is a NE-neighbor of N_j. Consider the set $\{x : x > \ell, \sigma(j) < \sigma(x) < \sigma(\ell)\}$. This set is not empty since it contains m. Let p be the smallest element in this set. Then N_p is a NE-neighbor of N_j. Thus, N_j has at least two NE-neighbors, N_ℓ and N_p. Therefore, I_j is vertical by Observation 3.10. In a similar way one can show that I_i, I_ℓ, I_m are also vertical.

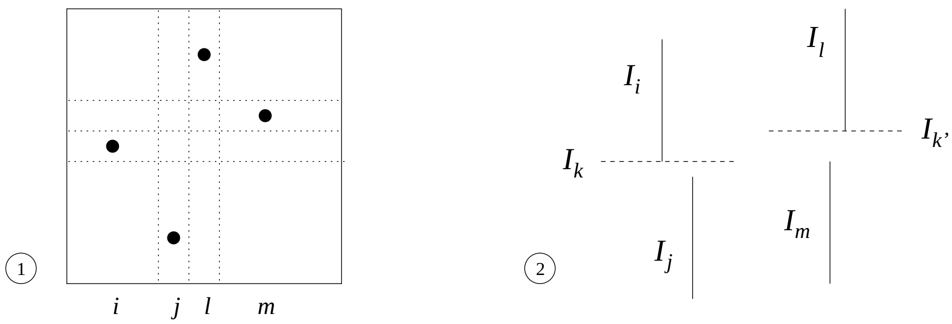


Figure 13: The pattern 2-14-3 never occurs in an S-permutation.

By Observation 3.3 we have that: $I_j \downarrow I_i$, $I_m \downarrow I_\ell$; $I_i \leftarrow I_\ell$, $I_j \leftarrow I_m$, $I_i \leftarrow I_m$, $I_j \leftarrow I_\ell$. Moreover, the last two relations are neighborhood relations. Let I_k be the below-neighbor of I_i, and let $I_{k'}$ be the below-neighbor of I_ℓ (see Fig. 13 (2)). The segments I_k and $I_{k'}$ are horizontal. If the line supporting I_k is (weakly) lower than the line supporting $I_{k'}$, then I_j (which is below I_i) cannot be a left-neighbor of I_ℓ since the interiors of their vertical projections do not intersect. Similarly, if the line supporting I_k is higher than the line supporting $I_{k'}$, then I_i cannot be a left-neighbor of I_m. We have thus reached a contradiction, and σ cannot contain 2-14-3. ☐

Proposition 3.12. *For each* $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$*-avoiding permutation* σ *of* $[n]$*, there exists a floorplan* P *with* n *segments such that* $S(P) = \sigma$*.*

Proof. We construct P on the graph of σ. The boundary of the graph is also the boundary of P. For each point $N_i = (i, \sigma(i))$ of the graph, we draw a segment K_i passing through N_i according to certain rules. We first determine the direction of the segments K_i (Paragraph A below), and then the coordinates of their endpoints (Paragraph B). We prove that we indeed obtain a floorplan (Paragraph C), and that its S-permutation is σ (Paragraph D). This is probably one of the most involved proofs of the paper.

A. Directions of the segments K_i

Let $N_i = (i, \sigma(i))$ be a point in the graph of σ. Our first two rules are forced by Observation 3.10 (see Fig. 14, where the the shaded areas contain no point):

- If N_i has several NW-neighbors or several SE-neighbors, then K_i is horizontal;
- If N_i has several SW-neighbors or several NE-neighbors, then K_i is vertical.

By Lemma 3.8, these two rules never apply simultaneously to the same point N_i. If one of them applies, we say that N_i is a *strong point*. Otherwise, N_i is a *weak point*. This means that N_i has at most one neighbor in each direction.

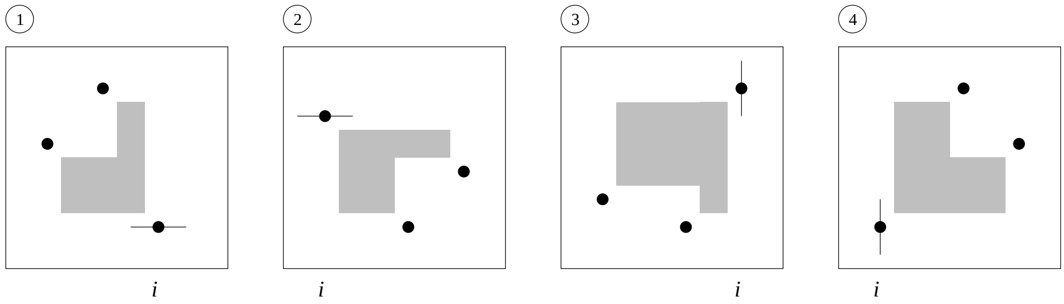


Figure 14: The direction of the segment K_i passing through a strong point.

We claim that if N_i and N_j are weak points, then they are in adjacent rows if and only if they are in adjacent columns. Due to symmetry, it suffices to show the *if* direction. Let

N_i and N_{i+1} be weak points, and assume without loss of generality that $\sigma(i) < \sigma(i+1)$. If $\sigma(i+1) - \sigma(i) > 1$, then there are points of the graph of σ between the rows that contain N_i and N_{i+1}; thus, either N_i has at least two NE-neighbors or N_{i+1} has at least two SW-neighbors, which means that one of them at least is strong. Hence $\sigma(i+1) = \sigma(i) + 1$.

Thus, weak points appear as ascending or descending sequences of adjacent neighbors: $N_i, N_{i+1}, \dots, N_{i+\ell}$ with $\sigma(i) = \sigma(i+1)-1 = \cdots = \sigma(i+\ell)-\ell$ or $\sigma(i) = \sigma(i+1)+1 = \cdots = \sigma(i+\ell)+\ell$. Note that a weak point N_i can be isolated.

For weak points, the direction of the corresponding segments is determined as follows:

- If $N_i, N_{i+1}, \dots, N_{i+\ell}$ is a maximal *ascending* sequence of weak points, then the directions of $K_i, K_{i+1}, \dots, K_{i+\ell}$ are chosen in such a way that K_j and K_{j+1} are never both horizontal, for $i \leq j < i+\ell$. Hence several choices are possible (this multiplicity of choices is consistent with the fact that all S-equivalent floorplans give the same permutation).

- If $N_i, N_{i+1}, \dots, N_{i+\ell}$ is a maximal *descending* sequence of weak points, then the directions of $K_i, K_{i+1}, \dots, K_{i+\ell}$ are chosen in such a way that K_j and K_{j+1} are never both vertical, for $i \leq j < i+\ell$.

In particular, for an isolated weak point N_i, the direction of K_i can be chosen arbitrarily.

B. Endpoints of the segments $\boldsymbol{K_i}$

Once the directions of all K_i's are chosen, their endpoints are set as follows (see Fig. 15 for an illustration):

- If K_i is vertical (which implies that N_i has at most one NW-neighbor and at most one SE-neighbor):

  - If N_i has a NW-neighbor N_j, then the upper endpoint of K_i is set to be $(i, \sigma(j))$. We say that N_j *bounds* K_i *from above*. Otherwise (if N_i has no NW-neighbor), K_i reaches the upper side of the boundary.

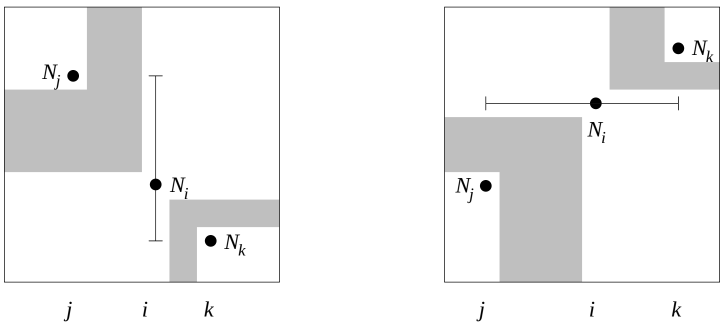


Figure 15: The points N_j and N_k bound the segment K_i.

– If N_i has a SE-neighbor N_k, then the lower endpoint of K_i is $(i, \sigma(k))$. We say that N_k *bounds* K_i *from below*. Otherwise, K_i reaches the lower side of the boundary.

- If K_i is horizontal (which implies that N_i has at most one SW-neighbor and at most one NE-neighbor):

  – If N_i has a SW-neighbor N_j, then the left endpoint of K_i is $(j, \sigma(i))$. We say that N_j *bounds* K_i *from the left*. Otherwise, K_i reaches the left side of the boundary.

  – If N_i has a NE-neighbor N_k, then the right endpoint of K_i is $(k, \sigma(i))$. We say that N_k *bounds* K_i *from the right*. Otherwise, K_i reaches the right side of the boundary.

Fig. 16 presents an example of the whole construction: in Part 1, the directions are determined for strong (black) points, and chosen for weak (gray) points; in Part 2, the endpoints are determined and a floorplan is obtained. Notice that σ is the S-permutation associated with the floorplan P of Fig. 11, but here we have obtained a different floorplan, P'. We leave it to the reader to check that another choice of directions of segments passing through weak points leads to P.

The question of when $S(P) = S(P')$ will be studied in Section 4.2.

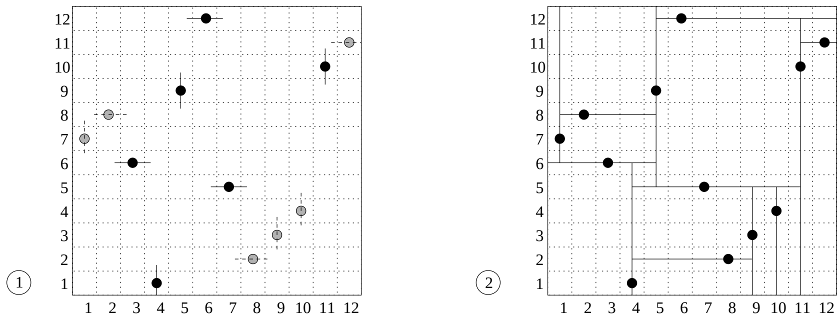


Figure 16: Constructing a floorplan from a $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$-avoiding permutation.

Remark. Using dynamic programming, one can determine in linear time the values

$$m_i = \max\{k < i : \sigma(k) < \sigma(i)\} \cup \{0\} \quad \text{and} \quad M_i = \max\{k < i : \sigma(k) > \sigma(i)\} \cup \{0\}.$$

By applying this procedure to σ and to the permutations obtained by applying to σ a symmetry of the square, one can decide in linear time, for each point N_i of σ, if it has one or several NW-neighbours and locate one of them. This implies that the above construction of a floorplan starting from a $(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$-avoiding permutation can be done in linear time.

C. The construction indeed determines a floorplan

In order to prove this, we need to show that two segments never cross, and that the endpoints of any segment K_i are contained in segments perpendicular to K_i (unless they lie on the boundary). The following observation will simplify some of our proofs.

Observation 3.13. *Let σ be a* (2-14-3, 3-41-2)*-avoiding permutation, and let σ' be obtained by applying a rotation ρ (by $\pm 90°$ or $180°$) to (the graph of) σ. If P is a configuration of segments obtained from σ by applying the rules of Paragraphs A and B above, then $\rho(P)$ can be obtained from σ' using those rules.*

It suffices to check that the rules are invariant by a 90° rotation, which is immediate[7].

C.1. Let K_i be a vertical (respectively, horizontal) segment, and let N_j and N_k be the points that bound it. Then the segments K_j and K_k are horizontal (respectively, vertical).

By Observation 3.13, it suffices to prove this claim for a *vertical* segment K_i and for the point N_j that bounds it *from above*. We need to prove that K_j is a horizontal segment.

We have $j < i$ and $\sigma(i) < \sigma(j)$, and, since N_j is a NW-neighbor of N_i, there is no ℓ such that $j < \ell < i$ and $\sigma(i) < \sigma(\ell) < \sigma(j)$. Furthermore, there is no ℓ such that $j < \ell < i$, $\sigma(j) < \sigma(\ell)$, or such that $\ell < j$, $\sigma(i) < \sigma(\ell) < \sigma(j)$: otherwise N_i would have several NW-neighbors and, therefore, K_i would be horizontal. Now, if $i - j > 1$, then there exists ℓ such that $j < \ell < i$, $\sigma(\ell) < \sigma(i)$; and if $\sigma(j) - \sigma(i) > 1$, then there exists m such that $i < m$, $\sigma(i) < \sigma(m) < \sigma(j)$. In both cases N_j has several SE-neighbors, and, therefore, K_j is horizontal as claimed.

It remains to consider the case where $j = i - 1$ and $\sigma(j) = \sigma(i) + 1$. If the point N_i is strong, then (since K_i is vertical) it has several NE-neighbors or several SW-neighbors. Assume without loss of generality that N_i has several NE-neighbors. Let ℓ be the minimal number such that N_ℓ is a NE-neighbor of N_i, and let N_m be another NE-neighbor of N_i. Then we have $\sigma(i-1) < \sigma(m) < \sigma(\ell)$ and $\sigma(\ell - 1) \le \sigma(i)$. However, then $i-1, \ell-1, \ell, m$ form a forbidden pattern 2-14-3. Therefore, N_i is a weak point. Clearly, N_{i-1} as a unique SE-neighbor (which is N_i). Its NE- and SW-neighbors coincide with those of N_i, so that there is at most one of each type. Thus if N_{i-1} is strong, it has several NW-neighbors, and K_{i-1} is horizontal, as claimed. If N_{i-1} is weak, then the rules that determine the direction of the segments passing through (descending) weak points implies that K_{i-1} and K_i cannot be both vertical. Therefore, $K_j = K_{i-1}$ is horizontal, as claimed.

C.2. If N_j and N_k are the points that bound the segment K_i, then the segments K_j and K_k contain the endpoints of K_i.

Thanks to Observation 3.13, it suffices to show that if K_i is a vertical segment and N_j bounds it from above, then K_j (which is horizontal as shown in Paragraph C.1 above) contains the point $(i, \sigma(j))$. We saw in Paragraph C.1 that in this situation there is no ℓ such that $j < \ell < i$, $\sigma(j) < \sigma(\ell)$. This means that there is no point N_ℓ that could bound K_j from the right before it reaches $(i, \sigma(j))$.

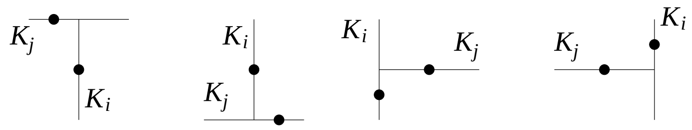
[7] That the construction has the other symmetries of Proposition 3.2 is also true, but less obvious. We shall only use Observation 3.13.

C.3. Two segments K_i and K_j cannot cross.
Assume that K_i and K_j cross. Assume without loss of generality that K_i is vertical and K_j is horizontal, so that their crossing point is $(i, \sigma(j))$. We have either $i < j$ or $j < i$, and $\sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i)$. Assume without loss of generality $j < i$ and $\sigma(i) < \sigma(j)$. Then N_j is to the NW of N_i. The ordinate of the (unique) NE-neighbor of N_i is hence at most $\sigma(j)$. By construction, the upper point of K_i has ordinate at most $\sigma(j)$, while K_j lies at ordinate $\sigma(j)$, and thus K_i and K_j cannot cross.

We have thus proved that our construction indeed gives a floorplan. Let us finish with an observation on joins of segments of this floorplan, which follows from Paragraph C.2 and is illustrated below.

Observation 3.14. *Suppose that a vertical segment K_i and a horizontal segment K_j join at the point $(i, \sigma(j))$. Then:*

- *If the join of K_i and K_j is of the type $\top$, then $i > j$.*
- *If the join of K_i and K_j is of the type $\perp$, then $i < j$.*
- *If the join of K_i and K_j is of the type $\vdash$, then $\sigma(i) < \sigma(j)$.*
- *If the join of K_i and K_j is of the type $\dashv$, then $\sigma(i) > \sigma(j)$.*



D. For any floorplan P obtained by the construction described above, $S(P) = \sigma$

This (concluding) part of the proof is given in Appendix A. □

4 Relations between the R- and S-permutations

In this section we prove that if two floorplans are R-equivalent, they are S-equivalent. In fact, we give a simple graphical way to construct $S(P)$ from $R(P)$, which also shows that $S(P)$ and $R(P)$ taken together form the *complete* Baxter permutation associated with the reduced Baxter permutation $R(P)$. Finally, we characterize the R-equivalence classes that belong to the same S-equivalence class.

4.1 Constructing $S(P)$ from $R(P)$

Let P be a floorplan of size $n+1$. We draw the graphs of $\rho = R(P)$ and $\sigma = S(P)$ on the same diagram in the following way (Fig. 17). For the graph of ρ we use an $(n+1) \times (n+1)$ square whose columns and rows are numbered by $1, 2, \ldots, n+1$. The points of the graph of ρ are black and placed at the centers of these squares. For the graph of σ we use the grid lines of the same drawing, when the ith vertical (respectively, horizontal) line is the grid line between the ith and the $(i+1)$st columns (respectively, rows). The point $(i, \sigma(i))$ is white and placed at the intersection of the ith vertical grid line and the jth horizontal grid line, where $j = \sigma(i)$. The whole drawing is called the *combined diagram* of P. The extreme (rightmost, leftmost, etc.) grid lines are not used.

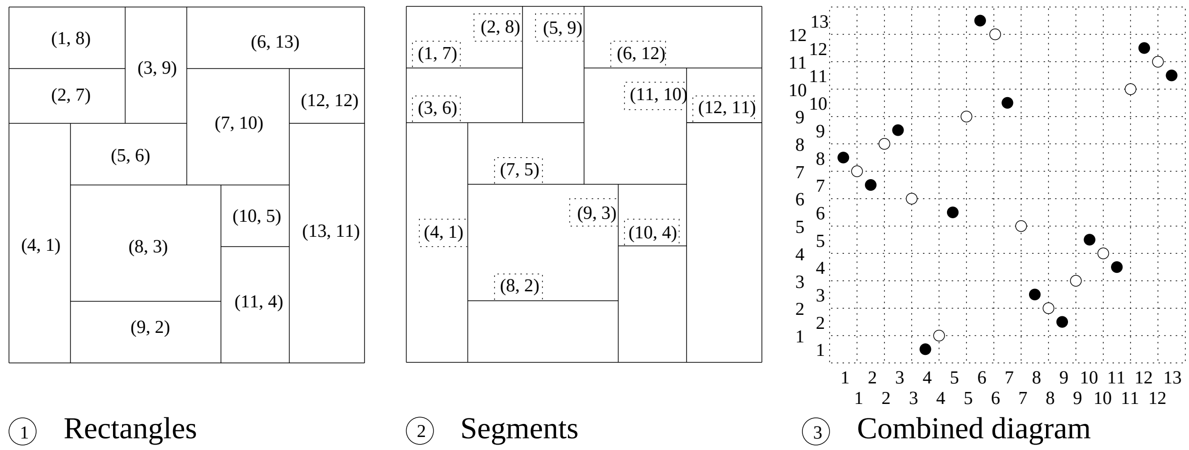


Figure 17: The floorplan P from Fig. 11: (1) The labeling of rectangles; (2) The labeling of segments; (3) The combined diagram: $R(P) = 8\ 7\ 9\ 1\ 6\ 13\ 10\ 3\ 2\ 5\ 4\ 12\ 11$ (black points) together with $S(P) = 7\ 8\ 6\ 1\ 9\ 12\ 5\ 2\ 3\ 4\ 10\ 11$ (white points).

Definition 4.1. Let ρ be a Baxter permutation of $[n+1]$. For $i \in [n]$, define j_i as follows:

- if $\rho(i) < \rho(i+1)$, then $j_i = \max\{\rho(k), k \le i \text{ and } \rho(k) < \rho(i+1)\}$,
- if $\rho(i) > \rho(i+1)$, then $j_i = \max\{\rho(k), k \ge i+1 \text{ and } \rho(k) < \rho(i)\}$.

The definition of Baxter permutations implies that

- if $\rho(i) < \rho(i+1)$, $k \ge i+1$ and $\rho(k) > \rho(i)$, then $\rho(k) > \rho(j_i)$,
- if $\rho(i) > \rho(i+1)$, $k \le i$ and $\rho(k) > \rho(i+1)$, then $\rho(k) > \rho(j_i)$.

Theorem 4.2. *Let P be a floorplan of size $n+1$, and let $\rho = R(P)$. Then $S(P) = (j_1, j_2, \ldots, j_n)$, where j_i is defined in Definition 4.1. In particular, R-equivalent floorplans are also S-equivalent.*

Returning to the original papers on Baxter permutations (see for instance [12, Thm. 2], or the definition of complete permutations in [13, p. 180]) this means that the combined diagram forms a (complete) Baxter permutation π. The points of $R(P)$ form the reduced Baxter permutation π_o associated with π, and the points of $S(P)$ are those that are deleted from π when constructing π_o.

Proof. Let $i \in [n]$. Denote $\sigma = S(P)$ and $j = \sigma(i)$. Then the segment I_i labeled i in the $\nwarrow$ order, is labeled j in the $\swarrow$ order. We denote by A_k (resp. B^k) the kth *rectangle* in the $\nwarrow$- (resp. $\swarrow$-) order. We wish to prove that $j = j_i$.

Assume first that I_i is horizontal. By Observation 2.11, the rightmost rectangle whose lower side is contained in I_i is A_i, and the leftmost rectangle whose upper side is contained in I_i is A_{i+1} (Fig. 18).

By definition of ρ, we have $A_k = B^{\rho(k)}$ for all k. By symmetry, since I_i is the jth segment in the $\swarrow$ order, the rightmost rectangle whose upper side is contained in I_i is B^j, and the leftmost rectangle whose lower side is contained in I_i is B^{j+1}. There holds $A_{i+1} \swarrow B^j \swarrow B^{j+1} \swarrow A_i$ and $B^{j+1} \nwarrow A_i \nwarrow A_{i+1} \nwarrow B^j$. By definition of $\rho = R(P)$, this means $\rho(i+1) \le j < j+1 \le \rho(i)$ and $\rho^{-1}(j+1) \le i < i+1 \le \rho^{-1}(j)$. This shows that j coincides with the value j_i of Definition 4.1 (for the case $\rho(i) > \rho(i+1)$).

The case where I_i is vertical is similar, and corresponds to an ascent in ρ. $\square$

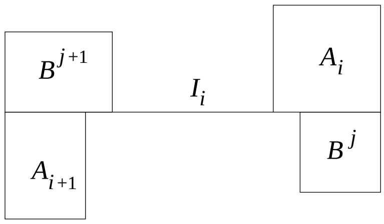


Figure 18: Illustration of the proof of Theorem 4.2.

The symmetry in the definition of j_i makes the following property obvious, without going through floorplans.

Corollary 4.3. *Let P be a floorplan and let $\rho = R(P)$ be the corresponding Baxter permutation. Let us abuse notation by denoting $S(\rho) := S(P)$. If ρ' is obtained by applying to ρ a symmetry of the square, then the same symmetry, applied to $S(\rho)$, gives $S(\rho')$.*

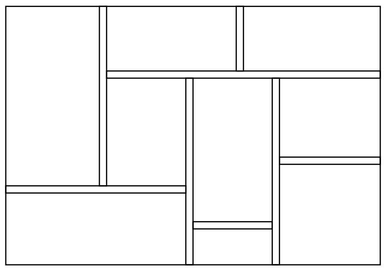


Figure 19: Inflating the segments of a floorplan.

Remark. The combined diagram is actually the R-permutation of a floorplan of size $2n+1$. Indeed, let P be a floorplan of size $n+1$. If we inflate segments of P into narrow rectangles, we obtain a new floorplan of size $2n+1$, which we denote by $\tilde{P}$ (Fig. 19). Observe that a rectangle of $\tilde{P}$ corresponding to a rectangle A of P has a unique above (respectively, right, below, left) neighbor, which corresponds to the segment of P that contains the above (respectively, right, below, left) side of A.

It follows from Observation 2.11 and Fig. 8 that the $\nwarrow$ order in $\tilde{P}$ is $A_1 I_1 \cdots A_n I_n A_{n+1}$. It is thus obtained by shuffling the $\nwarrow$ orders for rectangles and segments of P. Symmetrically, the $\swarrow$ order in $\tilde{P}$ is $A_{\rho^{-1}(1)} I_{\sigma^{-1}(1)} \cdots A_{\rho^{-1}(n)} I_{\sigma^{-1}(n)} A_{\rho^{-1}(n+1)}$. Thus the combined diagram of $R(P)$ and $S(P)$, as in Fig. 17, coincides with the graph of $R(\tilde{P})$.

4.2 Floorplans that produce the same S-permutation

We now characterize in terms of their R-permutations the floorplans that have the same S-permutation. This will play a central role in the enumeration of S-permutations.

We first describe the floorplans whose S-permutation is $123\ldots n$. We call them *ascending F-blocks*[8]. It is easy to see that in an ascending F-block, all vertical segments extend from the lower to the upper side of the boundary, and there is at most one horizontal segment between a pair of adjacent vertical segments (this can be shown inductively, by noticing that at most one horizontal segment starts from the left side of the bounding rectangle). See Fig. 20. Conversely, every floorplan of this type has S-permutation $123\ldots n$. Therefore, an ascending F-block consists of several rectangles that extend from the lower to the upper side of the boundary, some of them being split into two sub-rectangles by a horizontal segment. The corresponding R-permutations are those that satisfy $|\rho(i) - i| \leq 1$ for all $1 \leq i \leq n+1$. The number of ascending F-blocks of size $n+1$ (and, therefore, the number of such permutations) is the Fibonacci number F_{n+1} (where $F_0 = F_1 = 1$).

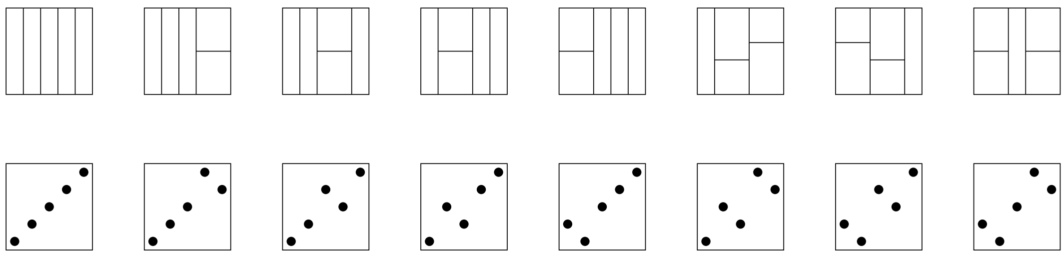


Figure 20: The 8 ascending F-blocks for $n = 4$, and their R-permutations.

A similar observation holds for the floorplans whose S-permutation is $n \ldots 321$, which we call *descending F-blocks*. In descending F-blocks, all horizontal segments extend from the left side to the right side of the boundary, and there is at most one vertical segment between a pair of adjacent horizontal segments. In other words, descending F-blocks consist of several rectangles that extend from the left to the right side of the boundary, some of them being split into two sub-rectangles by a vertical segment. The corresponding R-permutations are characterized by the condition $|\rho(i) - (n+2-i)| \leq 1$ for all $i \in [n+1]$.

For an F-block F, the size of F (that is, the number of rectangles) will be denoted by $|F|$. If $|F| = 1$, we say that F is a *trivial* F-block. Note that if $|F| \leq 2$, then F is

[8] The letter F refers to Fibonacci, for reasons that will be explained further down.

both ascending and descending, while if $|F| \geq 3$, then its type (ascending or descending) is uniquely determined.

Let P be a floorplan. We define an *F-block in P* as a set of rectangles of P whose union is an F-block, as defined above. In other words, their union is a rectangle, and the S-permutation of the induced subpartition is either $123\ldots$ or $\ldots 321$. The F-blocks of P are partially ordered by inclusion. Since segments of P do not cross, a rectangle in P belongs precisely to one maximal F-block (which may be of size 1). So there is a uniquely determined partition of P into maximal F-blocks (Fig. 21, left).

A *block* in a permutation ρ is an interval $[i, j]$ such that the values $\{\rho(i), \ldots, \rho(j)\}$ also form an interval [3]. We also call *block* the corresponding set of points in the graph of ρ. Consider ℓ rectangles in P that form an ascending (respectively, descending) F-block. By Observation 2.11 and the analogous statement for the $\swarrow$ order, these ℓ rectangles form an interval in the $\nwarrow$ and $\swarrow$ orders. Hence the corresponding ℓ points of the graph of $R(P)$ form a block, and their inner order is isomorphic to a permutation τ of $[\ell]$ that satisfies $|\tau(i) - i| \leq 1$ (respectively, $|\tau(i) - (\ell + 1 - i)| \leq 1$) for all $i \in [\ell]$.

The converse is also true: If ℓ points of the graph of $R(P)$ form an $\ell \times \ell$ block, and their inner order is isomorphic to a permutation τ of $[\ell]$ that satisfies $|\tau(i) - i| \leq 1$ (respectively, $|\tau(i) - (\ell + 1 - i)| \leq 1$) for all $1 \leq i \leq \ell$, then the corresponding rectangles in P form an ascending (respectively, descending) F-block. Indeed, let H be such an ascending block in the graph of $R(P)$. Let us partition the points of H in singletons (formed of points that lie on the diagonal) and pairs (formed of transposed points at adjacent positions). Let $Q_1, Q_2, \ldots$ be the parts of this partition, read from the SW to the NE corner of H. For each $i = 1, 2, \ldots$, the point(s) of Q_{i+1} are the only NE-neighbors of the point(s) of Q_i, and, conversely, the point(s) of Q_i are the only SW-neighbors of the point(s) of Q_{i+1}. Therefore, by the remark that follows Observation 3.3, the left side of the rectangle(s) corresponding to the point(s) of Q_{i+1} coincides with the right side of the rectangle(s) corresponding to the point(s) of Q_i. If Q_i consists of two points then we have two rectangles whose union is a rectangle split by a horizontal segment. The argument is similar for a descending block.

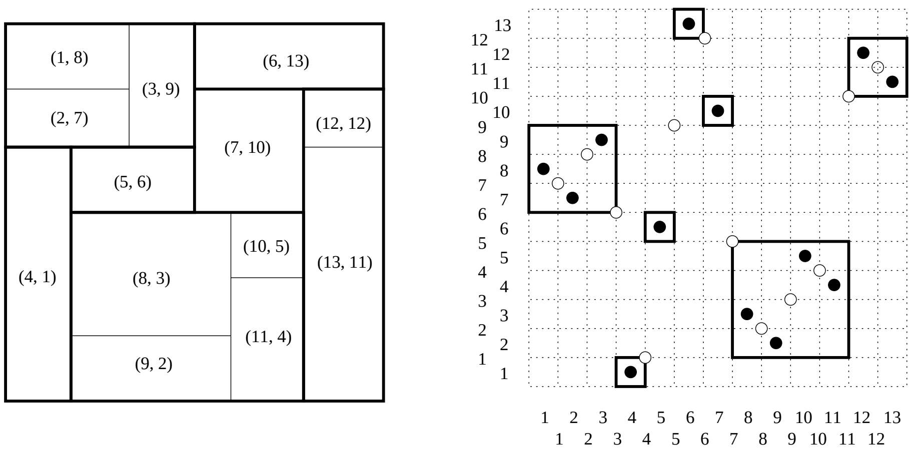


Figure 21: Maximal F-blocks in floorplans and in permutations.

Therefore, such blocks in the graph of ρ will be also called ascending (respectively, descending) F-blocks. Fig. 21 shows a floorplan P with maximal F-blocks denoted by bold lines, and the F-blocks in the permutation $R(P)$ (the graph of $S(P)$ is also shown).

Let $F_1, F_2, \dots$ be all the maximal F-blocks in the graph of ρ (ordered from left to right). For $i \geq 1$, let $[y_i, y_i']$ be the interval of values $\rho(j)$ occurring in F_i, and define $d_i := +$ if F_i is ascending, and $d_i := -$ if F_i is descending (d_i is left undefined if F_i has size 1 or 2). The *F-structure* of ρ is the sequence $\hat{F}_1, \hat{F}_2, \dots$, where $\hat{F}_i = ([y_i, y_i'], d_i)$. For example, the F-structure of the permutation in Fig. 21 is

$$([7,9],+)\,,\ \ ([1])\,,\ \ ([6])\,,\ \ ([13])\,,\ \ ([10])\,,\ \ ([2,5],+)\,,\ \ ([11,12])\,.$$

Theorem 4.4. *Let P_1 and P_2 be two floorplans with n segments. Then $S(P_1) = S(P_2)$ if and only if $R(P_1)$ and $R(P_2)$ have the same F-structure.*

In other words, $S(P_1) = S(P_2)$ if and only if $R(P_1)$ and $R(P_2)$ may be obtained from each other by replacing some F-blocks $F_1, F_2, \dots$ with, respectively, F-blocks $F_1', F_2', \dots$, where F_i is S-equivalent to F_i' for all i.

Proof. The "if" direction is easy to prove. Assume $R(P_1)$ and $R(P_2)$ have the same F-structure. In view of the way one obtains $S(P)$ from $R(P)$ (Theorem 4.2), we have $S(P_1) = S(P_2)$. Observe that inside a maximal F-block of $R(P)$, the points of $S(P)$ lie on the diagonal (in the ascending case) or the anti-diagonal (in the descending case).

In order to prove the "only if" direction, we will first relate, for a point of $S(P)$, the fact of being inside a maximal F-block to the property of being weak. (Recall that a point N_i in the graph of $S(P)$ is weak if it has at most one neighbor in each of the directions NW, NE, SE, SW, and strong otherwise.) If a maximal F-block of $R(P)$ occupies the area $[x, x'] \times [y, y']$, then the point $N_i = (i, j)$ is inside this block if $x \leq i < x'$ and $y \leq j < y'$. For example, in Fig. 21 six points in the graph of $S(P)$ (the white points in the combined diagram) are inside a maximal F-block: $(1,7)$, $(2,8)$, $(8,2)$, $(9,3)$, $(10,4)$ and $(12,11)$. Observe that the notion of "being inside" a maximal F-block is *a priori* relative to $R(P)$. The following proposition shows that it is an intrinsic notion, depending on $S(P)$ only.

Lemma 4.5. *Let N_i be a point in the graph of $\sigma = S(P)$. Then N_i is inside a maximal F-block of $R(P)$ if and only if it is a weak point of $S(P)$.*

This lemma is proved in Appendix B, and the rest of the theorem in Appendix C. □

5 Counting (2-14-3, 3-41-2)-avoiding permutations

It follows from Theorem 4.4 that S-permutations of size n are in bijection with Baxter permutations of size $n+1$ in which all maximal ascending F-blocks are increasing (that is, order isomorphic to a permutation of the form $123\dots m$), and all maximal descending F-blocks of size at least 3 are decreasing. A Baxter permutation that does not satisfy these conditions has at least one *improper pair*.

Definition 5.1. Let ρ be a Baxter permutation. Two points of the diagram of ρ that lie in adjacent rows and columns form an *improper pair* if they form a descent in a maximal ascending F-block, or an ascent in a maximal descending F-block of size at least 3.

This definition is illustrated in Fig. 22. Observe that a point belongs to at most one improper pair. In particular, a permutation of size $n+1$ has at most $\lfloor \frac{n+1}{2} \rfloor$ improper pairs.

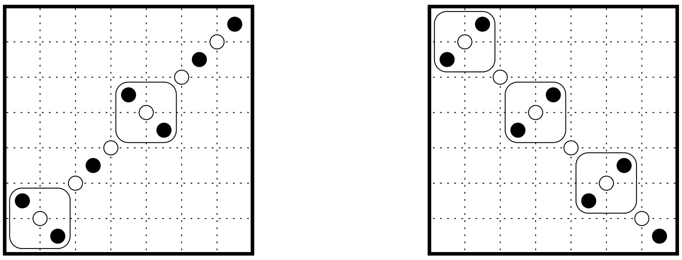


Figure 22: Improper pairs in maximal F-blocks.

Proposition 5.2. *Let*

$$b_n = \sum_{m=0}^{n} \frac{2}{n(n+1)^2} \binom{n+1}{m} \binom{n+1}{m+1} \binom{n+1}{m+2}$$

be the number of Baxter permutations of size n (see [16]*). The number a_n of* (2-14-3, 3-41-2)*-avoiding permutations of size n is*

$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i}.$$

Proof. We have just explained that (2-14-3, 3-41-2)-avoiding permutations of size n are in bijection with Baxter permutations of size $n+1$ having no improper pair. By the inclusion-exclusion principle,

$$a_n = \sum_{i \geq 0} (-1)^i b_{n+1,i},$$

where $b_{n+1,i}$ is the number of Baxter permutations of size $n+1$ with i marked improper pairs. Let ρ be such a permutation, and contract every marked improper pair into a single (marked) point: this gives a Baxter permutation ρ' of size $n+1-i$, with i marked points.

Observe that if two points of ρ are in the same maximal F-block, then their images, after contraction, are in the same maximal F-block of ρ'. (The converse is false: the permutation 1342 has two maximal F-blocks and one improper pair (consisting of the values 34). By contracting it, one obtains the permutation 132, which is an F-block.)

We claim that each Baxter permutation of size $n+1-i$ with i marked points is obtained exactly once in our construction, and that the unique way to expand each marked point into an improper pair is the following:

- if the marked point lies on the diagonal of an ascending maximal F-block (of size ≥ 1), replace it by a descending pair of adjacent points,
- if the marked point lies on the anti-diagonal of a descending maximal F-block of size ≥ 2, replace it by an ascending pair of adjacent points,
- otherwise, observe that the block has size at least 3; if it is ascending (resp. descending), and the marked point does not lie on the diagonal (resp. anti-diagonal), replace it by an ascending (resp. descending) pair of adjacent points.

Details are left to the reader.

This construction implies that the number of Baxter permutations of size $n+1$ having i marked improper pairs is $b_{n+1,i} = \binom{n+1-i}{i} b_{n+1-i}$, and the proposition follows. $\square$

Remarks

1. Let $A(t)$ be the generating function of (2-14-3, 3-41-2)-avoiding permutations, and let $B(t)$ be the generating function of (non-empty) Baxter permutations. The above result can be rewritten as

$$A(t) = \sum_{k\geq 0} t^k (1-t)^{k+1} b_{k+1} = \frac{1}{t} B(t(1-t)). \tag{1}$$

Observe that $t(1-t) = s$ is equivalent (in the world of formal power series) to $t = C(s)$, where $C(s) = \frac{1-\sqrt{1-4s}}{2}$ is the (shifted) generating function of Catalan numbers. Hence,

$$B(s) = C(s)A(C(s)) = \sum_{k\geq 0} a_k C(s)^{k+1}. \tag{2}$$

This can be interpreted as follows. Consider a Baxter permutation. As in the proof of Proposition 5.2, contract its improper pairs. If the resulting permutation (which is also a Baxter pemutation) has again improper pairs, contract them, and repeat this process until a Baxter permutation without improper pairs is obtained.

This process can be reversed by starting with a Baxter permutation without improper pairs and splitting, at each step, some point into an improper pair. It can be checked that the generating function of permutations obtained from a single point is $C(s)$, and therefore, the generating function of Baxter permutations produced from any fixed Baxter permutation of size $k+1$ without improper pairs is $C(s)^{k+1}$, which implies (2). The details are left to the reader.

2. Our first proof of Proposition 5.2 used a generating tree of (2-14-3, 3-41-2)-avoiding permutations, obtained by inserting/deleting the rightmost entry. We had to consider separately three types of permutations, and our trees involved (like the generating tree of Baxter permutations [11]) two integer labels. We thus obtained a system of three equations with two *catalytic variables*, which was reminiscent of the corresponding equation for Baxter permutations, and were finally able to relate both families of permutations through this system. The advantage of this method was to be completely independent from the rest of the paper, and in particular from Theorem 4.4. It also allowed us to refine (1) by

finding the counterparts in (2-14-3, 3-41-2)-avoiding permutations of the numbers of left-to-right and right-to-left maxima in Baxter permutations (but not of the descent number). However, this proof was much longer, and also less combinatorial.

3. The form of a_n and b_n implies that $A(t)$ and $B(t)$ are *D-finite*, that is, satisfy a linear differential equation with polynomial coefficients [29, 30]. In fact,

$$-12t + 6(1-2t)B(t) - 2t\left(-3+14t+8t^2\right)B'(t) - t^2(t+1)(8t-1)B''(t) = 0$$

and

$$\begin{aligned} 12(t-1)(2t-1)^3 + (104t - 338t^2 + 512t^3 - 294t^4 - 110t^5 + 192t^6 - 48t^7 - 12)A(t) \\ - 2t(t-1)\left(40t^6 - 128t^5 + 89t^4 + 53t^3 - 88t^2 + 35t - 4\right)A'(t) \\ - t^2(2t-1)\left(8t^2-8t+1\right)\left(t^2-t-1\right)(t-1)^2 A''(t) = 0. \end{aligned}$$

This implies that the asymptotic behavior of the numbers a_n and b_n can be determined almost automatically (see for instance [21, Sec. VII.9]). For Baxter permutations, it is known [35] that $b_n \sim 8^n n^{-4}$ (up to a multiplicative constant, which can be determined thanks to standard techniques [32]). For a_n, we find $a_n \sim (4+2\sqrt{2})^n\, n^{-4}$.

3. For $1 \le n \le 30$, the number of (2-14-3, 3-41-2)-avoiding permutations of $[n]$ is given in the following table, which we have sent to the OEIS [33, A214358].

1	1668	25274088	709852110576	27277772831911348
2	7744	135132886	4053103780006	161762725797343554
6	37182	732779504	23320440656376	963907399885885724
22	183666	4023875702	135126739754922	5769548815574513550
88	929480	22346542912	788061492048436	34679563373252224012
374	4803018	125368768090	4623591001082002	209275178482957838142

6 The case of guillotine floorplans

In this section we study the restriction of the map S to an important family of floorplan called *guillotine floorplans* [15, 25, 39].

Definition 6.1. A floorplan P is a *guillotine floorplan* (also called *slicing floorplan* [27]) if either it consists of just one rectangle, or there is a segment in P that extends from one side of the boundary to the opposite side, and splits P into two guillotine floorplans.

The restriction of the map R to guillotine floorplans induces a bijection between R-equivalence classes of guillotine floorplans and *separable* permutations (defined below) [1]. Here, we first characterize permutations that are obtained as S-permutations of guillotine floorplans, and then enumerate them.

6.1 Guillotine floorplans and separable-by-point permutations

A nonempty permutation σ is *separable* if it has size 1, or its graph can be split into two nonempty blocks H_1 and H_2, which are themselves separable. Then, either all the points in H_1 are to the SW of all the points of H_2 (then σ, as a separable permutation, has an *ascending structure*), or all the points in H_1 are to the NW of all the points of H_2 (then σ, as a separable permutation, has a *descending structure*). Separable permutations are known to coincide with (2-4-1-3, 3-1-4-2)-avoiding permutations [10]. In particular, they form a subclass of Baxter permutations. The number g_n of separable permutations of $[n]$ is the $(n-1)$st *Schröder number* [33, A006318], and the associated generating function is:

$$G(t) := \sum_{n\geq 1} g_n t^n = \frac{1-t-\sqrt{1-6t+t^2}}{2}. \tag{3}$$

Definition 6.2. A permutation σ of $[n]$ is *separable-by-point* if it is empty, or its graph can be split into three blocks H_1, H_2, H_3 such that

- H_2 consists of one point N,
- H_1 and H_3 are themselves separable-by-point (thus, they may be empty), and
- either all the points of H_1 are to the SW of N, and all the points of H_3 are to the NE of N (then σ has an *ascending structure*), or all the points of H_1 are to the NW of N and all the points of H_3 are to the SE of N (then σ has a *descending structure*).

The letter N refers to the fact that we have denoted by N_i the point $(i, \sigma(i))$ of an S-permutation σ. Observe that N corresponds to a fixed point of σ if σ is ascending, and to a point such that $\sigma(i) = n+1-i$ is σ is descending and has size n.

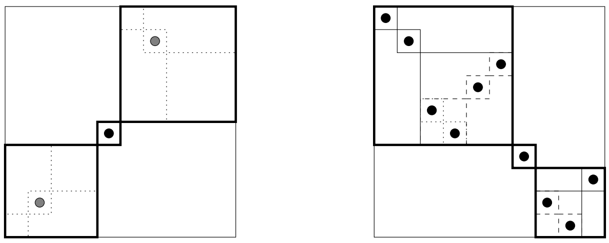


Figure 23: Separable-by-point permutations.

An example is shown in Fig. 23. For $n \leq 3$, all permutations are separable-by-point. Clearly, a (nonempty) separable-by-point permutation is separable. The permutations 2143 and 3412 are separable, but not separable-by-point. The following result characterizes separable-by-point permutations in terms of forbidden patterns. In particular, it implies that these permutations are S-permutations.

Proposition 6.3. *Let σ be a permutation of $[n]$. Then σ is separable-by-point if and only if it is* (2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2)*-avoiding.*

Proof. Assume that σ is separable-by-point. Then it is separable, and therefore avoids 2-4-1-3 and 3-1-4-2. Assume for the sake of contradiction that σ contains an occurrence of 2-14-3, corresponding to the points N_i, N_j, N_{j+1} and N_k, and has a minimal size for this property. Then the points forming the pattern must be spread in at least two of the three blocks. This forces σ to have an ascending structure, with N_i and N_j in one block, N_{j+1} and N_k in the following one (because N_j and N_{j+1} are adjacent). But this is impossible as the central block of σ contains a unique point. Similarly one shows that σ avoids 3-41-2.

Conversely, we argue by induction on the size of σ. Let σ be a (2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2)-avoiding permutation of $[n]$. For $n \leq 3$ there is nothing to prove. Let $n \geq 4$. Since σ is (2-4-1-3, 3-1-4-2)-avoiding, it is separable. Assume without loss of generality that σ (as a separable permutation) has an ascending structure: the first block is $[1,i] \times [1,i]$, the second block is $[i+1,n] \times [i+1,n]$ where $1 \leq i < n$. If $\sigma(i) \neq i$ and $\sigma(i+1) \neq i+1$, then $\sigma^{-1}(i), i, i+1, \sigma^{-1}(i+1)$ form a forbidden pattern 2-14-3. Thus, $\sigma(i) = i$ or $\sigma(i+1) = i+1$, and one obtains a three-block decomposition of σ by choosing for the central block N one of these two fixed points. The remaining two blocks avoid all four patterns, and, therefore are separable-by-point by the induction hypothesis. Then so is σ. □

Theorem 6.4. *A floorplan P is guillotine if and only if $S(P)$ is separable-by-point.*

Proof. Let P be a guillotine floorplan. We argue by induction on the size of P. If P consists of a single rectangle, then $S(P)$ is the empty permutation, and is separable-by-point. Otherwise, consider a segment that splits P into two rectangles. Assume that this segment is I_i (that is, the ith segment in the $\nwarrow$ order) and that it is vertical. All the segments to the left (respectively, right) of I_i come before (respectively, after) I_i in the $\nwarrow$ and $\swarrow$ orders. Consequently:

– I_i is also the ith segment in the $\swarrow$ order, so that $N_i = (i,i)$,

– by Observation 3.3, all the points of the graph of σ that correspond to segments located to the left (respectively, right) of I_i are to the SW (respectively, NE) of N_i.

Thus, we have three blocks H_1, H_2 and $H_3$with an ascending structure. The blocks H_1 and H_3 are the S-permutations of the two parts of P, which are themselves guillotine: by the induction hypothesis, H_1 and H_3 are separable-by-point. Thus $S(P)$ is separable-by-point with an ascending structure. Similarly, if I_i is horizontal, we obtain a separable-by-point permutation with a descending structure.

Conversely, assume that $\sigma := S(P)$ is separable-by-point. We will prove by induction on the size n of σ that P is a guillotine floorplan.

The claim is clear for $n = 1$. For $n > 1$, assume without loss of generality that σ has an ascending structure. Let $H_2 = \{(i,i)\}$ be the second block in a decomposition of σ. Then for $j < i$, we have $I_j \leftarrow I_i$, and for $j > i$, we have $I_i \leftarrow I_j$. Therefore, if I_i is vertical, it has no below- or above-neighbors, and thus extends from the lower to the upper side of the boundary. The two sub-floorplans of P correspond respectively to the blocks H_1 and H_3: hence they are guillotine by the induction hypothesis. Suppose now that I_i is horizontal. Then we have $\sigma(i-1) = i-1$ (if $i > 1$) and $\sigma(i+1) = i+1$ (if $i < n$), since otherwise I_i has several left-neighbors or several right-neighbors (Observation 3.3), which never holds for a horizontal segment. Assume without loss of generality that $i > 1$. Then another block decomposition of σ is obtained with the central block $H'_2 = \{(i-1, i-1)\}$, corresponding to the vertical segment I_{i-1}. The previous argument then shows that P is guillotine. $\square$

6.2 Enumeration

In this section we enumerate S-equivalence classes of guillotine floorplans, or equivalently, separable-by-point permutations.

Proposition 6.5. *For $n \geq 1$, let g_n be the number of separable permutations of size n, and let $G(t)$ the associated generating function, given by* (3)*. The number h_n of separable-by-point permutations of size n is*

$$h_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} g_{n+1-i}.$$

Equivalently, the generating function of separable-by-point permutations is

$$H(t) = \sum_{n\geq 0} h_n t^n = \sum_{n\geq 0} t^n (1-t)^{n+1} g_{n+1} = \frac{1}{t} G(t(1-t)) = \frac{1-t+t^2-\sqrt{1-6t+7t^2-2t^3+t^4}}{2t}.$$

Proof. Recall that the R-permutations associated with guillotine floorplans are the separable permutations, and return to the proof of Proposition 5.2. The contraction/expansion of points used in this proof preserves separability, so that we can apply the same argument, which yields directly the proposition. $\square$

Remarks

1. The first values are $1, 1, 2, 6, 20, 70, 254, 948, 3618, 14058$. This sequence [33, A078482] also enumerates (2-4-3-1, 3-2-4-1, 2-4-1-3, 3-1-4-2)-avoiding permutations (or permutations sortable by a stack of queues), as found by Atkinson and Stitt [5, Thm. 17].
2. Our proof is trivial after the (much harder) proof of Proposition 5.2, but a more direct proof is possible using Definition 6.2. Indeed, denoting by $H_a(t)$ the generating function of ascending separable-by-point permutations of size at least 2, it is easy to see that

$$H(t) = 1 + t + 2H_a(t) \qquad \text{and} \qquad H_a(t) = \frac{t(1+H_a(t))^2}{1-t(1+H_a(t))} - t,$$

which gives the above expression of $H(t)$.
3. Using the *transfer theorems* from [21, Sec. VI.4], we can find the asymptotic behavior of the numbers h_n:

$$h_n \sim \left(\frac{2}{1-\sqrt{8\sqrt{2}-11}} \right)^n n^{-3/2},$$

up to a multiplicative constant.
4. See [4] for a generalization of Proposition 6.5 to d-dimensional guillotine partitions.

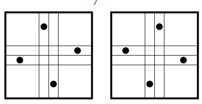
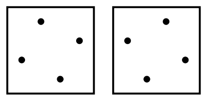
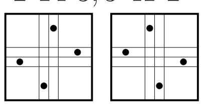
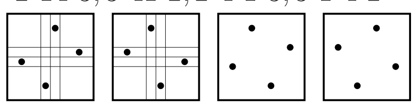
7 Final remarks

We have shown that many analogies exist between R- and S-equivalence. However, there also seems to be one important difference. Looking at Fig. 1 suggests that one can transform a floorplan into an R-equivalent one by some continuous deformation. In other words, R-equivalence classes appear as geometric planar objects. This is confirmed by the papers [9, 20, 23], which show that *bipolar orientations of planar maps* provide a convenient geometric description of R-equivalence classes of floorplans. However, S-equivalence is a coarser relation, and two S-equivalent floorplans may look rather different (Figs. 10 and 20). It would be interesting to find a class of geometric objects that captures the notion of S-equivalence classes, as bipolar orientations do for R-equivalence classes.

In Section 5 we have established a simple enumerative connection, involving Catalan numbers, between Baxter permutations and (2-14-3,3,41-2)-avoiding permutations (Proposition 5.2). Is there a direct combinatorial proof of (2), that would not use Theorem 4.4, nor the heavy generating trees alluded to in the remarks that follow Proposition 5.2? Recall that $C(s)$ is related to pattern avoiding permutations, since it counts τ-avoiding permutations, for any pattern τ of size 3.

Another question, raised by one of the referees, would be to determine the S-permutations corresponding to (2-4-1-3, 3-14-2)-avoiding permutations, which occur for instance in [18].

We conclude with a summary of the enumerative results obtained in [1] for R-equivalence classes and in the present paper for S-equivalence classes.

	All floorplans	Guillotine floorplans
R-equivalence classes	**Forbidden patterns:** 2-41-3, 3-14-2  **Enumerating sequence:** $1, 2, 6, 22, 92, 422, 2074, 10754, \ldots$ (Baxter numbers[33, A001181]) **Growth rate:** 8	**Forbidden patterns:** 2-4-1-3, 3-1-4-2  **Enumerating sequence:** $1, 2, 6, 22, 90, 394, 1806, 8558, \ldots$ (Schröder numbers [33, A006318]) **Growth rate:** $3 + 2\sqrt{2} \approx 5.8284$
S-equivalence classes	**Forbidden patterns:** 2-14-3, 3-41-2  **Enumerating sequence:** $1, 2, 6, 22, 88, 374, 1668, 7744, \ldots$ ([33, A214358]) **Growth rate:** $4 + 2\sqrt{2} \approx 6.8284$	**Forbidden patterns:** 2-14-3, 3-41-2, 2-4-1-3, 3-1-4-2  **Enumerating sequence:** $1, 2, 6, 20, 70, 254, 948, 3618, \ldots$ ([33, A078482]) **Growth rate:** $\frac{2}{1-\sqrt{8\sqrt{2}-11}} \approx 4.5465$

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A Proof of Proposition 3.12, Paragraph D

We prove that, given a permutation σ, (any) floorplan P obtained by the construction described in Paragraphs A-B of the proof satisfies $S(P) = \sigma$.

In order to prove this claim, we will show that for all $1 \leq i < n$, the segment K_{i+1} is the immediate successor of K_i in the $\nwarrow$ order, and that $K_{\sigma^{-1}(i+1)}$ is the immediate successor of $K_{\sigma^{-1}(i)}$ in the $\swarrow$ order.

Let us first prove that the first statement implies the second. Let σ' be obtained by applying a quarter-turn rotation ρ to σ in counterclockwise direction. Let us denote by K'_i the segment of $P' = \rho(P)$ containing the point $(i, \sigma'(i))$. By Observation 3.13, the floorplan P' is associated with σ' by our construction. That is, $\rho(K_{\sigma^{-1}(i)}) = K'_{n+1-i}$. By assumption, $K'_{n+1-i} = \rho(K_{\sigma^{-1}(i)})$ follows $K'_{n-i} = \rho(K_{\sigma^{-1}(i+1)})$ for the $\nwarrow$ order in P'. Applying the quarter turn clockwise rotation ρ^{-1} and the second remark following Proposition 2.12, this means that $K_{\sigma^{-1}(i+1)}$ follows $K_{\sigma^{-1}(i)}$ for the $\swarrow$ order in P.

Thus we only need to prove that K_{i+1} is the immediate successor of K_i in the $\nwarrow$ order. By Observation 2.10, the immediate successor of a horizontal (respectively, vertical) segment I in the $\nwarrow$ order is $\mathrm{R}(I)$, $\mathrm{LVB}(I)$ or $\mathrm{LHB}(I)$ (respectively, $\mathrm{B}(I)$, $\mathrm{UHR}(I)$ or $\mathrm{UVR}(I)$),[9] depending on the existence of these segments and the type of joins between them. There are 8 cases to consider, depending on whether $\sigma(i) < \sigma(i+1)$ or $\sigma(i) > \sigma(i+1)$, and on the directions of K_i and K_{i+i}.

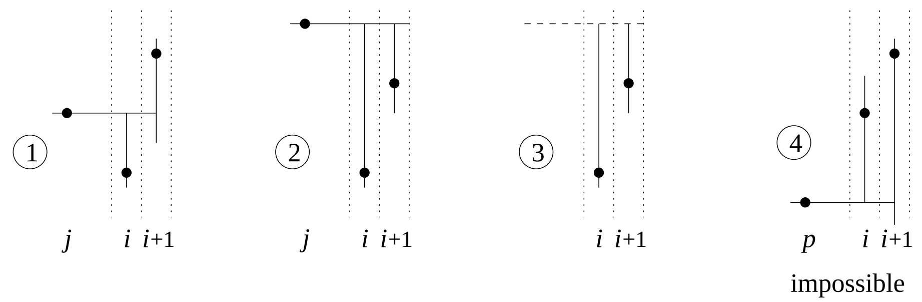

Figure 24: The first case.

Case 1: $\sigma(i) < \sigma(i+1)$, K_i and K_{i+1} are vertical.

Assume that N_j bounds K_i from above. Then, as shown in Paragraph C.1 above, K_j is horizontal; furthermore, K_i and K_j have a $\top$ join at the point $(i, \sigma(j))$. In particular, the rightmost point of K_j has abscissa at least $i+1$.

If $\sigma(j) < \sigma(i+1)$, then N_{i+1} bounds K_j from the right. There is a $\dashv$ join of K_j and K_{i+1} at the point $(i+1, \sigma(j))$ (Fig. 24(1)).

If $\sigma(j) > \sigma(i+1)$, then N_j bounds K_{i+1} from above and there is a $\top$ join of K_j and K_{i+1} (Fig. 24(2)).

If N_j does not exist and K_i reaches the upper side of the boundary, then no point can bound K_{i+1} from above, and, thus, K_{i+1} reaches the boundary as well (Fig. 24(3)).

In all these cases, it is readily seen that K_{i+1} is $\mathrm{UVR}(K_i)$.

[9] This notation is defined before Observation 2.10.

By Observation 2.10, $\mathrm{UVR}(K_i)$ is the successor of K_i, unless $\mathrm{UHR}(K_i)$ does not exist, $\mathrm{B}(K_i) \coloneqq K_p$ exists and its join with $\mathrm{UVR}(K_i)$ is of type $\dashv$ (Fig. 24(4)). But this would mean that $p < i$, and the positions of N_i and N_p would then contradict Observation 3.14.

Case 2: $\sigma(i) < \sigma(i+1)$, K_i *is vertical and* K_{i+1} *is horizontal.*

The point N_i bounds K_{i+1} from the left. Therefore, there is a $\vdash$ join of K_i and K_{i+1} at the point $(i, \sigma(i+1))$, and K_{i+1} is a horizontal right-neighbor of K_i. Moreover, if K_k is another horizontal right-neighbor of K_i, then $\sigma(k) < \sigma(i+1)$: otherwise N_i cannot be a SW-neighbor of N_k (Fig. 25, left). Therefore, $K_{i+1} = \mathrm{UHR}(K_i)$.

By Observation 2.10, $\mathrm{UHR}(K_i)$ is the successor of K_i, unless $\mathrm{UVR}(K_i) \coloneqq K_p$ exists (Fig. 25, right). If this were the case, K_p and K_{i+1} would have a $\perp$ join, and the position of N_{i+1} and N_p would then be incompatible with Observation 3.14.

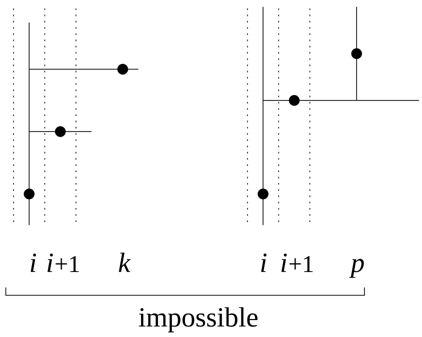

Figure 25: The second case.

Case 3: $\sigma(i) < \sigma(i+1)$, K_i *is horizontal and* K_{i+1} *is vertical.*

We claim that this case follows from the previous one. Let σ' be obtained by applying a half-turn rotation ρ to (the graph of) σ. By Observation 3.13, the floorplan $P' = \rho(P)$ is associated with σ'. The points and segments $\rho(N_i)$, $\rho(N_{i+1})$, $\rho(K_i)$, $\rho(K_{i+1})$ in P' are in the configuration described by Case 2, with $\rho(N_{i+1})$ to the left of $\rho(N_i)$. Consequently, $\rho(N_i)$ is the successor of $\rho(N_{i+1})$ in the $\nwarrow$ order in P'. By the first remark that follows Proposition 2.12, N_{i+1} is the successor of N_i in the $\nwarrow$ order in P.

Case 4: $\sigma(i) < \sigma(i+1)$, K_i *and* K_{i+1} *are horizontal.*

If this case, N_i bounds K_{i+1} from the left. Therefore, K_i must be vertical (see Paragraph C.1 above). Hence, this case is impossible.

Case 5: $\sigma(i) > \sigma(i+1)$, K_i *and* K_{i+1} *are vertical.*

Since K_{i+1} is vertical, N_{i+1} has at most one NW-neighbor, which is then N_i. By Paragraph C.1 above, K_i is then horizontal. Thus this case is impossible.

Case 6: $\sigma(i) > \sigma(i+1)$, K_i *is vertical and* K_{i+1} *is horizontal*

Since the segment K_i is vertical, the point N_i has at most one SE-neighbor, which is then N_{i+1}. Therefore, N_{i+1} bounds K_i from below, and there is a $\perp$ join of K_i and K_{i+1} at the point $(i, \sigma(i+1))$. In particular, $K_{i+1} = \mathrm{B}(K_i)$.

By Observation 2.10, $\mathrm{B}(K_i)$ is the successor of K_i, unless $\mathrm{UHR}(K_i) \coloneqq K_k$ exists (Case (3.2) in Fig. 7), or $\mathrm{UHR}(K_i)$ does not exist, but $\mathrm{UVR}(K_i) \coloneqq K_p$ does and forms with

$\mathrm{B}(K_i)$ a $\perp$ join (Case (4.1) in Fig. 7). In the former case, K_k reaches K_i and thus is bounded by N_i on the left, but then N_i and N_{i+1} are two SW-neighbors of N_k, and K_k cannot be horizontal. In the latter case, K_p and K_{i+1} would form a $\perp$ join, and the positions of N_p and N_{i+1} would contradict Observation 3.14.

Case 7: $\sigma(i) > \sigma(i+1)$, K_i is horizontal and K_{i+1} is vertical.

This case follows from Case 6 by the symmetry argument already used in Case 3.

Case 8: $\sigma(i) > \sigma(i+1)$, K_i and K_{i+1} are horizontal.

The point that bounds K_i from the right, if it exists, lies to the NE of N_{i+1}. Thus the abscissa of the rightmost point of K_i is greater than or equal to the abscissa of the rightmost point of K_{i+1}.

We will show that $K_{i+1} = \mathrm{LHB}(K_i)$. Once this is proved, Observation 2.10 implies that $\mathrm{LHB}(K_i)$ is the successor of K_i, unless $\mathrm{LVB}(K_i)$ does not exist, but $R(K_i)$ exists and forms with K_{i+1} a $\perp$ join (Case (4.2) in Fig. 7). But this would mean that K_{i+1} ends further to the right than K_i, which we have just proved to be impossible.

So let us prove that $K_{i+1} = \mathrm{LHB}(K_i)$. We assume that K_i does not reach the left side of the boundary, and that K_{i+1} does not reach the right side of the boundary (the other cases are proven similarly). Let N_k be the point that bounds K_i from the left, and let N_m be the point that bounds K_{i+1} from the right.

Consider A, the leftmost rectangle whose upper side is contained in K_i. The left side of A is clearly contained in K_k. We claim that the lower side of A is contained in K_{i+1}, and that the right side of A is contained in K_m. Note that this implies $K_{i+1} = \mathrm{LHB}(K_i)$.

Let K_p (respectively, K_q) be the segment that contains the lower (respectively, right) side of A. Clearly, $q > k$. If $q < i$, then K_q is a vertical below-neighbor of K_i, and the positions of N_q and N_i contradict Observation 3.14. Therefore, $q > i+1$.

Consider now the segment K_p. Clearly, $\sigma(p) \geq \sigma(i+1)$. One cannot have $p > i+1$: otherwise N_{i+1} (or a point located to the right of N_{i+1}) would bound K_p from the left, and K_p would not reach K_k. One cannot have either $p < i$: otherwise N_i (or a point located to the left of N_i) would bound K_p from the right, and K_p would not reach K_q. Since $p \neq i$, we have proved that $p = i+1$, and K_k is bounded by N_{i+1} from below.

Finally, K_q coincides with K_m: otherwise, $q < m$, and K_q is a vertical above-neighbor of K_{i+1}; however, in this case N_q would bound K_{i+1} from the right, and K_{i+1} would not reach K_m.

We have thus proved that $K_{i+1} = \mathrm{LHB}(K_i)$, and this concludes the study of this final case, and the proof of Proposition 3.12.

B Proof of Lemma 4.5

Let $N_i = (i, j)$ be inside a maximal F-block of $R(P)$. Assume for the sake of contradiction that N_i is strong, and for instance, has several NE-neighbors. Let N_k be the leftmost NE-neighbor of N_i, and let N_ℓ be the lowest NE-neighbor of N_i. If $k > i+1$, then $\sigma(k-1) < \sigma(i)$, and, therefore, $i, k-1, k, \ell$ form a forbidden pattern 2-14-3. Thus $k = i+1$. Symmetrically, $\sigma(\ell) = j+1$ (Fig. 26). Note also that $\sigma(i+1) > j+1$ and $\sigma^{-1}(j+1) > i+1$. Since the points of $S(P)$ inside an F-block are either on the diagonal or the anti-diagonal of this block, N_i is the highest (and rightmost) point of $S(P)$ inside the maximal F-block that contains it, and this F-block is of ascending type. In particular, either $\rho(i+1) = j+1$, or $\rho(i) = j+1$ and $\rho(i+1) = j$.

Since $\rho(i+1) \leq j+1$ and $\sigma(i+1) \geq j+2$, then $\rho(i+2) \geq j+3$ (Theorem 4.2). Symmetrically, $\rho^{-1}(j+2) \geq i+3$. But then the position of the point $(\rho^{-1}(j+2), j+2)$ is not compatible with the position of N_{i+1}: by Theorem 4.2, there cannot be a point of ρ located to the right of $\rho(i+2)$ and in the rows between those of $\rho(i+1)$ and N_{i+1}. Hence N_i cannot have several NE-neighbors. Symmetric statements hold for the other directions, and N_i is a weak point.

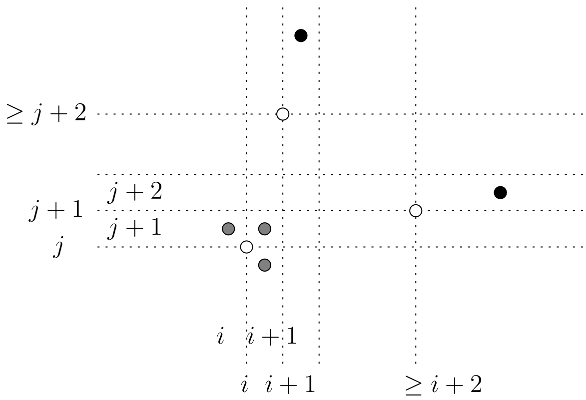

Figure 26: Some points of the combined diagram of ρ and σ. The grey points represent the two possibilities $\rho(i+1) = j+1$, or $\rho(i) = j+1$ and $\rho(i+1) = j$.

Now let $N_i = (i, j)$ be a point of the graph of σ, not inside a maximal F-block. Assume without loss of generality that ρ has an ascent at i: $\rho(i) \leq j < \rho(i+1)$ and (by Theorem 4.2) $\rho^{-1}(j) \leq i < \rho^{-1}(j+1)$. We shall show that N_i has several SW-neighbors or several NE-neighbors. We denote $M_i = (i, \rho(i))$.

First, if $\rho(i) = j$ and $\rho(i+1) = j+1$, then M_i and M_{i+1} form an F-block, and N_i is inside this block. Therefore, either $\rho(i) \neq j$ or $\rho(i+1) \neq j+1$, and we may assume without loss of generality that $\rho(i) \neq j$; hence $\rho(i) < j$ and $\rho^{-1}(j) < i$. Then it follows from the definition of Baxter permutations that $\rho(i-1) \leq j$ (otherwise, there is an occurrence of 2-41-3 at positions $\rho^{-1}(j), i-1, i, \rho^{-1}(j+1)$). Consequently, we have $\sigma(i-1) \leq j-1$. Symmetrically, $\rho^{-1}(j-1) \leq i$ and $\sigma^{-1}(j-1) \leq i-1$. There are two possibilities: either $\sigma(i-1) < j-1$ and $\sigma^{-1}(j-1) < i-1$, or $\sigma(i-1) = j-1$. In the former case, N_{i-1} and $N_{\sigma^{-1}(j-1)}$ are two SW-neighbors of N_i, and we have proved that N_i is strong. Let us go on with the latter case, where $\rho(i-1) = j$ and $\rho(i) = j-1$.

If $\rho(i+1) = j+1$, then M_{i-1}, M_i and M_{i+1} form an F-block, and N_i is inside this block, which contradicts our initial assumption. Otherwise, $\rho(i+1) \neq j+1$, and an argument similar to the one developed just above shows that either N_{i+1} and $N_{\sigma^{-1}(j+1)}$ are two NW-neighbors of N_i, or $\rho(i+1) = j+2$ and $\rho(i+2) = j+1$. In the latter case, M_{i-1}, M_i, M_{i+1} and M_{i+2} form an F-block containing N_i, which contradicts our initial assumption.

C Proof of Theorem 4.4, the "only if" direction

Let σ be a (2-14-3, 3-41-2)-avoiding permutation of size n. Let $\mathcal{B}$ be the set of Baxter permutations whose S-permutation (described by Theorem 4.2) is σ. Lemma 4.5 determines which points of the graph of σ are inside an F-block. These points are organized along the diagonal or anti-diagonal of their blocks. It follows that the location of all non-trivial F-blocks in the graph of ρ, for $\rho \in \mathcal{B}$, and their type (ascending or descending, for blocks of size at least 3), are also determined uniquely. It remains to show that the location of the trivial F-blocks (that is, F-blocks of size 1) is also determined by σ.

Assume for the sake of contradiction that $\mathcal{B}$ contains two distinct permutations ρ_1 and ρ_2. Let i be the abscissa of the leftmost trivial F-block that is not at the same ordinate in the graphs of ρ_1 and ρ_2. Denote $j = \sigma(i)$. By symmetry, we only have to consider two cases: (1) $\rho_1(i) < \rho_2(i) \leq j$; (2) $\rho_1(i) \leq j < \rho_2(i)$.

In the first case (Fig. 27), denote $k = \rho_2(i)$. Consider $\rho_1^{-1}(k)$. By assumption, $\rho_1^{-1}(k) \neq i$. Since $\rho_1(i) < k$ and $\sigma(i) = j \geq k$, we have $\rho_1^{-1}(k) < i$ by Theorem 4.2. However, this is impossible since the F-structures of ρ_1 and ρ_2 coincide to the left of the ith column.

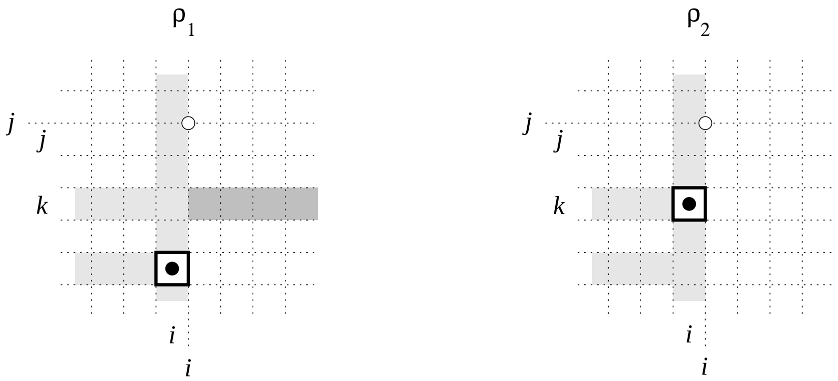

Figure 27: Proof of Theorem 4.4: the case $\rho_1(i) < \rho_2(i) \leq j$.

Consider the second case, $\rho_1(i) \leq j < \rho_2(i)$ (Fig. 28). Since $\rho_1(i) \leq j$ and $\sigma(i) = j$, the areas $[1, i] \times \{j + 1\}$ and $[i + 1, n] \times \{j\}$ are empty in the graph of ρ_1. Similarly, since $\rho_2(i) \geq j+1$, the areas $[1, i] \times \{j\}$ and $[i+1, n] \times \{j+1\}$ are empty in the graph of ρ_2. Since the F-structures of ρ_1 and ρ_2 coincide in $[1, i-1] \times [1, n]$ the areas $[1, i-1] \times \{j, j+1\}$ are empty in the graphs of both permutations. Given that rows cannot be empty, this forces $\rho_1(i) = j$ and $\rho_2(i) = j + 1$ (Fig. 29).

Assume without loss of generality that $\sigma(i + 1) < j$. Since $\rho_1(i) = j$ and $\sigma(i) = j$, we have, by Theorem 4.2, $\rho_1(i + 1) \geq j + 1$. In fact $\rho_1(i + 1) > j + 1$ since otherwise the point $(i, \rho_1(i))$ would not form a trivial F-block. Now, since $\sigma(i+1) < j$, the area $[i+2, n] \times \{j+1\}$ is empty in the graph of ρ_1. This area is also empty in the graph of ρ_2, since $\rho_2(i) = j+1$. Since the F-structures of ρ_1 and ρ_2 coincide in $[1, i-1] \times [1, n]$, the area $[1, i-1] \times \{j+1\}$ is also empty in the graph of ρ_1. Since $\rho_1(i) = j$ and $\rho_1(i+1) > j+1$, we have a contradiction: the whole row $j + 1$ is empty in the graph of ρ_1.

Thus, we have proved that all $\rho \in \mathcal{B}$ have the same F-structure.

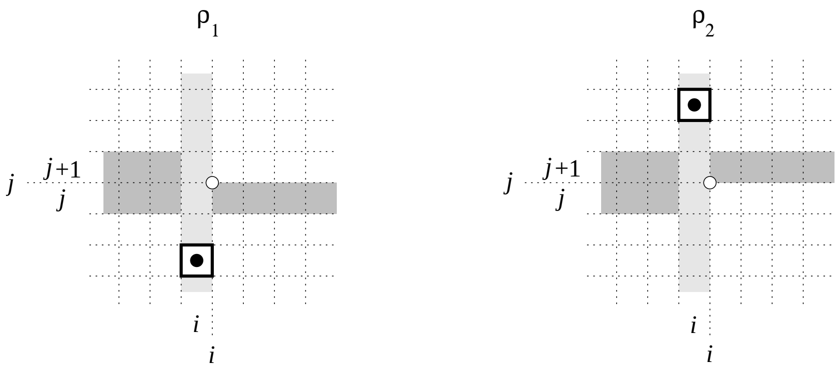

Figure 28: Proof of Theorem 4.4: the case $\rho_1(i) \leq j < \rho_2(i)$.

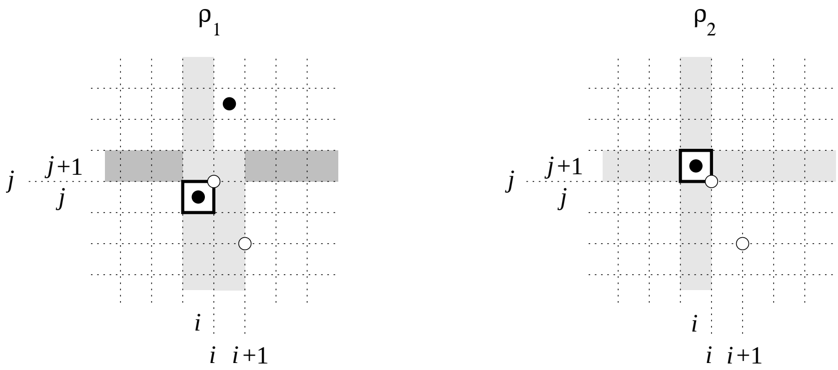

Figure 29: Proof of Theorem 4.4: the case $\rho_1(i) \leq j < \rho_2(i)$, continued.